Swing-Up of a Double, Triple, and Quadruple Pendulum via Nonlinear Normal Modes

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Summary. Identifying the nonlinear normal modes (NNMs) of a double, triple and quadruple pendulum, we found that all NNMs approach homoclinic orbits through the unstable upright equilibrium. The NNMs are computed by numerical continuation of conservative orbits initialized at small-amplitude linear modes around the downward stable equilibrium. Based on this insight we develop a swing-up control scheme bringing the pendula from the downward equilibrium to the upright position. As NNMs collect natural trajectories, sustaining a modal oscillation does not need control. The available control force is used to walk up the NNM, gradually injecting energy, until the homoclinic orbit is approached, and therefore the upright position is reached.

Background on Nonlinear Normal Modes

Nonlinear normal modes (NNMs) serve as one of many tools to analyze the dynamics of nonlinear mechanical conservative systems that have been applied especially in structural dynamics [2] and (soft-) robotics [1]. An NNM can be understood as the generalization of linear modes[1]. Consider a single linear mode. It can be understood as a subspace of the state space where the system oscillates back and forth along a line at a fixed frequency. Each of the modal oscillations is line-shaped, and when varying the energy, the amplitude of the oscillation changes continuously.

When generalizing to nonlinear normal modes, the property of line-shaped oscillations that smoothly change with energy is preserved. However, the lines become curved, may bend with energy and also may change the period time with energy. Conceptually, we can see a nonlinear normal mode as a family of brake orbits[2], where the orbit continuously changes with energy. As the NNMs emerge from the linearization around an equilibrium of a nonlinear system, there will be locally at least n NNMs for an n degrees of freedom system.

Model

Fig. 1 illustrates a sketch of one joint of the n-link pendulum. The joint angle q_i determines the angle between the links $i-1$ and i. Each link is assumed a thin rod of length l_i and total mass m_i uniformly distributed along the rod. The first joint q_1 connects link 1 to a fixed base and after the last link n is no further joint. We denote the end-points on the i-th link $\mathbf{p}_i$. These points are used to plot generators later.

Based on the thin rod model we compute the inertia of the links and the total kinetic energy is given by the sum of the individual links in motion. The kinetic energy is called $\mathbf{T}(\mathbf{q}, \dot{\mathbf{q}})$ and depends on the configuration and velocities. We assume the pendula in gravity. The potential energy is given by the height of the center of masses of the links. We call the potential energy $\mathbf{V}(\mathbf{q})$. Using the Lagrangian $\mathbf{L} = \mathbf{T} - \mathbf{V}$, and applying the Euler-Lagrange equations we obtain the dynamics in robotics standard form

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \mathbf{0}. \tag{1}$$

The total energy is given by $\mathbf{E}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{T}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{V}(\mathbf{q})$. The n-link pendulum is in stable equilibrium when it hangs straight down and all velocities are zero. In the following, we study NNMs around this stable equilibrium.

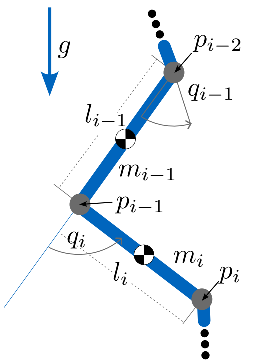

Figure 1: Joint i between links $i-1$ and i.

Results for Double, Triple and Quadruple Pendulum

Let us study the nonlinear normals modes of an exemplary double, triple and quadruple pendulum. All links are identical. We select thin rods of length $l = 0.5\mathrm{m}$ and a mass of $m = 0.662\mathrm{kg}$. Hence, the total length in stretched-out configuration is $1.0\mathrm{m}$ for the double, $1.5\mathrm{m}$ for the triple and $2.0\mathrm{m}$ for the quadruple pendulum, respectively. Using numerical continuation and collocation methods we compute nonlinear normal modes for increasing energy levels. We start the continuation using the linear eigenmodes of the linearized system around the stable downward equilibrium.

For all three models, we observe that the period time of the NNMs goes to infinity when approaching the energy level of maximal potential energy (first row in Fig. 3).

$$\lim_{E \to V_{\max}} T(E) = \infty. \tag{2}$$

The maximal potential energy $V_{\max} = \max_q V(\mathbf{q})$ corresponds to the configuration $\mathbf{q}_{\max}$ where the pendula point straight up. Therefore, we observe that for all three systems, all NNMs approach homoclinic[3] orbits through the unstable upright equilibrium. On the bottom row of Fig. 1 we show generators of the modes in Cartesian space: the curves trace out the position of end-points $\mathbf{p}_i$ of the links at the turning points for each energy level.

[1]*Linear modes:* eigendecomposition of complex linear dynamics into single oscillators of different frequencies in different directions.
[2]*Brake orbit:* periodic back and forth oscillation on a curve with exactly two turning points, where all joints simultaneously come to rest.
[3]*Homoclinic orbit:* limit case of a periodic orbit through an equilibrium approaching infinite period time.

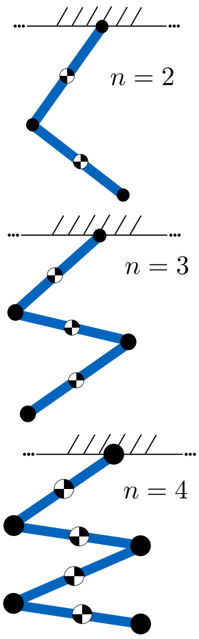

Figure 2: Pendula.

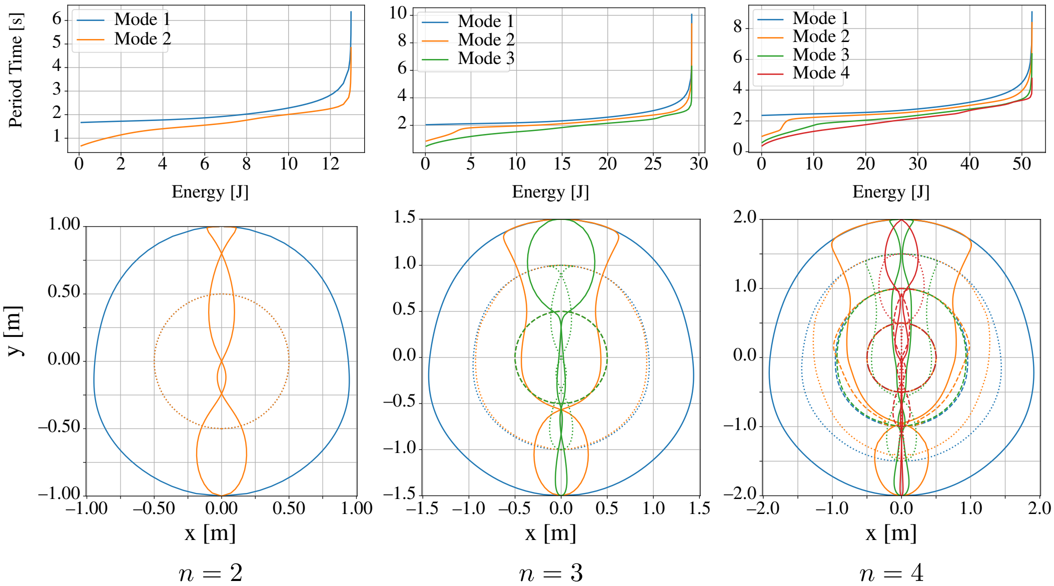

Figure 3: Top row: period times and generators of nonlinear normal modes for double, triple, and quadruple pendulum. Bottom row: generators (collection of all turning points) of the NNMs in Cartesian space. All nonlinear normal modes approach homoclinic orbits.

Application: Swing-Up of Weakly Actuated Pendula

The insight gained from the modal analysis offer the opportunity to develop a swing-up control approach bringing the pendula from the downward stable equilibrium to the upward unstable equilibrium. Because NNMs are natural oscillations, and they smoothly bend with energy the control effort is minimal. The idea is to start with either NNM, initiate a very small amplitude modal oscillation and slowly inject energy walking up the NNM until reaching the homoclinic orbit through the unstable equilibrium. For synthesizing the controller, the problem is separated into an eigenmanifold stabilizer and an energy injection controller and controllers for both objectives are superimposed [3].

Fig. 4 shows this exemplarily for *Mode 1* of the double pendulum. Panes (a) and (b) show the time evolution of joint angles and joint velocities and (c) the required control torques. The control torques are only capped at 10% of the maximal gravity torque, highlighting the feasibility of swing-up using NNMs even with very weak actuators. Pane (d) shows the modal phase (blue) and the energy (orange) over time and (e) shows the energy over modal phase. Finally, (f) visualizes a sequence of orbits during the swing-up, where the blue/orange curves show the paths the points p_i trace out at the respective energy level. The swing-up is also possible via the second mode (g).

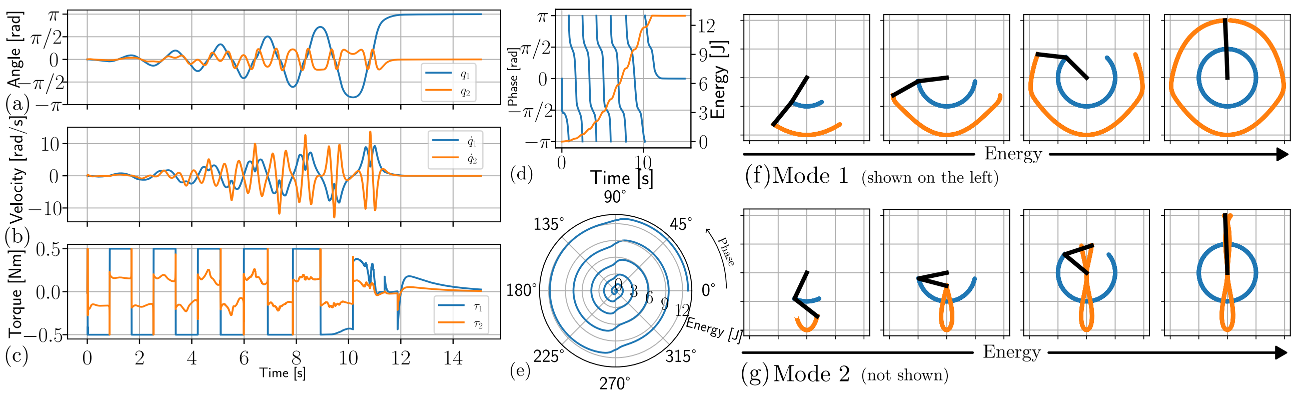

Figure 4: The insight that nonlinear normal modes approach nonlinear normal modes approach homoclinic orbits through the upright position can be used for swing-up control with weak actuators. *(a)* Joint angles over time. *(b)* Joint velocities over time. *(c)* Actuator torques over time. *(d)* Nonlinear phase and energy over time. *(e)* Phase over energy. *(f)* A sequence of stills for the mode used for swing-up. *(g)* A sequence of stills for the second mode. This mode is not shown on the left.

References

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[2] G. Kerschen, M. Peeters, J. Golinval, and A. Vakakis, "Nonlinear normal modes, part I: A useful framework for the structural dynamicist," *Mechanical Systems and Signal Processing,* vol. 23, no. 1, pp. 170-194, 2009.

[3] A. Sachtler, Davide Calzolari, Maximilian Raff, Annika Schmidt, Yannik P. Wotte, Cosimo Della Santina, David C. Remy, and Alin Albu-Schäffer, "Swing-Up of a Weakly Actuated Double Pendulum via Nonlinear Normal Modes," in *European Control Conference (ECC)*, pp. 2392–2398, Stockholm, Sweden, 2024.