Bounds on Sphere Sizes in the Sum-rank Metric and Coordinate-additive Metrics

Hugo Sauerbier Couvée[1], Thomas Jerkovits[2], and Jessica Bariffi[2,3]

[1] Department of Electrical and Computer Engineering, Technical University of Munich, Germany.
[2] Institute of Communication and Navigation, German Aerospace Center, Germany.
[3] Institute of Mathematics, University of Zurich, Switzerland

Abstract. This paper provides new bounds on the size of spheres in any coordinate-additive metric with a particular focus on improving existing bounds in the sum-rank metric. We derive improved upper and lower bounds based on the entropy of a distribution related to the Boltzmann distribution, which work for any coordinate-additive metric. Additionally, we derive new closed-form upper and lower bounds specifically for the sum-rank metric that outperform existing closed-form bounds.

Keywords: Sum-rank metric · Coordinate-additive metric · Sphere size · Combinatorics · Coding theory · Information theory

1 Introduction

The sum-rank metric [15], Hamming metric [7] and Lee metric [10] are examples of coordinate-additive metrics. Codes with distance properties in such metrics are of particular interest in various applications, such as linear network coding [12], quantum-resistant cryptography [8,17], coding for storage [13], space-time coding [19]. Bounds on the size of an ℓ-dimensional ball or sphere in such metrics are essential for deriving bounds like the sphere-packing bound or the Gilbert–Varshamov bound [4]. An information-theoretic approach for bounding the volume of an ℓ-dimensional ball concerning any coordinate-additive metric, via the entropy of an auxiliary probability distribution, was presented in [11]. Specifically addressing the sum-rank metric, closed-form upper and lower bounds for the sphere size were introduced in [17,16] and further discussed in [6]. However, these bounds are limited in their tightness, particularly noticeable in scenarios involving smaller sizes of the base field q and/or a larger number of blocks ℓ.

The exact value for the size of an ℓ-dimensional sphere $\mathcal{S}_t^\ell$ of radius t in any coordinate-additive metric can be derived by computing all its (ordered) integer partitions, where each part of the partition has at most a part size of the maximal possible weight in the corresponding metric. These will represent the decomposition of the nonzero entries of the elements in the sphere. To get the size of the sphere we sum over all integer partitions adding up the number of elements that have a weight decomposition corresponding to the integer partition. Although this procedure provides the exact value of $\left|\mathcal{S}_t^\ell\right|$, it often doesn't give an intuitive or practical understanding of the sphere size or how this size changes as the parameters change. For large parameters it is even impractical to compute the size in this way. Hence, the derivation of closed-form bounds on the exact formula are of major interest. A current method of obtaining both upper and lower bounds on $\left|\mathcal{S}_t^\ell\right|$ is, for instance, to consider only the partition attaining the maximum number of elements. This approach is utilized by [16,17,6]. Another method is to bound the size of an ℓ-dimensional ball $\mathcal{B}_t^\ell$ of radius t, since clearly every upper bound on $\left|\mathcal{B}_t^\ell\right|$ is a valid upper bound on $\left|\mathcal{S}_t^\ell\right|$, too. On a complex analytic side, sizes of spheres and balls can be described using generating functions, whose coefficients can be computed using the saddle-point technique and other techniques from analytic combinatorics (see [2,3]). We refer to [18] for a more detailed discussion and proofs of the results presented in this paper.

2 Preliminaries

Let q be a prime power and denote by $\mathbb{F}_q$ the finite field of q elements. The natural numbers $\mathbb{N}$ shall include 0. Given a random variable X over a finite alphabet $\mathcal{A}$ with probability distribution P, we define $P(a) := \operatorname{Prob}(X = a)$ with $a \in \mathcal{A}$. The entropy $H(P)$ of P with respect to the base q is defined as $H(P) := -\sum_{a\in\mathcal{A}, P(a)\neq 0} P(a)\log_q P(a)$.

2.1 Coordinate-Additive Metrics

Let $(\mathcal{A}, +)$ be a finite abelian group with identity element 0 called the **alphabet**. We define a **weight function** $\mathrm{wt}_\mathcal{A} : \mathcal{A} \to \mathbb{N}$ on $\mathcal{A}$ to be a function satisfying for all $a, b \in \mathcal{A}$:

1. $\mathrm{wt}_\mathcal{A}(a) = 0$ if and only if $a = 0$,
2. $\mathrm{wt}_\mathcal{A}(a) = \mathrm{wt}_\mathcal{A}(-a)$,
3. $\mathrm{wt}_\mathcal{A}(a + b) \leq \mathrm{wt}_\mathcal{A}(a) + \mathrm{wt}_\mathcal{A}(b)$.

This function can be extended to a **coordinate-additive weight function** on the cartesian product $\mathcal{A}^\ell$ (with group structure inherited coordinate-wise from $\mathcal{A}$) by defining the weight of an ℓ-tuple to be the sum of the weights of its coordinates, i.e., $\mathrm{wt}_{\Sigma\mathcal{A}}(a_1, \dots, a_\ell) = \sum_{i=1}^{\ell} \mathrm{wt}_\mathcal{A}(a_i)$. This coordinate-additive weight function naturally induces a **metric** $\mathrm{d}_{\Sigma\mathcal{A}} : \mathcal{A}^\ell \times \mathcal{A}^\ell \to \mathbb{N}$ as $\mathrm{d}_{\Sigma\mathcal{A}}(v, w) := \mathrm{wt}_{\Sigma\mathcal{A}}(v-w)$. Given a coordinate-additive weight function $\mathrm{wt}_{\Sigma\mathcal{A}}$ on $\mathcal{A}^\ell$, we define the ℓ-dimensional **sphere**, respectively **ball**, of radius $t \in \mathbb{N}$ by

$$\mathcal{S}_t^\ell := \{v \in \mathcal{A}^\ell \ : \ \mathrm{wt}_{\Sigma\mathcal{A}}(v) = t\} \quad \text{and} \quad \mathcal{B}_t^\ell := \{v \in \mathcal{A}^\ell \ : \ \mathrm{wt}_{\Sigma\mathcal{A}}(v) \leq t\}.$$

For the special case of the sum-rank metric, let m, η and ℓ be positive integers. Also define $\mu := \min\{m, \eta\}$ and $n := \eta\ell$. We write $\mathbb{F}_q^{m\times\eta\ell}$ for the space of $m\times(\eta\ell)$ matrices over the finite field $\mathbb{F}_q$. Every matrix $M \in \mathbb{F}_q^{m\times\eta\ell}$ is represented as a sequence of ℓ blocks, i.e., $M = (B_1 \,|\, B_2 \,|\, \dots \,|\, B_\ell)$ with each $B_i \in \mathbb{F}_q^{m\times\eta}$. The **sum-rank weight** of a matrix $M \in \mathbb{F}_q^{m\times\eta\ell}$ is defined as $\mathrm{wt}_{SR}(M) :=$

$\sum_{i=1}^{\ell} \mathrm{rk}_q(B_i)$ where $\mathrm{rk}_q(B_i)$ is the rank of B_i over $\mathbb{F}_q$. Analogously, we define for every $0 \leq t \leq \mu \cdot \ell$, the **sum-rank sphere** of radius t as

$$\mathcal{S}_t^{m,\eta,\ell,q} := \{M \in \mathbb{F}_q^{m\times\eta\ell} \ : \ \mathrm{wt}_{SR}(M) = t\}.$$

For fixed m, η, q, ℓ, the sum-rank sphere sizes $\left|\mathcal{S}_t^{m,\eta,\ell,q}\right|$ can be computed with a dynamic program described in [17].

2.2 Ordinary Generating Functions

The theory of ordinary generating functions (OGFs) is a useful branch of mathematics that lays connections between combinatorics, analysis, number theory, probability theory and other fields. In this paper we restrict ourselves to OGFs corresponding to weights in coordinate-additive metrics, which are polynomials with non-negative coefficients. Consider a finite abelian group $\mathcal{A}$ with weight function $\mathrm{wt}_{\mathcal{A}}$ and induced coordinate-additive weight function $\mathrm{wt}_{\Sigma\mathcal{A}}$ on $\mathcal{A}^\ell$. The OGF corresponding to $\mathrm{wt}_{\Sigma\mathcal{A}}$ is defined as the polynomial

$$F_{\mathcal{A}^\ell}(z) := \textstyle\sum_{v\in\mathcal{A}^\ell} z^{\mathrm{wt}_{\Sigma\mathcal{A}}(v)} = \sum_{i=0}^{\mu\ell} |\mathcal{S}_i^\ell| \ z^i.$$

The OGF for $\mathcal{A} = \mathcal{A}^1$ is denoted by $F_{\mathcal{A}}(z)$. For a polynomial $F(z) = F_0 + F_1 z + \ldots + F_d z^d$ we use the notation $[z^i]F(z)$ to refer to the i-th coefficient F_i of $F(z)$, with $[z^i]F(z) = 0$ for $i > \deg(F)$. The OGF for the sum-rank metric on $\mathbb{F}_q^{m\times\eta\ell}$ is denoted by $\mathcal{S}^{m,\eta,\ell,q}(z) = \sum_{i=0}^{\mu\ell} |\mathcal{S}_i^{m,\eta,\ell,q}| \ z^i$.

Definition 1 (Partial order on polynomials). *Let $F(z), G(z) \in \mathbb{R}[z]$ be two real polynomials. If $[z^i]F(z) \leq [z^i]G(z)$ for all $i \in \mathbb{N}$, we say $F(z)$ is coefficient-wise less-than-or-equal to $G(z)$, denoted as $F(z) \preccurlyeq_c G(z)$.*

Proposition 1 ([2, Theorem I.1]). *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be two finite alphabets with weight functions $\mathrm{wt}_{\mathcal{A}_1}, \mathrm{wt}_{\mathcal{A}_2}$ respectively. Then $\mathrm{wt}_{\mathcal{A}_1\times\mathcal{A}_2}(a,b) := \mathrm{wt}_{\mathcal{A}_1}(a) + \mathrm{wt}_{\mathcal{A}_2}(b)$ is a weight function on $\mathcal{A}_1 \times \mathcal{A}_2$ and*

$$F_{\mathcal{A}_1\times\mathcal{A}_2}(z) = F_{\mathcal{A}_1}(z)F_{\mathcal{A}_2}(z).$$

In particular, we have $F_{\mathcal{A}^\ell}(z) = F_{\mathcal{A}}(z)^\ell$, for $\ell \in \mathbb{N}$. Furthermore, the product of real polynomials with non-negative coefficients preserves the partial order: if $F(z) \preccurlyeq_c G(z)$ and $K(z) \preccurlyeq_c L(z)$, then $F(z)K(z) \preccurlyeq_c G(z)L(z)$.

Lemma 1. *Let $F(z)$ be a real polynomial of degree $d > 0$ with non-negative coefficients $F_i \geq 0$ and first derivative $F'(z)$. If $F(z)$ is not a monomial, then the function $G(z) = zF'(z)/F(z)$ is a strictly increasing smooth function on the positive reals $\mathbb{R}_{>0}$. In particular if $F(0) > 0$, which is the case with OGFs of finite alphabets with weight functions, $G(z)$ is a bijection from $[0, \infty)$ to $[0, d)$.*

Proof. Smoothness follows directly from smoothness of $F(z)$ and $1/z$ on $\mathbb{R}_{>0}$. Setting $K(a,b) := bF'(b)F(a) - aF'(a)F(b)$ with $0 < a < b$, we can show that $K(a,b) > 0$, thereby proving $G(z)$ is strictly increasing. Lastly, we have that $\lim_{z\to\infty} F'(z)/z^{d-1} = dF_d$ and $\lim_{z\to\infty} F(z)/z^d = F_d$, so $\lim_{z\to\infty} G(z) = d$. ☐

3 Information-Theoretic Bounds on Spheres

In [11] an asymptotically tight upper bound on the volume of an ℓ-dimensional ball $\left|\mathcal{B}_t^\ell\right|$ of radius t was introduced. This bound is valid for any arbitrary additive weight function $\mathrm{wt}_\mathcal{A}$ with respect to some finite abelian group $\mathcal{A}$ as described in Section 2.1. The bound was proved to hold for normalized weights ρ with $\rho := t/\ell$ up to the average weight $\overline{w} := |\mathcal{A}|^{-1} \sum_{a\in\mathcal{A}} \mathrm{wt}_\mathcal{A}(a)$ at which the volume of the ball is saturated. We extend the result from [11] to the size of spheres and also prove that the bound holds for $\rho \geq \overline{w}$ up to the maximum possible weight, i.e. $0 < \rho < \mu$ with $\mu := \max_{a\in\mathcal{A}} \{\mathrm{wt}_\mathcal{A}(a)\}$. Note that this notation coincides with $\mu = \min\{m, \eta\}$ for the sum-rank metric. For any $a \in \mathcal{A}$, $\ell \in \mathbb{N}$ and $0 < \rho < \mu$, we define the probability distribution

$$P_\beta(a) := \frac{q^{-\beta\,\mathrm{wt}_\mathcal{A}(a)}}{\mathcal{Z}(\beta)} \tag{1}$$

where β is the unique solution to the weight constraint

$$\textstyle\sum_{a\in\mathcal{A}} P_\beta(a)\,\mathrm{wt}_\mathcal{A}(a) = \rho \tag{2}$$

and $\mathcal{Z}(\beta)$ is chosen s.t. $\sum_{a\in\mathcal{A}} P_\beta(a) = 1$. Note that the normalized radius ρ and β are in one-to-one correspondence due to the weight constraint (2) and Lemma 1 (cf. (3)). For a $\beta \in \mathbb{R}$, the value ρ determined by this correspondence is denoted $\rho(\beta)$. Let us denote by $H_\rho := H(P_\beta)$ the entropy of the distribution in (1). Then, the following bound was proven in [11].

Theorem 1 ([11]). *For any* $0 < \rho \leq \overline{w}$ *and* $\ell \in \mathbb{N}$ *we have*

$$\frac{1}{\ell}\log_q \left|\mathcal{B}_{\rho\ell}^\ell\right| \leq H_\rho.$$

The following is an immediate consequence of Theorem 1 above.

Corollary 1. *For any* $0 < \rho < \overline{w}$ *and* $\ell \in \mathbb{N}$ *we have*

$$\frac{1}{\ell}\log_q \left|\mathcal{S}_{\rho\ell}^\ell\right| \leq H_\rho.$$

3.1 Upper Bounds

We show that Corollary 1 also holds for normalized weights s.t. $0 < \rho < \mu$. Recall the OGFs for $\mathcal{A}$ and $\mathcal{A}^\ell$

$$F_\mathcal{A}(z) = \textstyle\sum_{a\in\mathcal{A}} z^{\mathrm{wt}_\mathcal{A}(a)} \quad \text{and} \quad F_{\mathcal{A}^\ell}(z) = \sum_{v\in\mathcal{A}^\ell} z^{\mathrm{wt}_{\Sigma\mathcal{A}}(v)} = F_\mathcal{A}(z)^\ell.$$

We now can express $\mathcal{Z}(\beta)$, $\rho(\beta)$ and H_ρ in terms of these OGFs, i.e.

$$\mathcal{Z}(\beta) = F_\mathcal{A}\left(q^{-\beta}\right), \quad \rho(\beta) = q^{-\beta}\frac{F'_\mathcal{A}\left(q^{-\beta}\right)}{F_\mathcal{A}\left(q^{-\beta}\right)}, \quad H_\rho = \log_q\left(\frac{F_\mathcal{A}(q^{-\beta})}{(q^{-\beta})^\rho}\right). \tag{3}$$

Due to space constraints, we skip the proof for these equalities. We now make use of a technique explained in [2, Section VIII.2] where Flajolet and Sedgewick present the saddle-point bound, i.e., an upper bound on the coefficients of a OGF. For any real valued $y > 0$ we have

$$|\mathcal{S}_t^\ell| y^t = \left([z^t] F_{\mathcal{A}^\ell}(z)\right) y^t \leq F_{\mathcal{A}^\ell}(y) = F_{\mathcal{A}}(y)^\ell.$$

We can further rewrite this expression and take the infimum on the right-hand side and obtain

$$\tfrac{1}{\ell} \log_q |\mathcal{S}_t^\ell| \leq \inf_{y>0} \log_q \left(\tfrac{F_{\mathcal{A}}(y)}{y^\rho}\right). \tag{4}$$

We can, moreover, show that a global minimum of $F_{\mathcal{A}}(y)/y^\rho$ exists and therefore the infimum is a minimum: by setting the derivative of $F_{\mathcal{A}}(y)/y^\rho$ to zero and using (3) for ρ, we obtain a local minimum for $y = q^{-\beta}$. Then using Lemma 1, we can show that the derivative of $F_{\mathcal{A}}(y)/y^\rho$ is negative for $0 < y < q^{-\beta}$ and positive for $y > q^{-\beta}$. Therefore, the local minimum is also the global minimum, where the function $\log_q \left(\frac{F_{\mathcal{A}}(y)}{y^\rho}\right)$ takes the value H_ρ (cf. (3)).

To summarize, the saddle-point bound (4) coincides with the entropy bound (see also [5, Theorem 4.1], [1, Theorem IV.9]), but extends the range of ρ to $(0, \mu)$, as stated in the following theorem.

Theorem 2. *For any $0 < \rho < \mu$ and $\ell \in \mathbb{N}$ we have*

$$\frac{1}{\ell} \log_q |\mathcal{S}_{\rho\ell}^\ell| \leq H_\rho.$$

3.2 Lower Bounds

We now derive a lower bound based on the probability distribution in (1). Let $X_\beta, X_{\beta,1}, X_{\beta,2}, X_{\beta,3}, \ldots$ be i.i.d. random variables taking values in $\mathcal{A}$ with probability distribution P_β. Define the function $\varphi_\beta(a) := -\log_q (P_\beta(a))$ for $a \in \mathcal{A}$. As a consequence of Chebyshev's inequality [20], we have for any $\gamma > 0$

$$\mathrm{Prob}\left(\left|\tfrac{1}{\ell}\textstyle\sum_{i=1}^{\ell} \varphi_\beta(X_{\beta,i}) \;-\; H_\rho\right| \geq \gamma\right) \leq \tfrac{\mathrm{Var}(\varphi_\beta(X_\beta))}{\ell\gamma^2} = \tfrac{\beta^2\,\mathrm{Var}(\mathrm{wt}_{\mathcal{A}}(X_\beta))}{\ell\gamma^2}.$$

By setting $\gamma = |\beta|\delta/\ell$, where δ is chosen for some variable $0 < \varepsilon < 1$ as

$$\delta = \ell^{1/2} \frac{\mathrm{Var}(\mathrm{wt}_{\mathcal{A}}(X_\beta))^{1/2}}{(1-\varepsilon)^{1/2}}, \tag{5}$$

we can derive a lower bound with a similar technique used in [11].

Theorem 3. *Given $t = \ell\rho$ and $0 < \varepsilon < 1$, let β be defined by the weight constraint* (2) *and δ as in* (5). *Then*

$$\sum_{-\delta<j<\delta,\, j\in\mathbb{Z}} |\mathcal{S}_{t+j}^\ell| \geq \varepsilon\; q^{\ell H(P_\beta) - |\beta|\delta}.$$

Theorem 4 gives an alternative bound using the inequality

$$\max_{-\delta<j<\delta,\, j\in\mathbb{Z}} |\mathcal{S}_{t+j}^\ell| \geq \frac{1}{2\lceil\delta\rceil - 1} \sum_{-\delta<j<\delta,\, j\in\mathbb{Z}} |\mathcal{S}_{t+j}^\ell|.$$

Theorem 4. *Given $t = \ell\rho$ and $0 < \varepsilon < 1$, let β be defined by the weight constraint* (2) *and δ as in* (5)*. Then*

$$\max_{-\delta<j<\delta,\, j\in\mathbb{Z}} \frac{1}{\ell}\log_q |\mathcal{S}^{\ell}_{t+j}| \geq H(P_\beta) - \frac{|\beta|\delta}{\ell} - \frac{1}{\ell}\log_q\left(\frac{2\lceil\delta\rceil - 1}{\varepsilon}\right).$$

Empirically, good bounds seem to be obtained for ε close to 0. Moreover, for constant ε and ρ, the bound coincides asymptotically with Theorem 2 as $\ell \to \infty$ and is therefore asymptotically tight.

4 Bounds on Spheres in the Sum-rank Metric

In this section we derive improved closed-form upper and lower bounds on the size of a sphere in the sum-rank metric. Hence, we fix m,η and q and we use $\mathrm{NM}_q(m,\eta,t)$ to denote the number of matrices of rank t over $\mathbb{F}_q^{m\times\eta}$. For $a,b\in\mathbb{N}$ we define the ***q*-binomial coefficient** as $\begin{bmatrix} a\\ b\end{bmatrix}_q = \prod_{i=1}^{b}\frac{1-q^{a-b+i}}{1-q^i}$. Then, $\mathrm{NM}_q(m,\eta,t) = \begin{bmatrix} m\\ t\end{bmatrix}_q \prod_{i=0}^{t-1}(q^\eta - q^i)$ (see [14]). The ***q*-Pochhammer symbol** is defined as

$$(a;x)_\infty := \prod_{i=0}^{\infty}(1-ax^i), \quad \gamma_q := \left(\tfrac{1}{q};\tfrac{1}{q}\right)_\infty^{-1}.$$

Let $q\geq 2$, $\mu=\min\{m,\eta\}$, $\mathfrak{M}=\max\{m,\eta\}$ and $0\leq i\leq\mu$. Then the q-binomial coefficients and q-Pochhammer symbols satisfy the following inequalities, that follow from elementary arguments (see [9, Lemma 2.2])

$$1+\tfrac{1}{q} \geq \left(\tfrac{1}{q^2};\tfrac{1}{q^2}\right)_\infty^{-1} \quad \text{and} \quad \begin{bmatrix}\mu\\ i\end{bmatrix}_q \geq \begin{cases}(1+\frac{1}{q})q^{i(\mu-i)} & \text{if } 0<i<\mu\\ 1 & \text{if } i=0 \text{ or } i=\mu\end{cases}$$

and as a direct corollary of these two inequalities we obtain

$$\begin{bmatrix}\mu\\ i\end{bmatrix}_{1/q^2} q^{i(\mu-i)} \leq \begin{bmatrix}\mu\\ i\end{bmatrix}_q. \tag{6}$$

Now the inequality $\left(\prod_{j=0}^{a-1}(q^c-q^j)\right)^b > \left(\prod_{j=0}^{b-1}(q^c-q^j)\right)^a$ for $a,b,c\in\mathbb{N}$ with $0\leq a<b<c$ yields

$$\prod_{j=0}^{i-1}(q^{\mathfrak{M}}-q^j) > \left(\prod_{j=0}^{\mu-1}(q^{\mathfrak{M}}-q^j)\right)^{i/\mu} = q^{i\,\mathfrak{M}}\left(\gamma_{q,m,\eta}{}^{-1}\right)^{i/\mu} \tag{7}$$

where we introduce the notation $\gamma_{q,m,\eta}{}^{-1} := \prod_{j=\mathfrak{M}-\mu+1}^{\mathfrak{M}}(1-(1/q)^j)$. Combining (6) and (7) lead to a new lower bound on the number of matrices of rank t.

Proposition 2. *For $m,\eta,i\in\mathbb{N}$ with $i\leq\mu$, we have the lower bound*

$$\left(\gamma_{q,m,\eta}{}^{-1/\mu}\right)^i \begin{bmatrix}\mu\\ i\end{bmatrix}_{1/q^2} q^{i(m+\eta-i)} \leq \mathrm{NM}_q(m,\eta,i).$$

Next, we can obtain an upper bound for the number of matrices by introducing the function $\kappa_{q,m,\eta}(t) := \left(\frac{(1-q^{-m})(1-q^{-\eta})}{(1-q^{-1})}\right)^t$ and writing

$$\mathrm{NM}_q(m,\eta,t) = \left(\prod_{i=1}^{t}\frac{(1-q^{-m+i-1})(1-q^{-\eta+i-1})}{(1-q^{-i})}\right)q^{t(m+\eta-t)}.$$

Proposition 3. *For $m, \eta, t \in \mathbb{N}$ we have the following upper bound*

$$\mathrm{NM}_q(m, \eta, t) \ \le \ \kappa_{q,m,\eta}(t) q^{t(m+\eta-t)}.$$

In [17] an upper bound is derived using $\mathrm{NM}_q(m, \eta, t) \le \gamma_q q^{t(m+\eta-t)}$. By doing similar steps with $\kappa_{q,m,\eta}(t)$ instead of γ_q we obtain Theorem 5.

Theorem 5. *Given positive integers m, η, ℓ, t and a prime power q, it holds*

$$\left|\mathcal{S}_t^{m,\eta,\ell,q}\right| \le \kappa_{q,m,\eta}(t) \binom{\ell + 1 - 1}{\ell - 1} q^{t(m+\eta-\frac{t}{\ell})}.$$

Finally we state a strong form of log-concavity for $(\mathrm{NM}_q(m, \eta, i))_{i=0}^{\mu}$ that we apply later to Theorem 8.

Theorem 6. *For $0 < i < \mu$ we have*

$$\frac{\mathrm{NM}_q(m, \eta, i)^2}{\mathrm{NM}_q(m, \eta, i-1)\,\mathrm{NM}_q(m, \eta, i+1)} = \frac{(q^m - q^{i-1})}{(q^m - q^i)} \frac{(q^\eta - q^{i-1})}{(q^\eta - q^i)} \frac{q^i(q^{i+1} - 1)}{q^{i-1}(q^i - 1)} \ge q^2.$$

Moreover, since convolution preserves log-concavity, it holds that for all ℓ that the sequence $\left(\left|\mathcal{S}_i^{m,\eta,\ell,q}\right|\right)_{i=0}^{\mu\ell}$ is log-concave.

4.1 Integral Upper Bound

Let $f(x)$ and $g(x)$ be two real-valued functions defined on the natural numbers (or on a larger domain). We define the **discrete convolution** by $[f * g](t) := \sum_{i=0}^{t} f(i)g(t-i)$, for $t \in \mathbb{N}$. The ℓ-fold discrete convolution $[f * f * \cdots * f]$ (well-defined by associativity of $*$) is denoted as $f^{*\ell}$. Let $C(t)$ be a real-valued function depending on parameters m, η, q and satisfying

$$\left|\mathcal{S}_t^{m,\eta,1,q}\right| \le C(t) q^{t(m+\eta-t)} \quad \text{and} \quad C(t_1)C(t_2) = C(t_3)C(t_4)$$

whenever $t_1 + t_2 = t_3 + t_4$. By Proposition 3, examples of such functions are γ_q and $\kappa_{q,m,\eta}(t)$. The reason for looking at these functions is because they work well with discrete convolutions, i.e., $[C(x)f(x) * C(x)g(x)](t) = C(0)C(t)[f * g](t)$. Therefore, we can upper bound the sphere sizes as follows

$$\left|\mathcal{S}_t^{m,\eta,\ell,q}\right| \le \left(C(x) q^{x(m+\eta-x)}\right)^{*\ell}(t) = C(0)^{\ell-1} C(t) \left(q^{x(m+\eta-x)}\right)^{*\ell}(t).$$

Proposition 4 provides a formula to compute convolutions.

Proposition 4. *Consider $f_\ell(x) := q^{x(m+\eta-x/\ell)}$ for $x \in \mathbb{R}$ and $\ell \in \mathbb{N}$. Functions of this form satisfy the following relation on their discrete convolutions*

$$[f_{\ell_1} * f_{\ell_2}](t) \le \left(1 + \sqrt{\frac{\ell_1 \ell_2 \pi}{(\ell_1 + \ell_2) \ln q}}\right) f_{\ell_1+\ell_2}(t).$$

The bound is obtained by bounding summations by integrals and by noticing $[f_{\ell_1} * f_{\ell_2}](t) = f_{\ell_1+\ell_2}(t) \sum_{i=0}^{t} q^{-\left(\frac{1}{\ell_1}+\frac{1}{\ell_2}\right)\left(i-\frac{\ell_1}{\ell_1+\ell_2}t\right)^2}$. Setting $\ell_1 = 1$ and applying Proposition 4 inductively for $\ell_2 = 1, \ldots, \ell-1$ we obtain upper bounds on the sphere sizes.

Theorem 7. *Let m, η, ℓ, q, t be positive integers. Choosing $C(t)$ equal to γ_q or $\kappa_{q,m,\eta}(t)$, we observe the following bounds, respectively*

$$\left|\mathcal{S}_t^{m,\eta,\ell,q}\right| \leq \gamma_q^{\ell} \prod_{k=1}^{\ell-1} \left(1 + \sqrt{\frac{k\pi}{(k+1)\ln q}}\right) q^{t(m+\eta-t/\ell)}$$
$$\left|\mathcal{S}_t^{m,\eta,\ell,q}\right| \leq \kappa_{q,m,\eta}(t) \prod_{k=1}^{\ell-1} \left(1 + \sqrt{\frac{k\pi}{(k+1)\ln q}}\right) q^{t(m+\eta-t/\ell)}$$

where the further simplifications $\sqrt{\frac{k\pi}{(k+1)\ln q}} \leq \sqrt{\frac{(\ell-1)\pi}{\ell \ln q}} < \sqrt{\frac{\pi}{\ln q}}$ can be made.

4.2 Lower Bound via Ordinary Generating Functions

An alternative approach is not to bound the number of matrices first, but to bound the generating function $\mathcal{S}^{m,\eta,1,q}(z)$ coefficient-wise with another polynomial $\mathcal{F}(z)$ whose ℓ-th power can be computed more easily. The polynomial $\mathcal{F}$ that we use to obtain a lower bound can be factored nicely into linear parts by the q-binomial theorem.

Proposition 5. *Let $m, \eta \in \mathbb{N}$. Then,*

$$\mathcal{F}(z) := \sum_{i=0}^{\mu} q^{i(m+\eta-i)} \begin{bmatrix} \mu \\ i \end{bmatrix}_{1/q^2} z^i = \prod_{i=1}^{\mu} (1 + q^{m+\eta-2i+1} z).$$

This polynomial satisfies the following chain of coefficient-wise inequalities

$$\sum_{i=0}^{\mu} \gamma_q^{-1} q^{i(m+\eta-i)} z^i \preccurlyeq_c \gamma_q^{-1} \mathcal{F}(z) \preccurlyeq_c \mathcal{F}({\gamma_{q,m,\eta}}^{-1/\mu} z) \preccurlyeq_c \mathcal{S}^{m,\eta,1,q}(z).$$

The first inequality follows from $\begin{bmatrix} \mu \\ i \end{bmatrix}_{1/q^2} \geq 1$, the second from $\gamma_q^{-1} \leq {\gamma_{q,m,\eta}}^{-1} \leq {\gamma_{q,m,\eta}}^{-i/\mu} \leq 1$ for $0 \leq i \leq \mu$ and the third from Proposition 2. Since this coefficient-wise inequality is preserved under convolution, we obtain

$$\left(\sum_{i=0}^{\mu} \gamma_q^{-1} q^{i(m+\eta-i)} z^i\right)^{\ell} \preccurlyeq_c \mathcal{F}({\gamma_{q,m,\eta}}^{-1/\mu} z)^{\ell} \preccurlyeq_c \mathcal{S}^{m,\eta,\ell,q}(z). \tag{8}$$

If we look now at $\mathcal{F}(z)^{\ell} = \prod_{i=1}^{\mu} \left(\sum_{j=0}^{\ell} \binom{\ell}{j} q^{j(m+\eta-2i+1)} z^j\right)$, we can lower bound $[z^t]\mathcal{F}(z)^{\ell}$ as follows: let $t = t_*\ell + r$ with $t_* \in \mathbb{N}$ and $0 \leq r < \ell$. Then using, depending on i, the inequality

$$\left(\sum_{j=0}^{\ell} \binom{\ell}{j} q^{j(m+\eta-2i+1)} z^j\right) \succcurlyeq_c \begin{cases} q^{\ell(m+\eta-2i+1)} z^{\ell} & \text{for } 1 \leq i \leq t_* \\ \binom{\ell}{r} q^{r(m+\eta-2i+1)} z^r & \text{for } i = t_* + 1 \\ 1 & \text{for } t_* + 2 \leq i \leq \mu \end{cases} \tag{9}$$

we obtain

$$\sum_{t=0}^{\mu\ell} \binom{\ell}{r} q^{t(m+\eta-\frac{t}{\ell})+\frac{r^2}{\ell}-r} z^t \preccurlyeq_c \mathcal{F}(z)^\ell .$$

Finally, substituting ${\gamma_{q,m,\eta}}^{-1/\mu} z$ for z in this inequality and applying equation (8) we get the following result.

Theorem 8. *Let $t = t_*\ell + r$ with $t_* \in \mathbb{N}$ and $0 \leq r < \ell$. Then*

$$\left({\gamma_{q,m,\eta}}^{-1}\right)^{t/\mu} \binom{\ell}{r} q^{t(m+\eta-\frac{t}{\ell})+\frac{r^2}{\ell}-r} \leq \left|\mathcal{S}_t^{m,\eta,\ell,q}\right| .$$

Notice that remarkably, aside for the coefficient in front, we have obtained the same lower bound as [16, Lemma 2] via a completely different method. However, by choosing different inequalities in (9) there is still room for future optimization. Since $\left(\left|\mathcal{S}_i^{m,\eta,\ell,q}\right|\right)_{i=0}^{\mu\ell}$ is log-concave, we can take the smallest concave sequence that is coefficient-wise greater or equal to the sequence $\left(\log_q\left(\left({\gamma_{q,m,\eta}}^{-1}\right)^{i/\mu}\binom{\ell}{r}q^{i(m+\eta-\frac{i}{\ell})+\frac{r^2}{\ell}-r}\right)\right)_{i=0}^{\mu\ell}$ (i.e. its convex hull) for a slightly improved lower bound on $\log_q\left|\mathcal{S}_t^{m,\eta,\ell,q}\right|$.

5 Comparison of Bounds

In this section, we compare the new bounds presented in this paper with the existing bounds related to the sphere size in the sum-rank metric. In Figure 1 the relationship between the growth rate $\frac{1}{\ell}\log_q\left|\mathcal{S}_t^{m,\eta,\ell,q}\right|$ of the sphere size and the normalized radius ρ is shown. We observe that the upper bound using Theorem 2 and the lower bound using Theorem 4 are the tightest bounds and very close to the exact values. The computation of these bounds necessitates the evaluation of the entropy H_ρ. Computing H_ρ is straightforward for a specified β, whereas determining β for a given ρ cannot be achieved in a closed-form manner, as outlined in (2). For scenarios where prioritizing closed-form expressions dependent on ρ is essential, the derived alternative bounds may better suit the intended use-cases. In Figure 1, the upper bounds from Theorem 7 using $\kappa_{q,m,\eta}$, Theorem 7 using γ_q and Theorem 5 are consolidated into a single piece-wise function by selecting the minimum value among these bounds. The transition points are indicated by circles. We observe that for the new closed-form upper and lower bounds we improve significantly in comparison to the already existing closed-form bounds given in [17, Theorem 5] and [16, Lemma 2]. Furthermore, the new bounds are potentially useful tools for obtaining improved closed-form Gilbert-Varshamov or sphere-packing bounds, as introduced in [1] and [16].

In Figure 2 we show the tightness of the improved bounds for different numbers of blocks. We choose the same values for the parameters q, m, t and n as for the bounds given in [17]. Notably, the bounds proposed in [17] exhibit considerable looseness in scenarios where ℓ becomes substantially large (i.e., when the sum-rank metric converges to the Hamming metric). While superior bounds are already established for the Hamming metric (i.e., $\ell = n$), our analysis illustrates substantial enhancements for $\ell < 60$ compared to existing bounds.

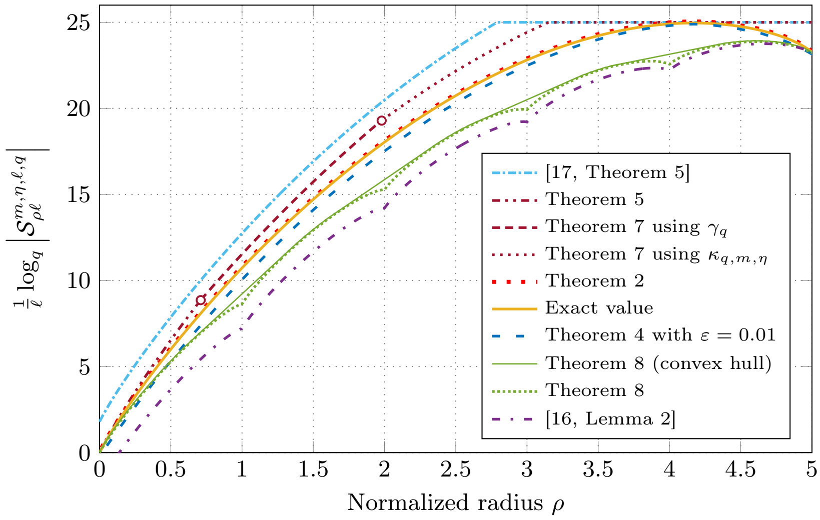

Fig. 1. Comparison of upper and lower bounds for the sphere $\mathcal{S}_{\rho\ell}^{m,\eta,\ell,q}$ as function of ρ with parameters $q = 2$, $m = 5$, $\eta = 5$, $\ell = 100$.

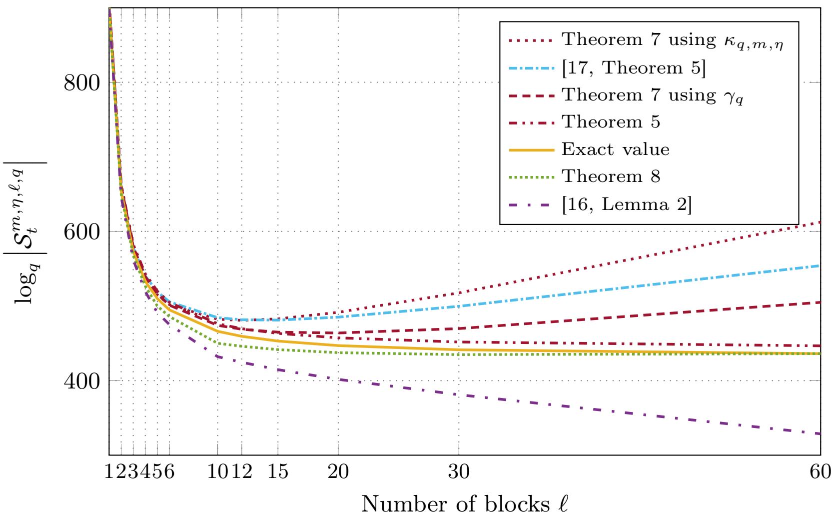

Fig. 2. Comparison of upper and lower bounds for the sphere $\mathcal{S}_{t}^{m,\eta,\ell,q}$ as function of ℓ with parameters $q = 2$, $m = 40$, $t = 10$ and keeping $n = \eta\ell = 60$ constant.

Acknowledgments. H. Sauerbier Couvée is supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 801434) and the Bavarian State Ministry of Science and Arts via the project EQAP. T. Jerkovits and J. Bariffi acknowledge the financial support by the Federal Ministry of Education and Research of Germany in the program of "Souverän. Digital. Vernetzt." Joint project 6G-RIC, project identification number: 16KISK022. The authors thank H. Bartz, A. Baumeister, G. Liva, and A. Wachter-Zeh for their insights and discussions.

References

1. Byrne, E., Gluesing-Luerssen, H., Ravagnani, A.: Fundamental properties of sum-rank-metric codes. IEEE Transactions on Information Theory **67**(10), 6456–6475 (2021). https://doi.org/10.1109/TIT.2021.3074190
2. Flajolet, P., Sedgewick, R.: Analytic combinatorics. Cambridge University Press (2009)
3. Gardy, D., Solé, P.: Saddle point techniques in asymptotic coding theory. In: Workshop on Algebraic Coding. pp. 75–81. Springer (1991)
4. Gilbert, E.N.: A comparison of signalling alphabets. The Bell System Technical Journal **31**(3), 504–522 (1952)
5. Greferath, M., O'Sullivan, M.E.: On bounds for codes over frobenius rings under homogeneous weights. Discrete Mathematics **289**(1), 11–24 (2004). https://doi.org/https://doi.org/10.1016/j.disc.2004.10.002
6. Gruica, A., Horlemann, A.L., Ravagnani, A., Willenborg, N.: Densities of codes of various linearity degrees in translation-invariant metric spaces. Designs, Codes and Cryptography (May 2023), https://doi.org/10.1007/s10623-023-01236-2
7. Hamming, R.W.: Error detecting and error correcting codes. The Bell system technical journal **29**(2), 147–160 (1950)
8. Horlemann-Trautmann, A.L., Weger, V.: Information set decoding in the Lee metric with applications to cryptography. Advances in Mathematics of Communications **15**(4) (2021). https://doi.org/10.3934/amc.2020089
9. Ihringer, F., Justus Liebig University Giessen: Finite geometry intersecting algebraic combinatorics : an investigation of intersection problems related to Erdös-Ko-Rado theorems on Galois geometries with help from algebraic combinatorics. Ph.D. thesis (2015). https://doi.org/10.22029/JLUPUB-9701
10. Lee, C.: Some properties of nonbinary error-correcting codes. IRE Transactions on Information Theory **4**(2), 77–82 (1958)
11. Loeliger, H.A.: An upper bound on the volume of discrete spheres. IEEE Transaction of Information Theory **40**(6), 2071–2073 (1994)
12. Lu, H.F., Kumar, P.V.: A unified construction of space-time codes with optimal rate-diversity tradeoff. IEEE Transactions on Information Theory **51**(5) (2005). https://doi.org/10.1109/TIT.2005.846403
13. Martínez-Peñas, U., Kschischang, F.R.: Universal and dynamic locally repairable codes with maximal recoverability via sum-rank codes. IEEE Transactions on Information Theory **65**(12), 7790–7805 (2019). https://doi.org/10.1109/TIT.2019.2924888
14. Migler, T., Morrison, K.E., Ogle, M.: Weight and rank of matrices over finite fields (2004)
15. da Nóbrega, R.W., Filho, B.F.U.: Multishot codes for network coding using rank-metric codes. 2010 Third IEEE International Workshop on Wireless Network Coding pp. 1–6 (2010), https://api.semanticscholar.org/CorpusID:10178357
16. Ott, C., Puchinger, S., Bossert, M.: Bounds and genericity of sum-rank-metric codes. In: 2021 17th International Symposium Problems of Redundancy in Information and Control Systems, REDUNDANCY 2021 (2021). https://doi.org/10.1109/REDUNDANCY52534.2021.9606442
17. Puchinger, S., Renner, J., Rosenkilde, J.: Generic decoding in the sum-rank metric. IEEE Transactions on Information Theory **68**(8), 5075–5097 (2022)
18. Sauerbier Couvée, H., Jerkovits, T., Bariffi, J.: Bounds on sphere sizes in the sum-rank metric and coordinate-additive metrics (2024), https://arxiv.org/abs/2404.10666

19. Shehadeh, M., Kschischang, F.R.: Space-time codes from sum-rank codes (2021)
20. Tchebichef, P.: Sur les valeurs limites des intégrales. Journal de mathématiques pures et appliquées **19**, 157–160 (1874)