Pontryagin's Minimum Principle for output-feedback systems: A proof using variation methods

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Executive Summary (Zusammenfassung)

Pontryagin's Maximum Principle and the Hamilton-Jacobi-Bellman equation are the most famous results characterization of solutions of optimal control problems. The first one, nowadays often called Pontryagin's Minimum Principle, provides necessary conditions utilizing variational calculus, while the later one also provides necessary conditions based on Bellman's Principle of Optimality. There exist many versions of the minimum principle. Among them are variations that consider constraints and sufficient conditions. Furthermore, a time-discrete formulation of the principle has been developed. Nevertheless, a generalization of the first order criteria to output-feedback systems has not been published. In this report the existing theory will be extended to allow these type of dynamical systems and a rigorous proof using variation methods will be given. The original minimum principle is then derived as a special case of the obtained criteria. Furthermore, general equality and inequality constraints will be taken into account.

Contents

1 Introduction

In the 1950s many famous results in the field of Optimal control (OC) [1, 2] have been developed. It has become and continued to be a very important topic. Nowadays it is indispensable in the fields of robotics, aviation, and even financial economics. Given ordinary differential equations which characterize the development of the system's states, OC aims to optimize a given objective function representing e.g. costs or gains. Historically the optimization problems were stated as maximization problems, while nowadays from application point of view it is more often desired to minimize costs or the deviation from a target state. Around 70 years ago two famous results, the Hamilton-Jacobi-Bellman equation (HJBE) based on Bellman's principle of optimality [3] regarding end pieces of optimal trajectories and Pontryagin's Maximum Principle (PMP) which characterizes optimal solutions, have been developed. The nowadays more often called Pontryagin's Minimum Principle has been derived by Lev Semenovich Pontryagin and his students [4–6]. Their idea was to consider small variations of an optimal trajectory and characterize its optimality using the resulting change in the objective [1, 7] or in other words they utilized the idea of differential calculus from $\mathbb{R}^n$ and applied it to function spaces. In contrast to the HJBE, a free final time and free final state can be taken into account using PMP. Unfortunately, PMP can not directly be used to obtain a feedback-law but is rather utilized for numerical solutions.

Using PMP not only constraints in form of a dynamical system can be considered but also equality and inequality constraints regarding the state and input variables [8–10]. Additionally, terminal state [11–13] can be included.

In contrast to the HJBE the minimum principle only delivers necessary conditions. However, if second order derivatives are taken into account PMP can be extended to deliver sufficient conditions [11, 14]. Furthermore, PMP and the HJBE must characterize the same optimal trajectory. Thus, they can be transformed into each other [15]. Despite all of the mentioned extensions and applications, a version of PMP which is applicable to output-feedback systems is still missing in the literature. Consequently, PMP needs to be extended to allow such systems in Section 2. In the following Section 3 the newly obtained results will be combined with known ones from the literature, i.e. including time-dependent equality and inequality constraints for the input and output. Later in Section 4, the in Section 2 derived equations are transformed into an output-feedback version of the HJBE. Finally, concluding remarks and further research perspectives will be given in Section 5.

2 Pontryagin's Minimum Principle for output-feedback

When criteria for optimal control laws and optimal trajectories are derived, typically optimization problems like

$$
\begin{aligned}
&\min_{u(\cdot)} \int_0^{t_f} \ell\big(\tau, x(\tau), u(\tau)\big)\,\mathrm{d}\tau + L\big(t_f, x(t_f)\big) \\
\text{s.t. } \forall t \in \mathbb{R}_{\geq 0}: \ &\dot{x}(t) = f\big(t, x(t), u(t)\big) \\
&x(0) = x_0 \in \mathbb{R}^{n_x}
\end{aligned}
\tag{1}
$$

are considered. Here $f \in C^1\big(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}; \mathbb{R}^{n_x}\big)$ is the function representing the systems dynamics, $\ell \in C^1\big(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}; \mathbb{R}\big)$ is the stage cost, and $L \in C^1\big(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x}; \mathbb{R}\big)$ the terminal cost function. The numbers $n_x, n_u \in \mathbb{N}$ are the dimensions of the state and input space, respectively, while the final time is denoted with t_f. For this setup Lev Semenovich Pontryagin and his students derived the well known Pontryagin's Minimum Principle, which can be split up into four cases letting the final time t_f and the final state $x(t_f) = x_f$ be free or fixed. In the following, the minimum principle which is obtained via variational calculus will be extended to also include input to output systems, i.e. systems where not all state variables x can be measured. To prepare for the actual derivation, the following lemma is given.

Lemma 1 (Variation approximation)**.**
For some $k, n, m \in \mathbb{N}$ let $f\colon \operatorname{Abb}(\mathbb{R};\mathbb{R}^n) \to \mathbb{R}^m$, $g\colon \operatorname{Abb}(\mathbb{R};\mathbb{R}^n)^k \to \mathbb{R}^m$ and $x, x_1, \ldots, x_,, \Delta x, \Delta x_1, \ldots, \Delta x_, \colon \mathbb{R} \to \mathbb{R}^n$ be C^1-functions and $t, \Delta t \in \mathbb{R}$. Then the following identities hold.

a) $f(x + \Delta x) - f(x) = Df(x) \cdot \Delta x + o(\Delta x)$

b) $$g(x_1 + \Delta x_1, \ldots, x_k + \Delta x_k) - g(x_1, \ldots, x_k) = \sum_{i=1}^{k} D_{x_i} g(x_1, \ldots, x_k) \cdot \Delta x_i + o(\Delta x_1, \ldots, \Delta x_k)$$

c) $$f(x + \Delta x)(t + \Delta t) - f(x)(t) = D_x f(x)(t) \cdot \dot{x}(t) \cdot \Delta t + D_x f(x)(t) \cdot \Delta x(t) + o\big(\Delta t, \Delta x(t)\big)$$

d) $$\int_{t}^{t+\Delta t} f(x)(\tau)\,\mathrm{d}\tau = f(x)(t) \cdot \Delta t + o(\Delta t)$$

Proof. (a) Follows from the definition of the derivative.

(b) The idea is to add clever zeroes and use the result of (a).

$$\begin{aligned}
&g(x_1 + \Delta x_1, \ldots, x_k + \Delta x_k) - g(x_1, \ldots, x_k) \\
&= g(x_1 + \Delta x_1, \ldots, x_k + \Delta x_k) - g(x_1, x_2 + \Delta x_2, \ldots, x_k + \Delta x_k) \\
&\quad + g(x_1, x_2 + \Delta x_2, \ldots, x_k + \Delta x_k) + \ldots - g(x_1, \ldots, x_k + \Delta x_k) \\
&\quad + g(x_1, \ldots, x_k + \Delta x_k) - g(x_1, \ldots, x_k) \\
&\overset{\text{(a)}}{=} \sum_{i=1}^{k} D_{x_i} g(x_1, \ldots, x_k) \cdot \Delta x_i + o(\Delta x_1, \ldots, \Delta x_k)
\end{aligned}$$

(c) Again a clever zero can be added and the result of (a) together with the chain rule is utilized.

$$\begin{aligned}
&f\big((x + \Delta x)(t + \Delta t)\big) - f\big(x(t)\big) \\
&= f\big((x + \Delta x)(t + \Delta t)\big) - f\big((x + \Delta x)(t)\big) + f\big((x + \Delta x)(t)\big) - f\big(x(t)\big) \\
&\overset{\text{(a)}}{=} D_x f\big(x(t)\big) \cdot \dot{x}(t) \cdot \Delta t + D_x f\big(x(t)\big) \cdot \Delta x(t) + o\big(\Delta t, \Delta x(t)\big)
\end{aligned}$$

(d) The claim follows via application of the fundamental theorem of calculus.

□

In the following, the optimal control problem (OCP)

$$\min_{u(\cdot)} \int_0^{t_f} \ell\big(\tau, y(\tau), u(\tau)\big)\,\mathrm{d}\tau + L\big(t_f, y(t_f)\big) \tag{2a}$$

$$\text{s.t. } \forall t \in \mathbb{R}_{\geq 0}: \ \dot{x}(t) = f\big(t, x(t), u(t)\big) \tag{2b}$$

$$\forall t \in \mathbb{R}_{\geq 0}: \ y(t) = h\big(t, x(t)\big) \tag{2c}$$

$$x(0) = x_0 \in \mathbb{R}^{n_x} \tag{2d}$$

is considered. In addition to the OCP in (1) the output function $h \in C^1\big(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x}; \mathbb{R}^{n_y}\big)$ is added. The stage cost function ℓ as well as the terminal cost L now depend on the output values $y \in \mathbb{R}^{n_y}$ instead of the states x but remain of class C^1. Furthermore, $n_y \in \mathbb{N}$ is the dimension of the output. To incorporate the constraints (2b) and (2c) Lagrange multipliers $\lambda \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_x}\big)$ and $\mu \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_y}\big)$ are introduced. In the field of control theory the multipliers λ are better known as co-states.
At this point it shall be noted that using this setup, i.e. the output function h does not depend on the input, one could simply substitute the output in the stage and terminal cost and applying the chain rule together with the classic version of PMP. The details can be found in an overview paper [16] that is currently being published. As outlined in [16], having the output function dependent on the input and the final time or the final state free, one can not use this approach so easily. The reasons for this will be seen in the following derivations (see Remark 1) using variational calculus and will be discussed in Remark 3.
Following well known strategies, the Hamiltonian of the OCP (2) states as

$$\mathcal{H}(t, x, y, u, \lambda, \mu) := \ell(t, y, u) + \lambda^\top \cdot f(t, x, u) + \mu^\top \cdot h(t, x) \tag{3}$$

and is of class C^1 as well. Following the steps outlined in [1] the total cost

$$J(\cdot) := \int_0^{t_f} \ell\big(\tau, y(\tau), u(\tau)\big)\,\mathrm{d}\tau + L\big(t_f, y(t_f)\big)$$

is defined. Utilizing the Hamiltonian (3) the total cost is rewritten as

$$\begin{aligned} J(t_f, y_f, x, y, u, \lambda, \mu) = &\int_0^{t_f} \mathcal{H}\big(\tau, x(\tau), y(\tau), u(\tau), \lambda(\tau), \mu(\tau)\big)\,\mathrm{d}\tau \\ &+ \int_0^{t_f} -\lambda(\tau)^\top \cdot \dot{x}(\tau) - \mu^\top(\tau) \cdot y(\tau)\,\mathrm{d}\tau + L\big(t_f, y(t_f)\big) \end{aligned} \tag{4}$$

Introducing functions $\Delta x \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_x}\big)$, $\Delta y \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_y}\big)$, $\Delta u \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_u}\big)$, $\Delta\lambda \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_\lambda}\big)$, and $\Delta\mu \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_\mu}\big)$ as well as $\Delta t_f \in \{t \in \mathbb{R} : t \geq -t_f\}$ and $\Delta y_f \in \mathbb{R}^{n_y}$ variations of a solution $(t_f, y_f, x, y, u, \lambda, \mu)$ of (2) are denoted with

$(t_f + \Delta t_f, y_f + \Delta y_f, x + \Delta x, y + \Delta y, u + \Delta u, \lambda + \Delta\lambda, \mu + \Delta\mu)$. An illustration of different trajectories in x- and y-space with the same start and end point is given in Fig. 1. To find criteria for optimality, the first variation ΔJ of the total cost function needs to be investigated.

$$
\begin{aligned}
\Delta J &= J(t_f + \Delta t_f, y_f + \Delta y_f, x + \Delta x, y + \Delta y, u + \Delta u, \lambda + \Delta\lambda, \mu + \Delta\mu) \\
&\quad - J(t_f, y_f, x, y, u, \lambda, \mu) \\
&= \int_0^{t_f} \mathcal{H}(\tau, x + \Delta x, y + \Delta y, u + \Delta u, \lambda + \Delta\lambda, \mu + \Delta\mu) - \mathcal{H}(\tau, x, y, u, \lambda, \mu) \,\mathrm{d}\tau \\
&\quad + \int_0^{t_f} -(\lambda + \Delta\lambda)^\top (x \dot{+} \Delta x) + \lambda^\top \dot{x} - (\mu + \Delta\mu)^\top (y + \Delta y) + \mu^\top y \,\mathrm{d}\tau \\
&\quad + \int_{t_f}^{t_f + \Delta t_f} \mathcal{H}(\tau, x + \Delta x, y + \Delta y, u + \Delta u, \lambda + \Delta\lambda, \mu + \Delta\mu) \,\mathrm{d}\tau \\
&\quad + \int_{t_f}^{t_f + \Delta t_f} -(\lambda + \Delta\lambda)^\top (x \dot{+} \Delta x) - (\mu + \Delta\mu)^\top (y + \Delta y) \,\mathrm{d}\tau \\
&\quad + L\big(t_f + \Delta t_f, (y + \Delta y)(t_f + \Delta t_f)\big) - L\big(t_f, y(t_f)\big) \\
&\overset{\text{Lemma 1}}{=} \int_0^{t_f} \nabla_x \mathcal{H}(\tau, x, y, u, \lambda, \mu) \cdot \Delta x + \nabla_y \mathcal{H}(\tau, x, y, u, \lambda, \mu) \cdot \Delta y \,\mathrm{d}\tau \\
&\quad + \int_0^{t_f} \nabla_u \mathcal{H}(\tau, x, y, u, \lambda, \mu) \cdot \Delta u + \nabla_\lambda \mathcal{H}(\tau, x, y, u, \lambda, \mu) \cdot \Delta\lambda \,\mathrm{d}\tau \\
&\quad + \int_0^{t_f} \nabla_\mu \mathcal{H}(\tau, x, y, u, \lambda, \mu) \cdot \Delta\mu - \lambda^\top \dot{\Delta x} - \dot{x}^\top \Delta\lambda - \mu^\top \Delta y - y^\top \Delta\mu \,\mathrm{d}\tau \\
&\quad + \mathcal{H}\big(t_f, x(t_f), y(t_f), u(t_f), \lambda(t_f), \mu(t_f)\big) \cdot \Delta t_f - \lambda^\top(t_f) \cdot \dot{x}(t_f) \cdot \Delta t_f \\
&\quad - \mu^\top(t_f) \cdot y(t_f) \cdot \Delta t_f + \frac{\partial}{\partial t} L\big(t_f, y(t_f)\big) \cdot \Delta t_f + \nabla_y L\big(t_f, y(t_f)\big) \cdot \dot{y}(t_f) \cdot \Delta t_f \\
&\quad + \nabla_y L\big(t_f, y(t_f)\big) \cdot \Delta y(t_f) + o\big(\Delta t_f, \Delta y(t_f), \Delta x, \Delta y, \Delta u, \Delta\lambda, \Delta\mu\big)
\end{aligned} \tag{5}
$$

If $(t_f, y_f, x, y, u, \lambda, \mu)$ is an optimal solution of (2), the first variation ΔJ vanishes as $(\Delta t_f, \Delta y_f, \Delta x, \Delta y, \Delta u, \Delta\lambda, \Delta\mu)$ approaches zero. To be able to obtain the optimality criteria from Equation (5) the terms $-\lambda^\top \cdot \dot{\Delta x}$, $\Delta y(t_f)$, and $\dot{y}(t_f)$ need to be evaluated further. The term $-\lambda^\top \cdot \dot{\Delta x}$ has to be reshaped such that there is no dependence on the derivative of Δx. Keeping in mind that $\Delta x(0) = 0$, since all trajectories start in x_0, integration by parts yields the desired outcome.

$$
\int_0^{t_f} -\lambda^\top(\tau) \cdot \dot{\Delta x}(\tau) \,\mathrm{d}\tau = -\lambda^\top(t_f) \cdot \Delta x(t_f) + \int_0^{t_f} \dot{\lambda}^\top(\tau) \cdot \Delta x(\tau) \,\mathrm{d}\tau \tag{6}
$$

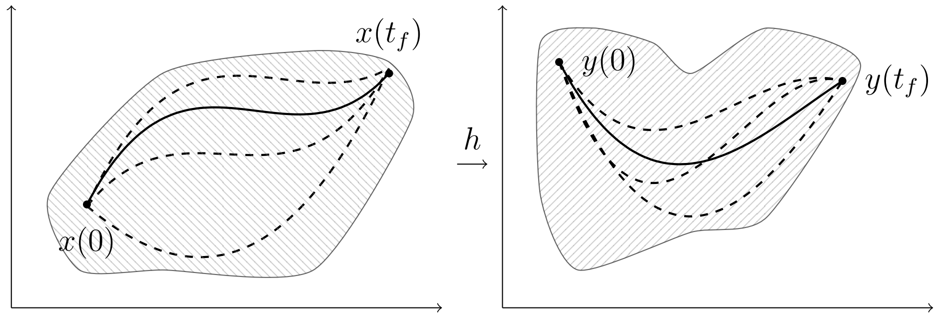

Figure 1: Optimal solution (solid line) and suboptimal solutions (dashed lines) in x- and y-space as well as the reachable region and its image (hatched area) when starting in $x(0)$

$\Delta y(t_f)$ is in a first step stated in terms of $\Delta x(t_f)$, since it is now already included in (5) and in general there is no inverse function of $h(t,\cdot)$.

$$\begin{aligned}\Delta y(t_f) &= h\big(t_f,(x+\Delta x)(t_f)\big) - h\big(t_f,x(t_f)\big) \\ &\overset{\text{Lemma 1 (a)}}{=} \nabla_x h\big(t_f,x(t_f)\big)\cdot\Delta x(t_f) + o\big(\Delta x(t_f)\big)\end{aligned} \tag{7}$$

The following Fig. 2 shows the relation between the variation of the endpoints $\Delta x_f = (x+\Delta x)(t_f+\Delta t_f) - x(t_f)$ of the state trajectories and the variation at the final time $\Delta x(t_f)$. Once again using Lemma 1 (c) with $f(\cdot)$ being the identity, $\Delta x(t_f)$ can be stated

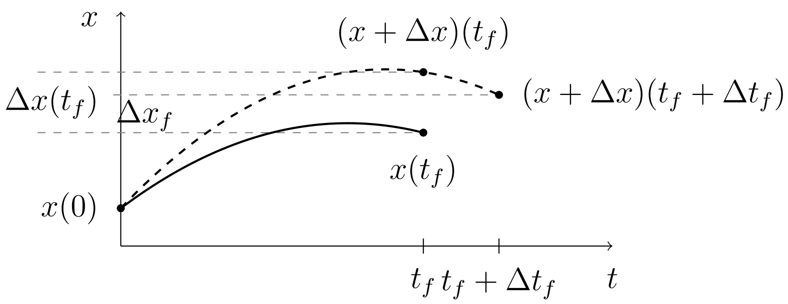

Figure 2: Variation of the endpoint in $x-$space

in terms of Δx_f and Δt_f.

$$\Delta x(t_f) = \Delta x_f - \dot{x}(t_f)\cdot\Delta t_f + o(\Delta t_f) \tag{8}$$

The last term that need to be taken care of is the derivative $\dot{y}(t_f)$. It can be calculated using (2c). Thus, later all terms containing the derivatives at the final time t_f are cancelled out.

$$\dot{y}(t_f) = \frac{\partial}{\partial t}h\big(t_f,x(t_f)\big) + \nabla_x h\big(t_f,x(t_f)\big)\cdot\dot{x}(t_f) \tag{9}$$

Remark 1. *(a) Equation (9) shows why the control input u can not be included in the output function h. It would lead to the additional term $\nabla_u h\big(t_f,x(t_f),u(t_f)\big)\cdot\dot{u}(t_f)$ and, therefore, the knowledge of as well as a condition for the derivative $\dot{u}$. Similarly, if the output function depends on the input, one obtains $\nabla_u h\big(t_f,x(t_f),u(t_f)\big)\cdot\Delta u(t_f)$ as additional term in (7). Stating $\Delta u(t_f)$ in terms of $\Delta u_f := (u+\Delta u)(t_f+\Delta t_f) - u(t_f)$ and Δt_f would again require the knowledge of $\dot{u}(t_f)$, see Equation (8).*

(b) *If on the other hand the final time t_f and the final output y_f respective final state x_f would be fixed in the OCP (2), Δt_f would vanish and, therefore, (5) does not depend on $\dot{y}(t_f)$. Additionally, if the final output y_f is fixed, $\Delta y(t_f)$ vanishes and $h\big(t, x(t)\big)$ can be replaced with $h\big(t, x(t), u(t)\big)$.*

Coming back to the deviation of PMP the Equations (6)-(9) are combined with (5) such that the first variation ΔJ of the total cost yields the following identity.

$$\begin{aligned}
\Delta J = &\int_0^{t_f} \nabla_x \mathcal{H} \cdot \Delta x + \nabla_y \mathcal{H} \cdot \Delta y + \nabla_u \mathcal{H} \cdot \Delta u + \nabla_\lambda \mathcal{H} \cdot \Delta \lambda + \nabla_\mu \mathcal{H} \cdot \Delta \mu \,\mathrm{d}\tau \\
&+ \int_0^{t_f} \dot{\lambda}^\top \cdot \Delta x - \dot{x}^\top \cdot \Delta \lambda - \mu^\top \cdot \Delta y - y^\top \cdot \Delta \mu \,\mathrm{d}\tau - \lambda^\top(t_f) \cdot \Delta x_f \\
&+ \mathcal{H}\big(t_f, x(t_f), y(t_f), u(t_f), \lambda(t_f), \mu(t_f)\big) \cdot \Delta t_f - \mu^\top(t_f) \cdot y(t_f) \cdot \Delta t_f \\
&+ \frac{\partial}{\partial t} L\big(t_f, y(t_f)\big) \cdot \Delta t_f + \nabla_y L\big(t_f, y(t_f)\big) \cdot \frac{\partial}{\partial t} h\big(t_f, x(t_f)\big) \cdot \Delta t_f \\
&+ \nabla_y L\big(t_f, y(t_f)\big) \cdot \nabla_x h\big(t_f, x(t_f)\big) \cdot \Delta x_f + o\big(\Delta t_f, \Delta x_f, \Delta x, \Delta y, \Delta u, \Delta \lambda, \Delta \mu\big)
\end{aligned}$$

ΔJ must vanish as $(\Delta t_f, \Delta x_f, \Delta x, \Delta y, \Delta u, \Delta \lambda, \Delta \mu)$ approaches zero. Since the variation is arbitrary, each component of $(\Delta t_f, \Delta x_f, \Delta x, \Delta y, \Delta u, \Delta \lambda, \Delta \mu)$ can be treated individually, i.e. their coefficients need to vanish as well.

$$\begin{aligned}
\Delta x:\ 0 &= \nabla_x \mathcal{H}\big(t, x(t), y(t), u(t), \lambda(t), \mu(t)\big) + \dot{\lambda}^\top(t) \\
&= \lambda^\top(t) \cdot \nabla_x f\big(t, x(t), u(t)\big) + \mu^\top(t) \cdot \nabla_x h\big(t, x(t)\big) + \dot{\lambda}^\top(t) \\
\Delta y:\ 0 &= \nabla_y \mathcal{H}\big(t, x(t), y(t), u(t), \lambda(t), \mu(t)\big) - \mu^\top(t) \\
&= \nabla_y \ell\big(t, y(t), u(t)\big) - \mu^\top(t) \\
\Delta u:\ 0 &= \nabla_u \mathcal{H}\big(t, x(t), y(t), u(t), \lambda(t), \mu(t)\big) \\
\Delta \lambda:\ 0 &= \nabla_\lambda \mathcal{H}\big(t, x(t), y(t), u(t), \lambda(t), \mu(t)\big) - \dot{x}^\top(t) \\
\Delta \mu:\ 0 &= \nabla_\mu \mathcal{H}\big(t, x(t), y(t), u(t), \lambda(t), \mu(t)\big) - y^\top(t) \\
\Delta x_f:\ 0 &= -\lambda^\top(t_f) + \nabla_y L\big(t_f, y(t_f)\big) \cdot \nabla_x h\big(t_f, x(t_f)\big) \\
\Delta t_f:\ 0 &= \mathcal{H}\big(t_f, x(t_f), y(t_f), u(t_f), \lambda(t_f), \mu(t_f)\big) - \mu^\top(t_f) \cdot y(t_f) \\
&\quad + \frac{\partial}{\partial t} L\big(t_f, y(t_f)\big) + \nabla_y L\big(t_f, y(t_f)\big) \cdot \frac{\partial}{\partial t} h\big(t_f, x(t_f)\big)
\end{aligned}$$

Replacing the Hamiltonian $\mathcal{H}$ now leads to the generalization of PMP.

Theorem 2 (Pontryagin's Minimum Principle for output-feedback)**.**
Suppose an OCP as in (2) with ℓ, L, f, and h continuously differentiable. Then for (x, y, u, λ, μ) to be an optimal solution the following conditions must hold for all $t \geq 0$.

$$\dot{\lambda}^\top(t) = -\lambda^\top(t) \cdot \nabla_x f\big(t, x(t), u(t)\big) - \nabla_y \ell\big(t, y(t), u(t)\big) \cdot \nabla_x h\big(t, x(t)\big) \tag{10a}$$

$$0 = \nabla_u \ell\big(t, y(t), u(t)\big) + \lambda^\top(t) \cdot \nabla_u f\big(t, x(t), u(t)\big) \tag{10b}$$

$$\dot{x}(t) = f\big(t, x(t), u(t)\big) \tag{10c}$$

$$y(t) = h\big(t, x(t)\big) \tag{10d}$$

$$x(0) = x_0$$

Depending on whether or not the final time and final state are fixed or free additional conditions apply.

(a) *Final time t_f and final state x_f are fixed:* $x(t_f) = x_f$

(b) *Final time t_f is fixed and final state x_f is free:*

$$\lambda^\top(t_f) = \nabla_y L\big(t_f, y(t_f)\big) \cdot \nabla_x h\big(t_f, x(t_f)\big) \tag{11}$$

(c) *Final time t_f is free and final state x_f is fixed:* $x(t_f) = x_f$

$$\begin{aligned} 0 = &\ \ell\big(t_f, y(t_f), u(t_f)\big) + \lambda^\top(t_f) \cdot f\big(t_f, x(t_f), u(t_f)\big) \\ &+ \frac{\partial}{\partial t} L\big(t_f, y(t_f)\big) + \nabla_y L\big(t_f, y(t_f)\big) \cdot \frac{\partial}{\partial t} h\big(t_f, x(t_f)\big) \end{aligned} \tag{12}$$

(d) *Final time t_f and final state x_f are free:* (11) *and* (12)

Remark 2 (Parametric version)**.**
In Theorem 2 all functions could also depend on some parameters $p \in \mathbb{R}^{n_p}$ ($n_p \in \mathbb{N}$) leading to a parametric version of PMP and a family of solutions $(t_{f,p}, x_{f,p}, x_p, y_p, u_p, \lambda_p, \mu_p)$. The parametric version of PMP for an infinite horizon can be found in the appendix of [17].

Corollary 3 (Input in the output function)**.**
As outlined in Remark 1, the output function h can be dependent on the input u if the final time t_f and final output y_f are fixed. In this case PMP states as the following:

$$\begin{aligned} \dot{\lambda}^\top(t) &= -\lambda^\top(t) \cdot \nabla_x f\big(t, x(t), u(t)\big) - \nabla_y \ell\big(t, y(t), u(t)\big) \cdot \nabla_x h\big(t, x(t), u(t)\big), \\ 0 &= \nabla_u \ell\big(t, y(t), u(t)\big) + \lambda^\top(t) \cdot \nabla_u f\big(t, x(t), u(t)\big) \\ &\quad + \nabla_y \ell\big(t, y(t), u(t)\big) \cdot \nabla_u h\big(t, x(t), u(t)\big), \\ \dot{x}(t) &= f\big(t, x(t), u(t)\big), \\ y(t) &= h\big(t, x(t), u(t)\big), \\ x(0) &= x_0 \text{ and } x(t_f) = x_f. \end{aligned}$$

The major difference is the additional term in the second condition which is necessary for an extremum regarding u.

Naturally, the original version of PMP can be obtained as a special case of the output-feedback version given in Theorem 2.

Corollary 4 (Pontryagin's Minimum Principle)**.**
Given an OCP as in (2) with ℓ, L, and f continuously differentiable. Then if $y = h(t, x) = x$ the optimality conditions stated in Theorem 2 simplify to

$$\begin{aligned} \dot{\lambda}^\top(t) &= -\lambda^\top(t) \cdot \nabla_x f\big(t, x(t), u(t)\big) - \nabla_x \ell\big(t, x(t), u(t)\big), \\ 0 &= \nabla_u \ell\big(t, x(t), u(t)\big) + \lambda^\top(t) \cdot \nabla_u f\big(t, x(t), u(t)\big), \\ \dot{x}(t) &= f\big(t, x(t), u(t)\big), \\ x(0) &= x_0 \end{aligned}$$

and one of the following depending on whether or not the final time and state are free.

(a) *Final time t_f and final state x_f are fixed: $x(t_f) = x_f$*

(b) *Final time t_f is fixed and final state x_f is free:*

$$\lambda^\top(t_f) = \nabla_x L\big(t_f, x(t_f)\big) \tag{13}$$

(c) *Final time t_f is free and final state x_f is fixed: $x(t_f) = x_f$ and*

$$0 = \ell\big(t_f, x(t_f), u(t_f)\big) + \lambda^\top(t_f) \cdot f\big(t_f, x(t_f), u(t_f)\big) + \frac{\partial}{\partial t} L\big(t_f, x(t_f)\big) \tag{14}$$

(d) *Final time t_f and final state x_f are free:* (13) *and* (14)

Remark 3 (Chain rule)**.**
Comparing the classical version of PMP stated in Corollary 4 with the generalized results from Theorem 2 one would obtain the same by simply applying the chain rule for $y(t) = h\big(t, x(t)\big)$. Unfortunately, this argument is in general not bulletproof, since without further considerations, i.e. restrictions regarding the input, this would lead to wrong results if the output function additionally depends on the input, i.e. $y(t) = h\big(t, x(t), u(t)\big)$. The chain rule would lead to Corollary 3 with extra conditions in case of free final time and free final state/output similar to the Equations 11 and 12. Since this is in contrast to Remark 1, applying the chain rule still requires insight in how the PMP conditions are originally derived.

3 Equality and inequality constraints

The goal in this section is to extend the OCP (2) by the following constraints and again derive necessary conditions for a minimum.

$$\begin{aligned} \forall t \in \mathbb{R}_{\geq 0}: \quad g_1\big(t, y(t), u(t)\big) &\leq 0 && \text{(15a)}\\ g_2\big(t, y(t), u(t)\big) &= 0 && \text{(15b)}\\ G_1\big(t_f, y(t_f)\big) &\leq 0 && \text{(15c)}\\ G_2\big(t_f, y(t_f)\big) &= 0 && \text{(15d)} \end{aligned}$$

As all functions in (2), $g_i\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_{g_i}}$, $G_i\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_{G_i}}$ $(i \in \{1,2\})$ need to be at least continuously differentiable. The numbers of constraints of each type are denoted with n_{g_i} and n_{G_i} $(i \in \{1,2\})$. Such rather general inequality and equality constraints have been already considered in [9, 14] and others. Therefore, their incorporation is not new. Nevertheless, other authors mostly consider constraints containing only the states or input or final state, see [5, 11, 13]. To incorporate the constraints (15) Lagrange multipliers and variations $\alpha_i, \Delta\alpha_i \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_{g_i}}\big)$ with $\alpha_1(\cdot) \geq 0$ and $\alpha_1(\cdot) + \Delta\alpha_1(\cdot) \geq 0$ as well as $\beta_i, \Delta\beta_i \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}^{n_{G_i}}\big)$ $(i \in \{1,2\})$ with $\beta_1 \geq 0$ and $\beta_1 + \Delta\beta_1 \geq 0$ are introduced. The strategy and calculation is now very similar to what has been shown in detail in Section 2. Therefore, only the additional terms in the variation ΔJ (5) will be discussed next. First of all the variation of $\alpha_i^\top \cdot g_i$ in the time interval $[0, t_f]$ is considered.

$$\int_0^{t_f} (\alpha_i + \Delta\alpha_i)^\top \cdot g_i\big(\tau, (y + \Delta y), (u + \Delta u)\big) - \alpha_i^\top \cdot g_i(\tau, y, u)\,\mathrm{d}\tau$$

$$\overset{\text{Lemma 1 (b)}}{=} \int_0^{t_f} g_i^\top(\tau, y, u) \cdot \Delta\alpha_i + \alpha_i^\top \cdot \nabla_y g_i(\tau, y, u) \cdot \Delta y + \alpha_i^\top \cdot \nabla_u g_i(\tau, y, u) \cdot \Delta u\,\mathrm{d}\tau$$

From this formula one implies $g_i(\cdot)$ must equal zero if the choice of $\Delta\alpha_i(\cdot)$ is free. In case of g_1 it is more restrictive than (15a). Therefore, only Lagrange multipliers with the property $\alpha_1(t) = 0$ if the constraint g_1 is strictly fulfilled should be allowed. For sufficiently small variations $\Delta y(t)$ and $\Delta u(t)$ it follows $g_1\big(t, (y + \Delta y)(t), (u + \Delta u)(t)\big)$ strictly smaller than zero and, therefore, also $(\alpha_1 + \Delta\alpha_1)(t) = 0$ and consequently $\Delta\alpha_1(t) = 0$. Thus, the term $g_i^\top(\tau, y, u) \cdot \Delta\alpha_i$ ($i \in \{1, 2\}$) vanishes while the other two summands will change (10a) and (10b).
Lemma 1 (d) can be used to simplify the next term containing the variation in the additional time interval $[t_f, t_f + \Delta t_f]$.

$$\int_{t_f}^{t_f+\Delta t_f} (\alpha_i + \Delta\alpha_i)^\top \cdot g_i\big(\tau, (y + \Delta y), (u + \Delta u)\big)\,\mathrm{d}\tau = \alpha_i^\top(t_f) \cdot g_i\big(t_f, y(t_f), u(t_f)\big) \cdot \Delta t_f$$

Since $\alpha_1^\top \cdot g_1$ and g_2 have to vanish, these terms do not need to be considered further. The last terms that need to be taken care of are the variations of the terminal constraints (15c) and (15d). The calculation steps are the same as were needed for the terminal cost L in the previous section.

$$\begin{aligned}
&(\beta_i + \Delta\beta_i)^\top \cdot G_i\big(t_f + \Delta t_f, (y + \Delta y)(t_f + \Delta t_f)\big) - \beta_i^\top \cdot G_i\big(t_f, y(t_f)\big) \\
\overset{\text{Lemma 1 (b),(c)}}{\underset{(7),(8),(9)}{=}}\ & G_i^\top\big(t_f, y(t_f)\big) \cdot \Delta\beta_i + \beta_i^\top \cdot \nabla_y G_i\big(t_f, y(t_f)\big) \cdot \frac{\partial}{\partial t} h\big(t_f, x(t_f)\big) \cdot \Delta t_f \\
& + \beta_i^\top \cdot \frac{\partial}{\partial t} G_i\big(t_f, y(t_f)\big) \cdot \Delta t_f + \beta_i^\top \cdot \nabla_y G_i\big(t_f, y(t_f)\big) \cdot \nabla_x h\big(t_f, x(t_f)\big) \cdot \Delta x_f
\end{aligned}$$

As above, it makes more sense to require $\beta_1^\top \cdot G_1 \equiv 0$. Thus, with the same arguments $G_i^\top\big(t_f, y(t_f)\big) \cdot \Delta\beta_i$ vanishes, while the other three summands must be added to (12) and (11). To state the final theorem of this section the constraints g_1 and g_2 as well as G_1 and G_2 need one more property. A more general version of this property is stated in [9].

Definition 1 (Uniformly positively linear independence)**.**
Functions $A_i, B_j \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}\big)$ ($i \in \{1, \ldots, n_a\}$, $j \in \{1, \ldots, n_b\}$, $n_a, n_b \in \mathbb{N}$) are called uniformly in t positively linear independent if there exists $\delta > 0$ such that for any $a_i, b_j \in C^1\big(\mathbb{R}_{\geq 0}; \mathbb{R}\big)$ with $a_i(\cdot) \geq 0$ and

$$\forall t \geq 0 : \sum_{i=1}^{n_a} a_i(t) + \sum_{j=1}^{n_b} |b_j(t)| = 1$$

the following holds:

$$\left| \sum_{i=1}^{n_a} a_i(t) \cdot A_i(t) + \sum_{j=1}^{n_b} b_j(t) \cdot B_j(t) \right| \geq \delta.$$

Having this PMP for output-feedback including input and output as well as terminal equality and inequality constraints is formulated as follows.

Theorem 5 (Output-feedback with equality and inequality constraints)**.**
Suppose an OCP as in (2) together with the constraints (15), where all functions are at least continuously differentiable and $t \mapsto g_1\big(t, y(t), u(t)\big)$ *and* $t \mapsto g_2\big(t, y(t), u(t)\big)$ *as well as* $t \mapsto G_i\big(t, y(t)\big)$ *and* $t \mapsto G_i\big(t, y(t)\big)$ *(*$i \in \{1, 2\}$*) are uniformly in* t *positively linear independent. Then for* $(x, y, u, \lambda, \mu, \alpha_1, \alpha_2, \beta_1, \beta_2)$ *to be an optimal solution the following conditions must hold for all* $t \geq 0$*.*

$$\begin{aligned} \dot{\lambda}^\top(t) &= -\lambda^\top(t) \cdot \nabla_x f\big(t, x(t), u(t)\big) - \nabla_y \ell\big(t, y(t), u(t)\big) \cdot \nabla_x h\big(t, x(t)\big) \\ &\quad - \sum_{i=1}^{2} \alpha_i^\top(t) \cdot \nabla_y g_i\big(t, y(t), u(t)\big) \cdot \nabla_x h\big(t, x(t)\big) \end{aligned} \tag{16a}$$

$$\begin{aligned} 0 &= \nabla_u \ell\big(t, y(t), u(t)\big) + \lambda^\top(t) \cdot \nabla_u f\big(t, x(t), u(t)\big) \\ &\quad + \sum_{i=1}^{2} \alpha_i^\top(t) \cdot \nabla_u g_i\big(t, y(t), u(t)\big) \end{aligned} \tag{16b}$$

$$\dot{x}(t) = f\big(t, x(t), u(t)\big) \tag{16c}$$

$$y(t) = h\big(t, x(t)\big) \tag{16d}$$

$$0 = \alpha_1^\top(t) \cdot g_1\big(t, y(t), u(t)\big) \tag{16e}$$

$$0 \geq g_1\big(t, y(t), u(t)\big) \tag{16f}$$

$$\alpha_1(t) \geq 0, \quad \beta_1 \geq 0 \tag{16g}$$

$$0 = g_2\big(t, y(t), u(t)\big) \tag{16h}$$

$$x(0) = x_0 \tag{16i}$$

Depending on whether or not the final time and final state are fixed or free additional conditions apply.

(a) Final time t_f *and final state* x_f *are fixed:* $x(t_f) = x_f$

(b) Final time t_f *is fixed and final state* x_f *is free:*

$$\begin{aligned} \lambda^\top(t_f) &= \nabla_y L\big(t_f, y(t_f)\big) \cdot \nabla_x h\big(t_f, x(t_f)\big) \\ &\quad + \sum_{i=1}^{2} \beta_i^\top \cdot \nabla_y G_i\big(t_f, y(t_f)\big) \cdot \nabla_x h\big(t_f, x(t_f)\big) \end{aligned} \tag{17}$$

(c) *Final time t_f is free and final state x_f is fixed: $x(t_f) = x_f$*

$$\begin{aligned} 0 &= \ell\big(t_f, y(t_f), u(t_f)\big) + \lambda^\top(t_f) \cdot f\big(t_f, x(t_f), u(t_f)\big) \\ &\quad + \frac{\partial}{\partial t} L\big(t_f, y(t_f)\big) + \nabla_y L\big(t_f, y(t_f)\big) \cdot \frac{\partial}{\partial t} h\big(t_f, x(t_f)\big) \\ &\quad + \sum_{i=1}^{2} \left(\beta_i^\top \cdot \nabla_y G_i\big(t_f, y(t_f)\big) \cdot \frac{\partial}{\partial t} h\big(t_f, x(t_f)\big) + \beta_i^\top \cdot \frac{\partial}{\partial t} G_i\big(t_f, y(t_f)\big) \right) \end{aligned} \tag{18}$$

(d) *Final time t_f and final state x_f are free:* (17) *and* (18)

Remark 4. *If one of the functions ℓ, f, g_1, or g_2 is not differentiable with respect to the input u Equation* (16b) *can be replaced by the more general formulation*

$$u = \underset{\tilde{u}(\cdot)}{\operatorname{argmin}} \begin{cases} \ell\big(t, y(t), \tilde{u}(t)\big) + \lambda^\top(t) \cdot f\big(t, x(t), \tilde{u}(t)\big) \\ \textit{s.t. } (15a) \textit{ and } (15b). \end{cases}$$

An adaptation of Corollary 3 to include constraints as well as a parametric version are obvious and, therefore, omitted.

To illustrate how to use the obtained conditions (16) an example representing the optimization of the speed profile $v(\cdot)$ and slope $\gamma(\cdot)$ to minimize the fuel consumption and time during a rocket launch is utilized. The states are the altitude $h(\cdot)$ and total mass $m(\cdot)$, while it is assumed that only the consumed fuel $y(\cdot) = m(0) - m(\cdot)$ is measured. The final time and mass are free, while the final altitude should be $9144\,\mathrm{m}$. Given some constraints for the speed ($\geq 128.6\,\mathrm{m\,s^{-1}}$) and slope (in $[0\,\mathrm{deg}, 15\,\mathrm{deg}]$) as well as initial values the optimization problem states as the following.

$$\begin{aligned} &\min_{v(\cdot),\gamma(\cdot)} \alpha \cdot t_f + (1-\alpha) \cdot y(t_f) \\ &\qquad \dot{h} = v \cdot \sin(\gamma) \\ &\qquad \dot{m} = -C_{s1} \cdot \left(1 + \frac{v}{C_{s2}}\right) \cdot C_{T1} \cdot \left(1 - \frac{h}{C_{T2}} + C_{T3} \cdot h^2\right) \\ &\qquad y = m_0 - m \\ &128.6 - v \leq 0, \quad -\gamma \leq 0, \quad \gamma - 0.262 \leq 0 \\ &\qquad h_0 = 3480, \quad m_0 = 69\,000, \quad h_f = 9144, \quad m_f \text{ free} \end{aligned}$$

Here the coefficient $\alpha \in [0, 1]$ can be used to weight the two different objectives of minimizing, i.e. the time and the total fuel consumption. The values of the coefficients C_{s1}, C_{s2}, C_{T1}, C_{T2}, and C_{T3} can be found in [18], where this example was taken from. The running cost $\ell(\cdot)$ is zero, while three constraints regarding the input variables must be fulfilled. Furthermore, one final state is fixed while the other one is free. Additionally to the equalities and inequalities in the optimization problem, (16) leads to

$$\begin{aligned}
\begin{pmatrix}\dot\lambda_1 & \dot\lambda_2\end{pmatrix} &\overset{(16a)}{=} -\begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix}\cdot\begin{pmatrix}0 & 0\\ -C_{s1}\cdot\left(1+\frac{v}{C_{s2}}\right)\cdot C_{T1}\cdot\left(-\frac{1}{C_{T2}}+2C_{T3}\cdot h\right) & 0\end{pmatrix},\\
0 &\overset{(16b)}{=} \begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix}\cdot\begin{pmatrix}\sin(\gamma) & v\cdot\cos(\gamma)\\ -\frac{C_{s1}}{C_{s2}}\cdot C_{T1}\cdot\left(1-\frac{h}{C_{T2}}+C_{T3}\cdot h^2\right) & 0\end{pmatrix}\\
&\quad+\begin{pmatrix}-\alpha_1 & -\alpha_2+\alpha_3\end{pmatrix},\\
0 &\overset{(16e)}{=} \alpha_1\cdot(128.6-v),\quad 0\overset{(16e)}{=}\alpha_2\cdot\gamma,\quad 0\overset{(16e)}{=}\alpha_3\cdot(\gamma-0.262),\\
0 &\overset{(16g)}{\le} \alpha_1,\alpha_2,\alpha_3,\\
\lambda_2(t_f) &\overset{(17)}{=} \alpha-1,\\
-\alpha &\overset{(18)}{=} \begin{pmatrix}\lambda_1(t_f) & \lambda_2(t_f)\end{pmatrix}\cdot\begin{pmatrix}v(t_f)\cdot\sin\big(\gamma(t_f)\big)\\ -C_{s1}\cdot\left(1+\frac{v(t_f)}{C_{s2}}\right)\cdot C_{T1}\cdot\left(1-\frac{h(t_f)}{C_{T2}}+C_{T3}\cdot h^2(t_f)\right)\end{pmatrix},
\end{aligned}$$

which can only be solved numerically.

4 Relation to the Hamilton-Jacobi-Bellman equation

In this section the results of Theorem 2 will be utilized to derive the HJBE [2, 3] for output-feedback systems. The other direction, i.e. how the HJBE can be used to obtain PMP for state-feedback systems, is outlined in [15].
The main idea is to start from Equation (10a) and substitute $\lambda^\top(t)$ with

$$\nabla_y V\big(t,h(t,x(t))\big)\cdot\nabla_x h\big(t,x(t)\big)=\nabla_x V\big(t,h(t,x(t))\big),$$

where $V\in C^1\big(\mathbb{R}_{\ge 0}\times\mathbb{R}^{n_y};\mathbb{R}\big)$ is the so-called value function of the OCP in (2).

$$V\big(t,y(t)\big):=\min_{u(\cdot)}\int_t^{t_f}\ell\big(\tau,y(\tau),u(\tau)\big)\,\mathrm{d}\tau+L\big(t_f,y(t_f)\big) \tag{19}$$

Given (10c) and (10d) condition (10a) transforms into a gradient.

$$\begin{aligned}
0&=\frac{\mathrm{d}}{\mathrm{d}t}\Big(\nabla_x V\big(t,h(t,x(t))\big)\Big)+\nabla_x V\big(t,h(t,x(t))\big)\cdot\nabla_x f\big(t,x(t),u(t)\big)\\
&\quad+\nabla_y\ell\big(t,h(t,x(t)),u(t)\big)\cdot\nabla_x h\big(t,x(t)\big)\\
&=\nabla_x\left[\frac{\partial}{\partial t}V\big(t,h(t,x(t))\big)+\nabla_y V\big(t,h(t,x(t))\big)\cdot\frac{\partial}{\partial t}h\big(t,x(t)\big)\right]\\
&\quad+\nabla_x\Big[\nabla_y V\big(t,h(t,x(t))\big)\cdot\nabla_x h\big(t,x(t)\big)\cdot f\big(t,x(t),u(t)\big)\Big]+\nabla_x\ell\big(t,h(t,x(t)),u(t)\big)
\end{aligned}$$

The substitution of the derivative $\dot x(t)$ with the function $f\big(t,x(t),u(t)\big)$ leads to another dependency on the states. Therefore, the order of derivatives with respect to x and t can not be exchanged so easily. This problem can be overcome by incorporating the term

$\nabla_x V\big(t, h(t, x(t))\big) \cdot \nabla_x f\big(t, x(t), u(t)\big)$ into the gradient. Integration with respect to x together with the terminal condition (12) yields

$$\begin{aligned} 0 = \frac{\partial}{\partial t} V\big(t, y(t)\big) + \nabla_y V\big(t, y(t)\big) \cdot \frac{\partial}{\partial t} h\big(t, x(t)\big) \\ + \nabla_y V\big(t, y(t)\big) \cdot \nabla_x h\big(t, x(t)\big) \cdot f\big(t, x(t), u(t)\big) + \ell\big(t, y(t), u(t)\big), \end{aligned} \tag{20}$$

which is the HJBE for the output-feedback and time-dependent case. The second terminal condition (11) is already included in the definition of the value function (19) since $V\big(t_f, y(t_f)\big) = L\big(t_f, y(t_f)\big)$. Finally, the derivative of (20) with respect to u leads to

$$0 = \nabla_y V\big(t, y(t)\big) \cdot \nabla_x h\big(t, x(t)\big) \cdot \nabla_u f\big(t, x(t), u(t)\big) + \nabla_u \ell\big(t, y(t), u(t)\big) \tag{21}$$

the first order optimality condition which is identical with (10b).

Corollary 6 (Hamilton-Jacobi-Bellman equation for output-feedback)**.**
Consider an OCP (2) *with ℓ, L, f, and h continuously differentiable. Then the Hamilton-Jacobi-Bellman equation is given by* (20) *and a first order optimality criteria by* (21).

Remark 5. *A derivation of the HJBE for time-independent output-feedback systems based on Bellman's Principle of Optimality can be found in the appendix of [17].*

5 Conclusions

In this work, Pontryagin's Minimum Principle has been generalized to output-feedback systems. Depending on whether or not the final time and the final output respectively state are fixed or free optimality conditions have been found. Conditions under which the output function may include the systems input were investigated. Furthermore, time-depended equality and inequality constraints for the input and output variables as well as terminal equality and inequality constraints were taken into account and the optimality criteria were adjusted. To show a possible application, an example regarding the time and fuel optimal climb phase during a rocket launch has been given. In the end, an output-feedback version of the HJBE has been derived using the generalized PMP.
Future research potentially focuses on a discrete-time output-feedback version of Pontryagin's Minimum Principle and sufficient conditions for the optimality.

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