STIFFNESS MATRIX METHOD FOR MODELING OF GUIDED WAVES IN MULTILAYERED ANISOTROPIC PLATES: THE DISPERSION CALCULATOR

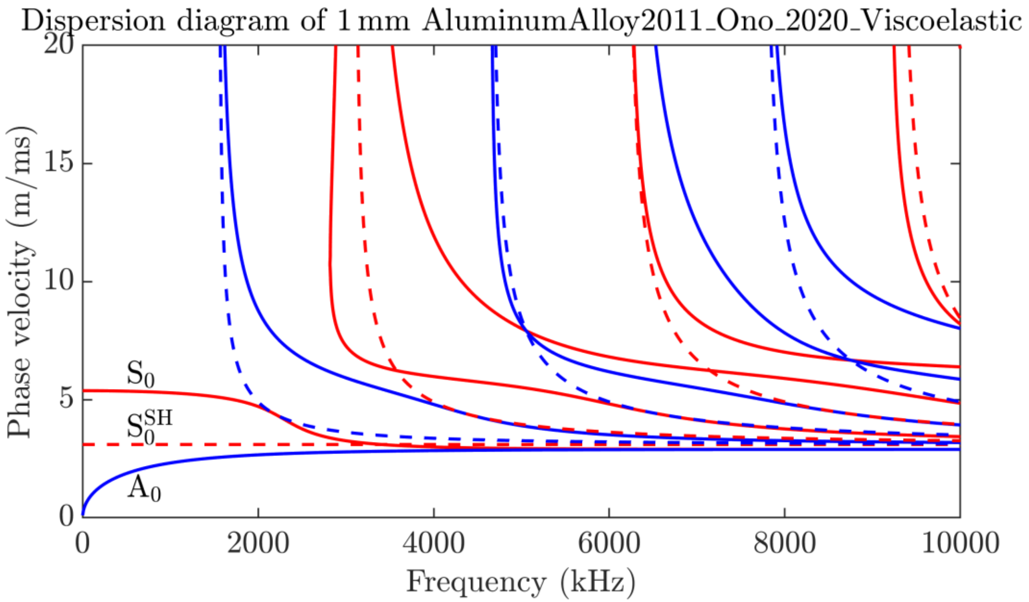
Dr. Armin Huber – 9 July 2024

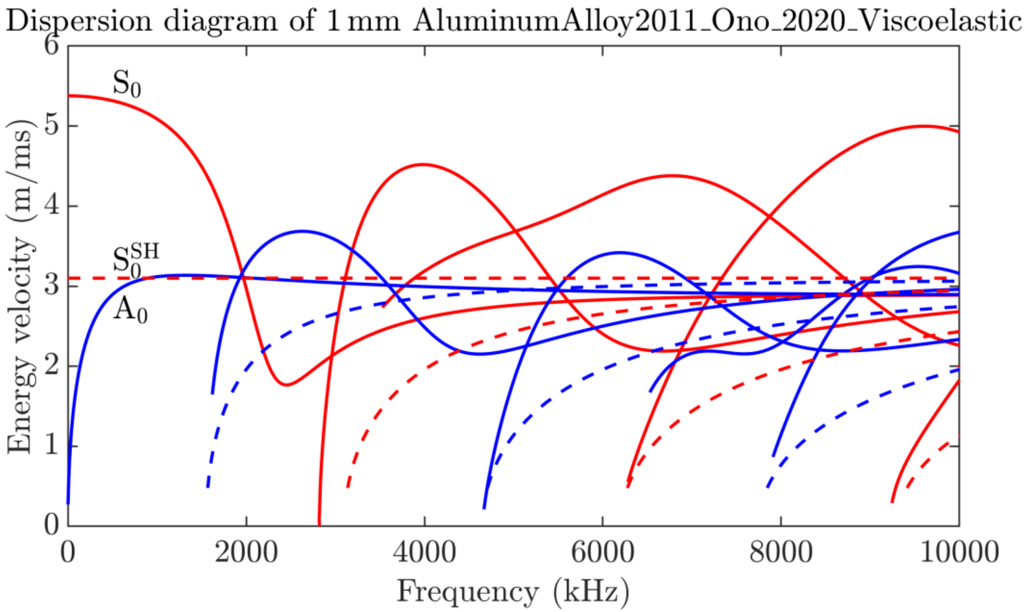
DLR

Outline

- Dispersion diagrams
- Overview of guided wave modeling methods and tools available
- The stiffness matrix method
- Mode family specific dispersion equations
- Live demonstration of the Dispersion Calculator (DC)
- Concluding remarks and future work

Modeling of guided waves → dispersion diagrams

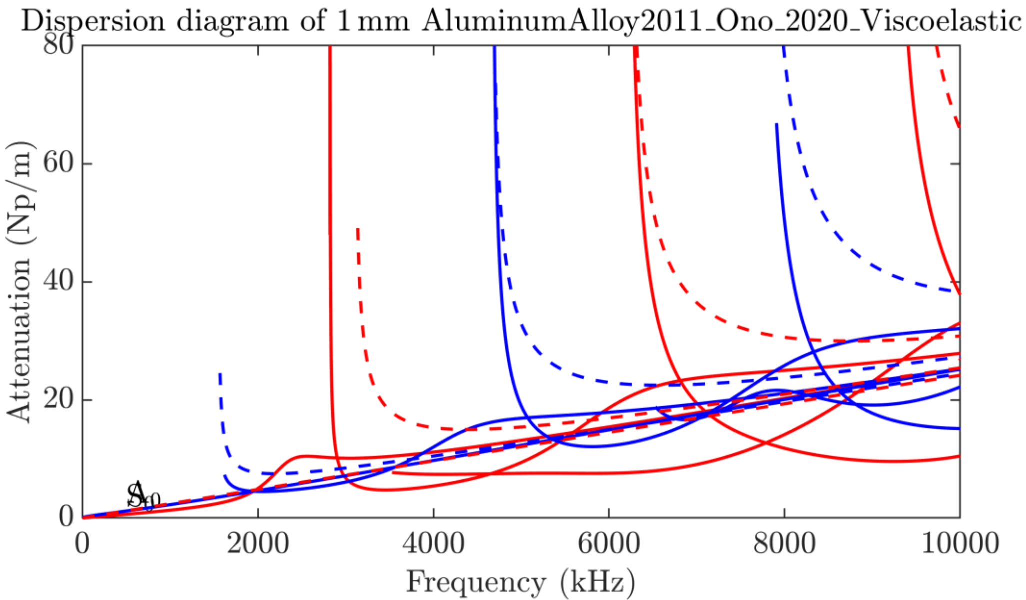




Primary goal:

Obtaining dispersion diagrams in the most efficient and robust way.

→ Using mode family specific dispersion equations

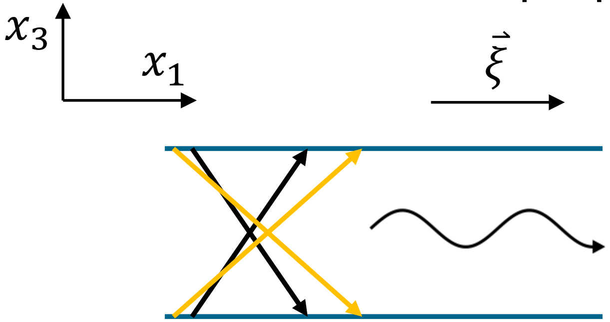


Overview of (selected) guided wave modeling methods

Root-finding methods
- Transfer Matrix M. (TMM)
- Global Matrix M. (GMM)
- Stiffness Matrix M. (SMM)

Discretizing methods
- FEM (Comsol Multiphysics)
- Semi-analytical FEM (SAFE)
- Spectral Collocation M. (SCM)
- Spectral Element M. (SEM)
- Legendre Polynomial M. (LPM)

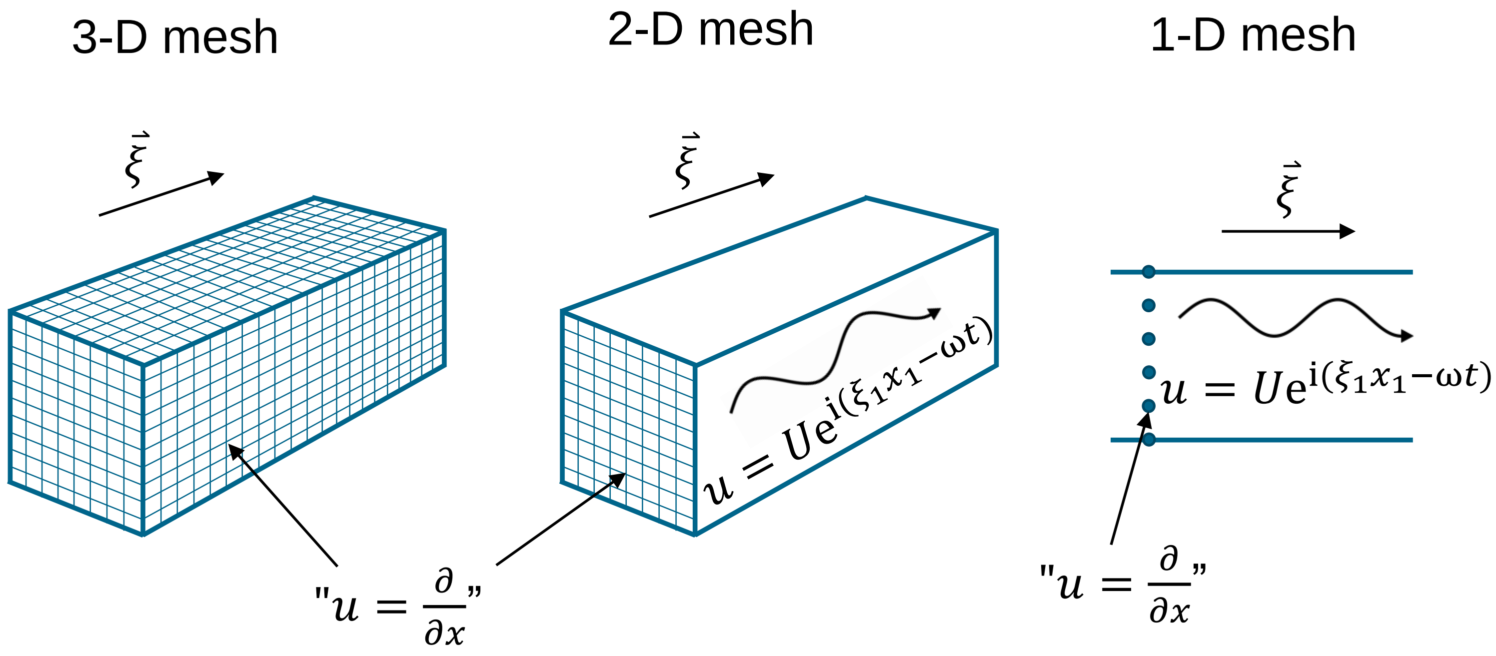
Lamb waves in an isotropic plate:



L+,L-,SV+,SV-

$$(u_1, u_3) = \sum_{q=1}^{4} (1, W_q) U_{1q} \mathrm{e}^{\mathrm{i}(\xi_1 x_1 + \xi_3 x_3 - \omega t)}$$

$$(\sigma_{33}, \sigma_{13}) = \sum_{q=1}^{4} (D_{33q}, D_{13q}) U_{1q} \mathrm{e}^{\mathrm{i}(\xi_1 x_1 + \xi_3 x_3 - \omega t)}$$



Which one is the best?

	Root-finding method: SMM	Discretizing method: SCM
Waveguide geometries	Only plates, rods, and pipes.	Arbitrary cross sections (e.g. rails).
Robustness	Can miss solutions!	Does not miss any solution!
Ease of implementation	A good dispersion curve tracing algorithm is needed.	It is a systematic calculation, without the need for dispersion curve tracing.
Accuracy	Accurate! We just make bisections until we reach the desired accuracy.	Depends on the number of collocation points! Accuracy decreases with increasing frequency.
Score	**1**	**3**

Most important tools currently available

Tool	Authors	Initial release	Computation method	GUI	Free
DISPERSE	Lowe, Pavlakovic	Early 1990s	Global Matrix Method (currently upgraded to the Spectral Collocation Method)	Yes	No
GUIGUW	Marzani, Bocchini	2011	Semi-analytical Finite Element Method	Yes	Yes
DC	Huber	2018	Stiffness Matrix Method	Yes	Yes
ElasticMatrix	Ramasawmy et al.	2020	Global Matrix Method	No	Yes
Dispersion Box	Orta et al.	2022	GMM, SMM, HCSMM, SAFE, LPM, 5-SDT	Yes	Yes
SAFEDC	Liu et al.	2022	Semi-analytical Finite Element Method	Yes	Yes
GEWtool	Kiefer	2023	Spectral Element Method	No	Yes

Choice of SMM for the Dispersion Calculator

In 2016, I only knew TMM, then finding SMM, unaware of SAFE, SCM, etc.

However:

- If we are interested in (multilayered anisotropic) plates,
- and we have a good (although never perfectly robust) dispersion curve tracing algorithm:

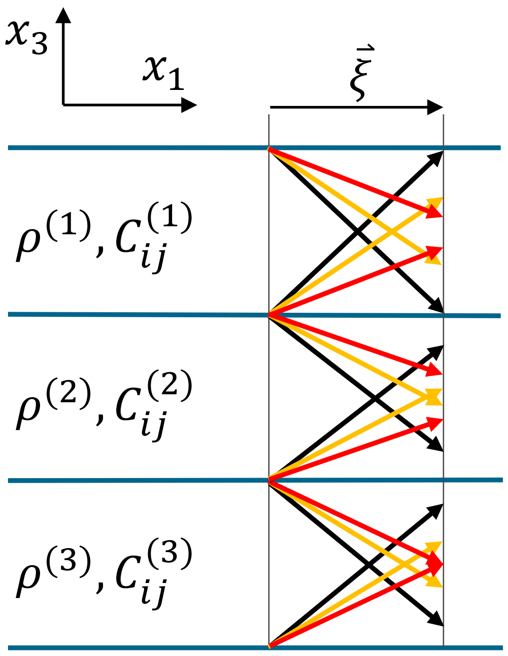
→SMM can be a good choice,

and performs very well!

	Root-finding method: SMM	Discretizing method: SCM
~~Waveguide geometries~~	~~Only plates, rods, and pipes.~~	~~Arbitrary cross sections (e.g. rails).~~
Robustness	Can miss solutions!	Does not miss any solution!
~~Ease of implementation~~	~~A good dispersion curve tracing algorithm is needed!~~	~~It is a systematic calculation, without the need for dispersion curve tracing.~~
Accuracy	Accurate! We just make bisections until we reach the desired accuracy.	Depends on the number of collocation points! Accuracy decreases with increasing frequency.
Score	**1 → 1**	**3 → 1**

Bulk waves in a multilayered, anisotropic plate

m –layered, anisotropic plate



6 bulk waves per layer

Quasi-
L+,L-,SV+,SV-,SH+,SH-

"

"

For **each layer**, solve Christoffel's equations

$$\rho \frac{\partial^2 u_i}{\partial t^2} = c_{ijkl} \frac{\partial^2 u_l}{\partial x_j \partial x_k}$$

to get 3 pairs of solutions for the out-of-plane wavenumber components

$$\xi_{3,2} = -\xi_{3,1}, \quad \xi_{3,4} = -\xi_{3,3}, \quad \xi_{3,6} = -\xi_{3,5}$$

and the corresponding polarizations

$$V_q, W_q \ , q = 1, 2, \ldots, 6$$

Transfer matrix and stiffness matrix

Express the displacements and stresses in each layer as a superpostion of the bulk waves

$$(u_1, u_2, u_3) = \sum_{q=1}^{6} (1, V_q, W_q) U_{1,q} \mathrm{e}^{\mathrm{i}\xi_{3,q} x_3}$$

$$(\sigma_{33}, \sigma_{13}, \sigma_{23}) = \sum_{q=1}^{6} (D_{33,q}, D_{13,q}, D_{23,q}) U_{1,q} \mathrm{e}^{\mathrm{i}\xi_{3,q} x_3}$$

The displacements and stresses at the top and bottom of each layer in Rokhlin's tensor notation:

$$\begin{bmatrix} \mathbf{u}_m \\ \boldsymbol{\sigma}_m \end{bmatrix} = \begin{bmatrix} \mathbf{P}^- & \mathbf{P}^+\mathbf{H} \\ \mathbf{D}^- & \mathbf{D}^+\mathbf{H} \end{bmatrix}_m \begin{bmatrix} \mathbf{U}_m^- \\ \mathbf{U}_m^+ \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{u}_{m+1} \\ \boldsymbol{\sigma}_{m+1} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^-\mathbf{H} & \mathbf{P}^+ \\ \mathbf{D}^-\mathbf{H} & \mathbf{D}^+ \end{bmatrix}_m \begin{bmatrix} \mathbf{U}_m^- \\ \mathbf{U}_m^+ \end{bmatrix}$$

The **transfer matrix** representation by Thomson/Haskell:

$$\begin{bmatrix} \mathbf{u}_{m+1} \\ \boldsymbol{\sigma}_{m+1} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^-\mathbf{H} & \mathbf{P}^+ \\ \mathbf{D}^-\mathbf{H} & \mathbf{D}^+ \end{bmatrix}_m \begin{bmatrix} \mathbf{P}^- & \mathbf{P}^+\mathbf{H} \\ \mathbf{D}^- & \mathbf{D}^+\mathbf{H} \end{bmatrix}_m^{-1} \begin{bmatrix} \mathbf{u}_m \\ \boldsymbol{\sigma}_m \end{bmatrix} = \mathbf{A}_m \begin{bmatrix} \mathbf{u}_m \\ \boldsymbol{\sigma}_m \end{bmatrix}$$

→ Numerically unstable (except for SH waves)!

The **stiffness matrix** representation by Rokhlin and Wang:

$$\begin{bmatrix} \boldsymbol{\sigma}_m \\ \boldsymbol{\sigma}_{m+1} \end{bmatrix} = \begin{bmatrix} \mathbf{D}^- & \mathbf{D}^+\mathbf{H} \\ \mathbf{D}^-\mathbf{H} & \mathbf{D}^+ \end{bmatrix}_m \begin{bmatrix} \mathbf{P}^- & \mathbf{P}^+\mathbf{H} \\ \mathbf{P}^-\mathbf{H} & \mathbf{P}^+ \end{bmatrix}_m^{-1} \begin{bmatrix} \mathbf{u}_m \\ \mathbf{u}_{m+1} \end{bmatrix} = \mathbf{K}_m \begin{bmatrix} \mathbf{u}_m \\ \mathbf{u}_{m+1} \end{bmatrix}$$

→ Numerically stable!

Global stiffness matrix and dispersion equation

Combine the layer stiffness matrices using the recursive algorithm

$$\begin{bmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11}^A + \mathbf{K}_{12}^A \mathbf{X} \mathbf{K}_{21}^A & -\mathbf{K}_{12}^A \mathbf{X} \mathbf{K}_{12}^B \\ \mathbf{K}_{21}^B \mathbf{X} \mathbf{K}_{21}^A & \mathbf{K}_{22}^B - \mathbf{K}_{21}^B \mathbf{X} \mathbf{K}_{12}^B \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_3 \end{bmatrix}$$

with

$$\mathbf{X} = (\mathbf{K}_{11}^B - \mathbf{K}_{22}^A)^{-1}$$

Apply the stress free boundary conditions to get the dispersion equation

$$\sigma_{i3}^{0,-d} = 0, \quad i = 1, 2, 3 \quad \rightarrow \quad \begin{bmatrix} \boldsymbol{\sigma}^0 \\ \boldsymbol{\sigma}^{-d} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{u}^0 \\ \mathbf{u}^{-d} \end{bmatrix} = 0$$

to get the dispersion equation

$$\det \mathbf{K} = 0$$

S. I. Rokhlin and L. Wang, "Stable recursive algorithm for elastic wave propagation in layered anisotropic media: Stiffness matrix method," J. Acoust. Soc. Am. **112**(3), 822-834 (2002).

Mode family specific dispersion equations

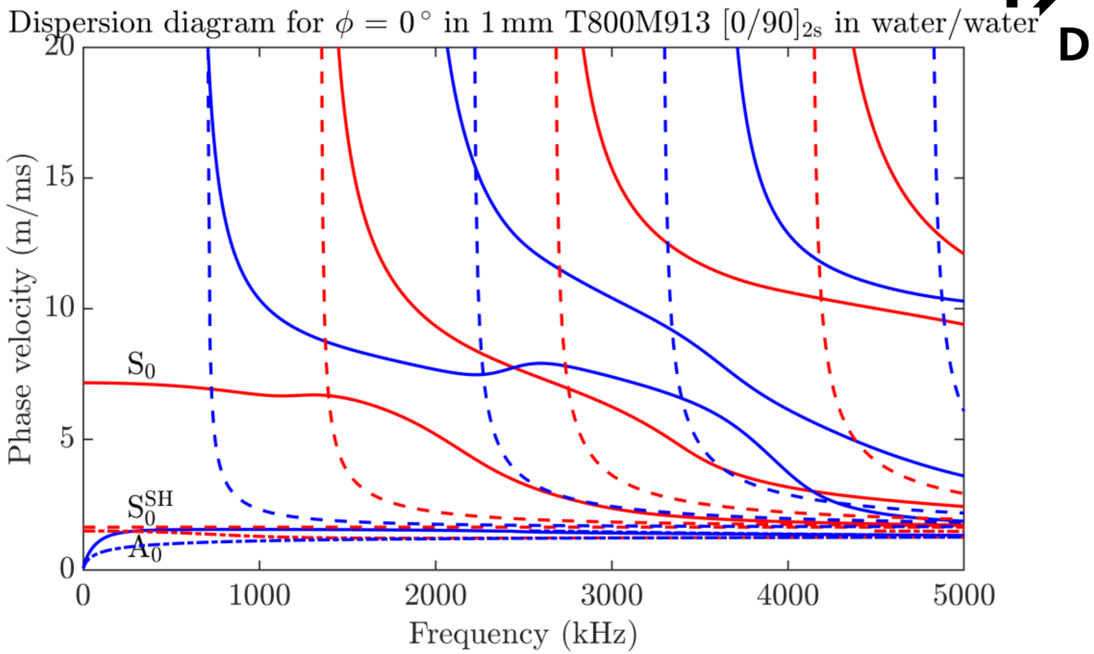
Benefits from mode family specific dispersion equations:

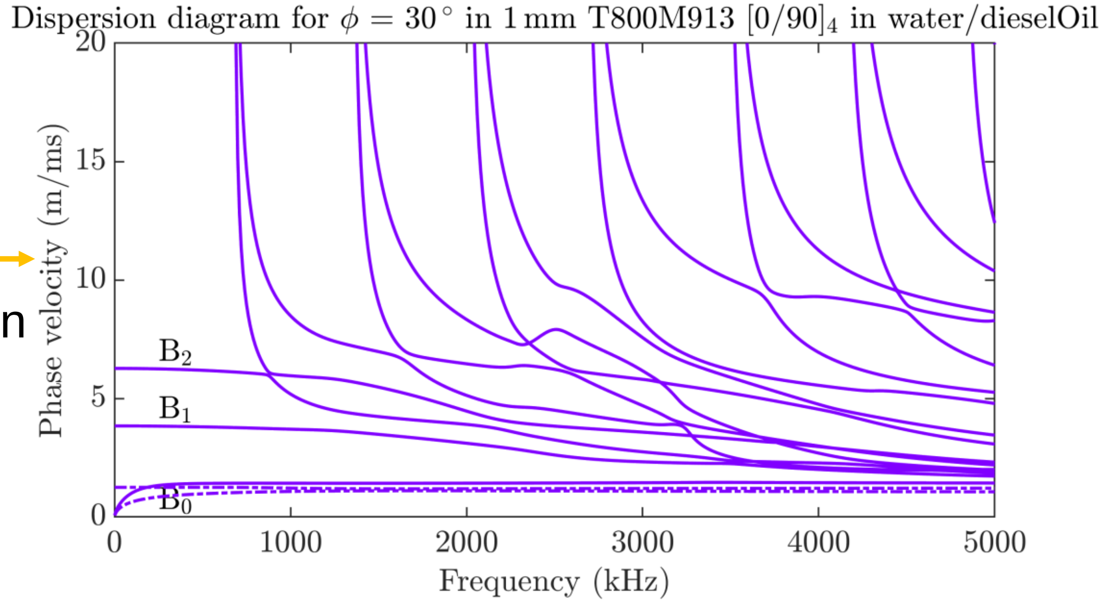
- Faster computation
- More robust dispersion curve tracing

6 dispersion equations

symmetric	decoupled	modes
✓	✓	S, A, S^{SH}, A^{SH}, $S^{Scholte}$, $A^{Scholte}$
✓	✗	S, A, $S^{Scholte}$, $A^{Scholte}$
✗	✓	B, B^{SH} (S^{SH}, A^{SH})^a, $B^{Scholte}$
✗	✗	B, $B^{Scholte}$

2 dispersion equations





Decoupling into Lamb and SH waves

Coupled polarization: $(u_1, u_2, u_3) = \sum_{q=1}^{6} (1, V_q, W_q) U_{1,q} \mathrm{e}^{\mathrm{i}\xi_{3,q} x_3}$

Decoupled polarization: $(u_1, u_3) = \sum_{q=1}^{4} (1, W_q) U_{1,q} \mathrm{e}^{\mathrm{i}\xi_{3,q} x_3}$ $u_2 = \sum_{q=1}^{2} U_{2,q} \mathrm{e}^{\mathrm{i}\xi_{3,q} x_3}$

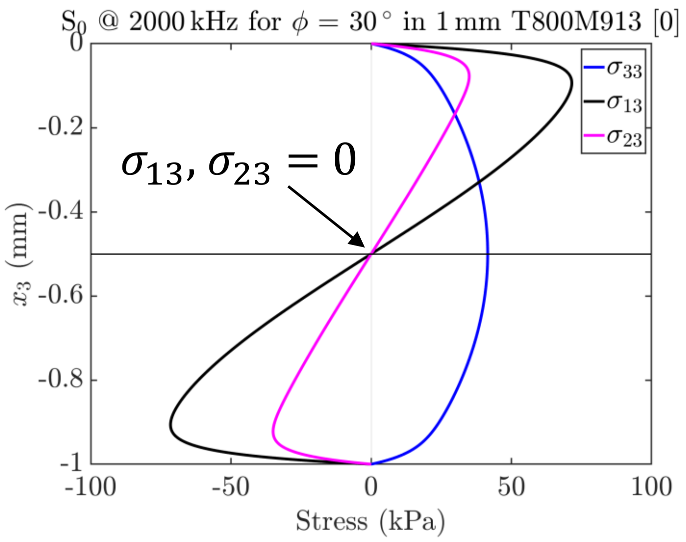
Lamb waves

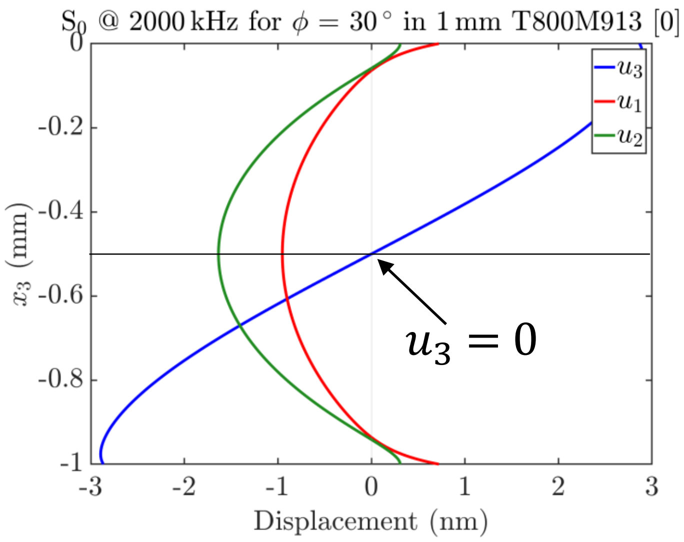
SH waves

Symmetric and antisymmetric mode shapes

Symmetric modes:

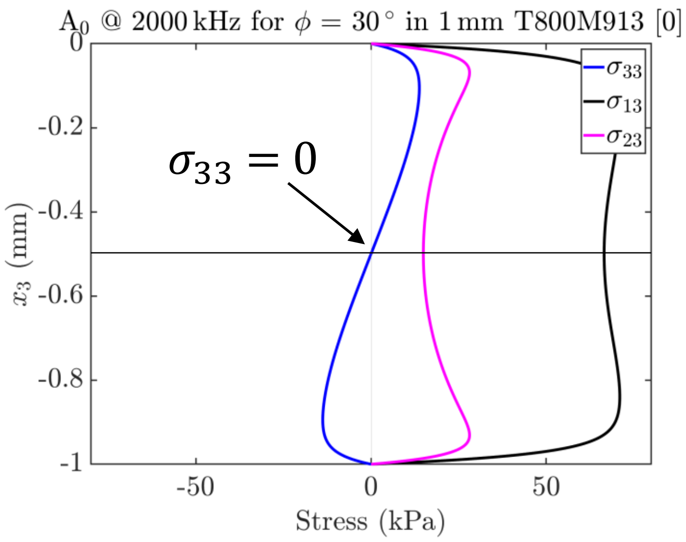
$$\sigma_{13}^{-d/2}, \sigma_{23}^{-d/2}, u_3^{-d/2} = 0$$

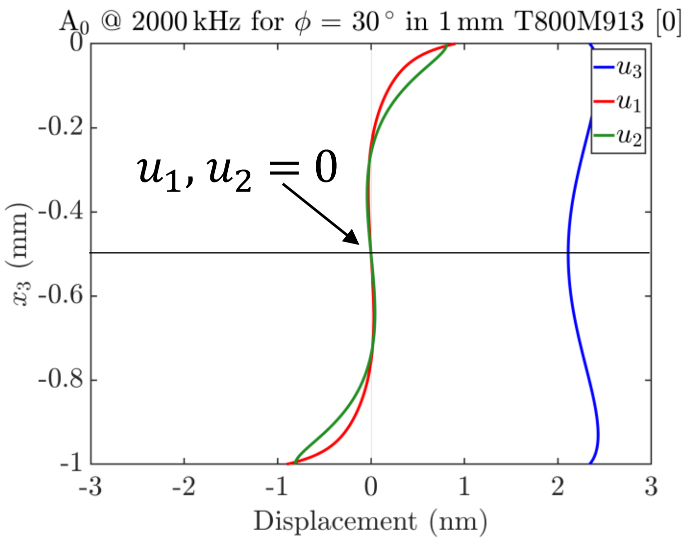




Antisymmetric modes:

$$\sigma_{33}^{-d/2}, u_1^{-d/2}, u_2^{-d/2} = 0$$





Symmetric and antisymmetric dispersion equations

Stiffness matrix relation for the upper half of the symmetric layup:

$$\begin{bmatrix} \boldsymbol{\sigma}^0 \\ \boldsymbol{\sigma}^{-d/2} \end{bmatrix} = \mathbf{K}^{\text{half}} \begin{bmatrix} \mathbf{u}^0 \\ \mathbf{u}^{-d/2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma_{33}^{-d/2} \\ \sigma_{13}^{-d/2} \\ \sigma_{23}^{-d/2} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix}^{\text{half}} \begin{bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \\ u_1^{-d/2} \\ u_2^{-d/2} \\ u_3^{-d/2} \end{bmatrix}$$

S

$$\sigma_{13}^{-d/2}, \sigma_{23}^{-d/2}, u_3^{-d/2} = 0$$

↓

$$\det \mathbf{K}_S = \det \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} \end{bmatrix}^{\text{half}} = 0$$

A

$$\sigma_{33}^{-d/2}, u_1^{-d/2}, u_2^{-d/2} = 0$$

↓

$$\det \mathbf{K}_A = \det \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{46} \end{bmatrix}^{\text{half}} = 0$$

Benefits:

- Calculate only $\mathbf{K}^{\text{half}}$ of the upper half
- $\mathbf{K}$(6x6) reduce to $\mathbf{K}_S$(5x5) and $\mathbf{K}_A$(4x4)
- No mode tracing jumps between S- and A-modes

Decoupled case (Lamb waves):

S

$$\det \mathbf{K}_S = \det \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{41} & k_{42} & k_{43} \end{bmatrix}^{\text{half}} = 0$$

A

$$\det \mathbf{K}_A = \det \begin{bmatrix} k_{11} & k_{12} & k_{14} \\ k_{21} & k_{22} & k_{24} \\ k_{31} & k_{32} & k_{34} \end{bmatrix}^{\text{half}} = 0$$

- $\mathbf{K}$(4x4) reduces to $\mathbf{K}_S$(3x3) and $\mathbf{K}_A$(3x3)

Symmetric and antisymmetric dispersion equations for (symmetric) fluid-loading

Compliance matrix relation for the upper half of the symmetric layup:

$$\begin{bmatrix} \mathbf{u}^0 \\ \mathbf{u}^{-d/2} \end{bmatrix} = {}^{\text{half}}\mathbf{S} \begin{bmatrix} \boldsymbol{\sigma}^0 \\ \boldsymbol{\sigma}^{-d/2} \end{bmatrix} = \begin{bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \\ u_1^{-d/2} \\ u_2^{-d/2} \\ u_3^{-d/2} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{bmatrix}^{\text{half}} \begin{bmatrix} \sigma_{33}^0 \\ 0 \\ 0 \\ \sigma_{33}^{-d/2} \\ \sigma_{13}^{-d/2} \\ \sigma_{23}^{-d/2} \end{bmatrix}$$

$$\mathbf{S} = \mathbf{K}^{-1}$$

Refl. coeffs.

S

$$\sigma_{13}^{-d/2}, \sigma_{23}^{-d/2}, u_3^{-d/2} = 0$$

↓

$$R_\text{S} = \frac{\Lambda - X_\text{S}}{\Lambda + X_\text{S}} = 0$$

$$X_\text{S} = s_{31} - \frac{s_{34}s_{61}}{s_{64}}$$

A

$$\sigma_{33}^{-d/2}, u_1^{-d/2}, u_2^{-d/2} = 0$$

↓

$$R_\text{A} = \frac{\Lambda - X_\text{A}}{\Lambda + X_\text{A}} = 0$$

$$X_\text{A} = s_{31} - \frac{s_{35}s_{51}}{s_{55}} - \left(\frac{s_{35}s_{56}}{s_{55}} - s_{36}\right)\frac{s_{41}s_{55} - s_{45}s_{51}}{s_{45}s_{56} - s_{46}s_{55}}$$

Decoupled case (leaky “Lamb” waves):

S

$$X_\text{S} = s_{21} - \frac{s_{23}s_{41}}{s_{43}}$$

A

$$X_\text{A} = s_{21} - \frac{s_{24}s_{31}}{s_{34}}$$

A. M. A. Huber, "Classification of solutions for guided waves in fluid-loaded viscoelastic composites with large numbers of layers," J. Acoust. Soc. Am. **154**(2), 1073–1094 (2023).

Live demonstration of the Dispersion Calculator (DC)

Concluding remarks

- The discretizing SCM is superior to the root-finding SMM in general:
  - arbitrary cross section waveguides
  - does not miss solutions
  - no dispersion curve tracing algorithm needed
- However, SMM used with mode family specific dispersion equations is a powerful alternative in case
  - plates and cylinders are considered,
  - and a good dispersion curve tracing algorithm is available.

Future work

- 2025: DC v3.0
  - Guided waves in isotropic rods and pipes including fluid-loading
- DC v3.x
  - (?) Guided waves in multilayered anisotropic rods and pipes
  - (?) Piezoelectric effects
  - (?) Backpropagating modes (c_p and c_e point in opposite directions → *negative* damping)

Dispersion Calculator

Thank you!

Dr. Armin Huber
German Aerospace Center (DLR)
armin.huber@dlr.de