An Intrinsic McAulay-Seidman Bound for Parameters Evolving on Matrix Lie Groups

Samy Labsir
DEOS/NAVIR^{2}ES
ISAE-SUPAERO
Toulouse, France

Daniel Medina
Institute of Comm. and Navigation
German Aerospace Center (DLR)
Neustrelitz, Germany

Jordi Vilà-Valls
DEOS/NAVIR^{2}ES
ISAE-SUPAERO
Toulouse, France

Eric Chaumette
DEOS/NAVIR^{2}ES
ISAE-SUPAERO
Toulouse, France

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***Abstract*—Lower bounds on the mean square error (MSE) are of fundamental importance to know the ultimate achievable estimation performance of any unbiased estimator. Even if the Cramér-Rao bound (CRB) is the most popular one, mainly due to its simplicity of calculation, other bounds are of interest in several applications. In this communication, we derive a new intrinsic McAulay-Seidman bound (IMSB) for the estimation of unknown deterministic parameters lying on Lie groups, which generalize known results on the intrinsic CRB. The validity of the proposed IMSB is shown for the Gaussian observation model with unknown deterministic parameters belonging to $SO(3)$ by comparing the IMSB with the intrinsic MSE.**

I. INTRODUCTION

It is well-known that for the characterization of estimation techniques it is fundamental to know the ultimate achievable performance in the the mean square error (MSE) sense. This information can be brought by lower bounds on the MSE [1]. The Cramér-Rao bound (CRB) [2], [3] is the most popular, mainly due to its simplicity of calculation, and because it gives an accurate estimation of the MSE of the maximum likelihood estimator (MLE) in the asymptotic region of operation, i.e., in the large sample regime and/or high signal-to-noise (SNR) regime of the Gaussian conditional signal model [4], [5].

In the last decade there has been an increasing interest in the derivation of intrinsic lower bounds [6]–[10], where the parameters of interest live in a manifold, which appears in many signal processing applications. For instance, in vision-based problems the transformation between two images belongs to the manifold $SO(3)$, in simultaneous localization and mapping one must estimate the pose transformation belonging to $SE(3)$, and in several applications the parameters of interest belong to the manifold of Hermitian positive definite matrices [10]. For all these problems, it is crucial to obtain a lower bound that takes into account the manifold properties. Notice that the contributions cited above focus on the derivation of intrinsic CRBs (ICRBs). In the context of Lie groups (LGs), the corresponding ICRB has also been studied in some contributions. For instance, [9] proposed an inequality on the intrinsic MSE for LGs, and provided a tractable approximated CRB on $SO(3)$.

In this article, we propose a new intrinsic bound on LGs, so-called intrinsic McAulay-Seidman bound (IMSB), which is a generalization of both the classical MSB and CRB. Indeed, in the Euclidean formalism, the MSB is an approximation of the general Barankin bound [11], [12], then we first generalize this bound for parameters evolving on LGs. This is achieved by rigorously defining the notion of intrinsic MSE (IMSE) and bias for estimators on LGs, that leads to an intrinsic Barankin bound (IBB) on LGs. Then, by specifying the notion of test points with respect to the group operations, we can obtain an approximation of such IBB that yields the IMSB on LGs. Because the IMSB only depends on the group exponential and logarithm operators, it admits a closed-form expression for Gaussian observation models where the parameters of interest belong to a matrix LG. Thus, the validity of the proposed IMSB is shown for the Gaussian observation model with unknown deterministic parameters belonging to $SO(3)$ by comparing the IMSB with the IMSE obtained through Monte Carlo simulations.

The communication is organized as follows: in Sec. II we introduce some background of LG theory. In Sec. III we detail the expression of the proposed IMSB. Furthermore, we compute an analytical expression for the LG $SO(3)$. Then, in Sec. IV we confirm by numerical simulations the consistency of the proposed bound.

II. BACKGROUND ON LIE GROUPS

A. Definition

A matrix LG $G \subset \mathbb{R}^{n\times n}$ is a matrix space with a structure of smooth manifold and group [13] [14].

- Its smooth manifold nature means that it is possible to define the operations of integration and derivation. Particularly, we can specify the notion of tangent space according to each element of G.
- Its group nature involves the existence of an internal law connecting between each element of G. Thus, it exists a neutral element (identity matrix) allowing the inversion of each element. Moreover, its internal law allows to link each element of the neutral element tangent space to the tangent space of any element, as shown in Fig. 1.

B. Lie algebra

The tangent space $T_{\mathbf{I}}G$ is denoted Lie algebra and can be written as $\mathfrak{g}$. Each element of the Lie group can be projected to an element of the Lie algebra by using the

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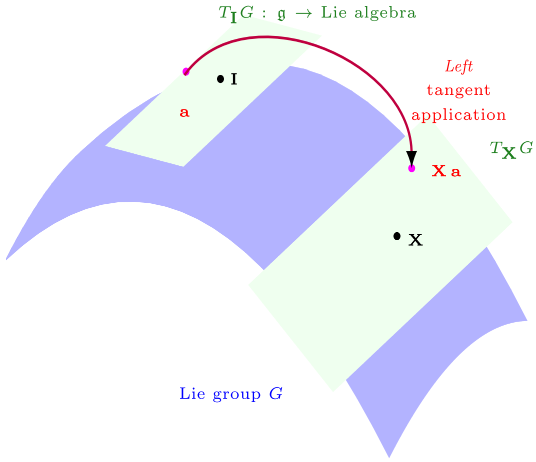

Fig. 1. Relation between the space tangent to $\mathbf{X} \in G$ and the space tangent to the neutral element $\mathbf{I}$. The element $\mathbf{a}$ belonging to $\mathfrak{g} = T_{\mathbf{I}}G$ is transported to $T_{\mathbf{X}}G$ thanks to the left application defined by $\mathbf{Xa}$.

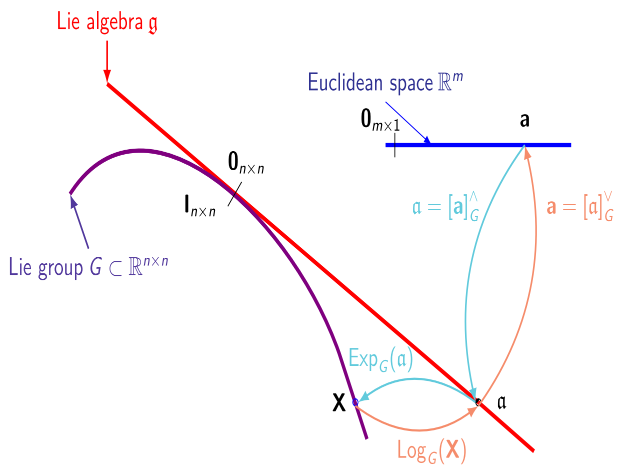

Fig. 2. Relation between $\mathbb{R}^m$, G and $\mathfrak{g}$

logarithm and exponential applications defined, respectively, by $\text{Exp}_G : \mathfrak{g} \to G$ and $\text{Log}_G : G \to \mathfrak{g}$, as illustrated in Fig. 2. What's more, as $\mathfrak{g}$ is isomorph to $\mathbb{R}^m$, we can specify two bijections $[.]^\wedge : \mathbb{R}^m \to \mathfrak{g}$ and $[.]^\vee : \mathfrak{g} \to \mathbb{R}^m$. In this way, we can denote the exponential and logarithm applications such as: $\forall\, \mathbf{a} \in \mathbb{R}^m$, $\text{Exp}_G^\wedge(\mathbf{a}) = \text{Exp}([\mathbf{a}]_G^\wedge)$ and $\forall\, \mathbf{X} \in G$, $[\text{Log}_G(\mathbf{X})]_G^\vee = \text{Log}_G^\vee(\mathbf{X})$.

C. *Case of the LG $SO(3)$*

As previously emphasized, we focus our attention on the LG $SO(3)$. It is the group of rotation matrices in 3D dimension. $\mathbf{X} \in SO(3)$ verifies the following properties : $\mathbf{X}\mathbf{X}^\top = \mathbf{I}$ and $|\mathbf{X}| = 1$. Its Lie algebra corresponds to the set of skew-symmetric matrices and $\mathfrak{so}(3) = \{[\mathbf{w}]_\times | \mathbf{w} \in \mathbb{R}^3\}$ where $[.]_\times$ denotes the operator transforming a vector to a skew-symmetric matrix. $\forall \mathbf{X}$ close to $\mathbf{I}$, $\mathbf{X}$ can be written as $\text{Exp}_{SO(3)}^\wedge(\mathbf{w})$ (with $\mathbf{w} \in \mathbb{R}^3$) and can be developed with the Rodrigues formula [15]:

$$\text{Exp}_{SO(3)}^\wedge(\mathbf{w}) = \mathbf{I}_{3\times 3} + \frac{[\mathbf{w}]_\times}{\|\mathbf{w}\|}\sin(\|\mathbf{w}\|) + \frac{[\mathbf{w}]_\times^2}{\|\mathbf{w}\|^2}\left(1-\cos(\|\mathbf{w}\|)\right). \quad (1)$$

Conversely, the logarithm operator computed in $\mathbf{X}$ is provided by:

$$\text{Log}_{SO(3)}^\vee(\mathbf{X}) = \frac{\|\mathbf{w}\|\left[\mathbf{X}-\mathbf{X}^\top\right]^\vee}{2\sin(\|\mathbf{w}\|)}. \quad (2)$$

where $[.]^\vee$ is the operator transforming a skew-symmetric matrix to a vector.

D. *Estimation on Lie groups*

In the Euclidean framework, an estimator $\hat{\mathbf{x}}$ of the unknown parameter $\mathbf{x} \in \mathbb{R}^p$, gathered from the likelihood of observations $\mathbf{z}$, $p(\mathbf{z}|\mathbf{x})$, can be described by three relevant statistical indicators[1]: its intrinsic mean $\mathbf{m}_{\hat{\mathbf{x}}}$ verifying $\int_{\mathbf{z}\in\mathbb{R}^m}(\hat{\mathbf{x}} - \mathbf{m}_{\hat{\mathbf{x}}})p(\mathbf{z}|\mathbf{x})d\mathbf{z} = \mathbf{0}$, its bias $\int_{\mathbf{z}\in\mathbb{R}^m}(\mathbf{x}-\hat{\mathbf{x}})p(\mathbf{z}|\mathbf{x})\,d\mathbf{z}$ and its estimation error covariance $\int_{\mathbf{z}\in\mathbb{R}^m}(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^\top p(\mathbf{z}|\mathbf{x})d\,\mathbf{z}$.

Now, consider the LG framework. Let us assume that a random observation $\mathbf{Z}$, belonging to a LG G', is disponible and is function of an unknown parameter $\mathbf{X} \in G$. Both are linked by the likelihood $p(\mathbf{Z}|\mathbf{X})$.

Intrinsically, the gap between $\mathbf{X}$ and some LG estimator $\widehat{\mathbf{X}}$ can be evaluated by the error norm $\|\text{Log}_G^\vee\left(\mathbf{X}^{-1}\widehat{\mathbf{X}}\right)\|$. It is worth noticing that this error is not a geodesic distance mathematically speaking since it is not built from a LG metric. Nevertheless, it specifies a good way to assess the intrinsic path traveled from $\mathbf{X}$ to $\widehat{\mathbf{X}}$, and it is classically used in the LG estimation literature [16] [17].

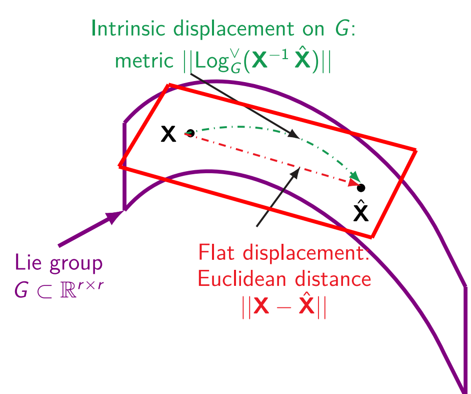

Fig. 3. Illustration of the intrinsic gap between $\mathbf{X}$ and $\hat{\mathbf{X}}$, which takes into account the curvature of the group.

Analogously to the Euclidean case, three intrinsic indicators can be designed from the estimator $\widehat{\mathbf{X}} \triangleq \widehat{\mathbf{X}}(\mathbf{Z})$:

- Its mean $\mathbf{M}_{\widehat{\mathbf{X}}} \in G$ such that:

$$\int_{G'} l_G\left(\widehat{\mathbf{X}}, \mathbf{M}_{\widehat{\mathbf{X}}}\right) p(\mathbf{Z}|\mathbf{X})\lambda_G(d\mathbf{Z}) = \mathbf{0} \quad (3)$$

[1]Note that other indicators could be used (for instance the median or the consistency).

- its intrinsic bias $\mathbf{b}_{\mathbf{Z}|\mathbf{X}} \in \mathbb{R}^m$ given by [16]:

$$\mathbf{b}_{\mathbf{Z}|\mathbf{X}}(\mathbf{X}, \widehat{\mathbf{X}}) = \int_{G'} \boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}}\right) p(\mathbf{Z}|\mathbf{X}) \lambda_G(d\mathbf{Z}) \quad (4)$$

$$\triangleq \mathbb{E}_{p(\mathbf{Z}|\mathbf{X})} \left(\boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}}\right)\right) \quad (5)$$

- its intrinsic estimation error covariance $\mathbf{C}_{\mathbf{Z}|\mathbf{X}} \in \mathbb{R}^{m \times m}$ defined by [18]:

$$\mathbf{C}_{\mathbf{Z}|\mathbf{X}}(\mathbf{X}, \widehat{\mathbf{X}}) = \int_{G'} \boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}}\right) \boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}}\right)^\top p(\mathbf{Z}|\mathbf{X}) \, \lambda_G(d\, \mathbf{Z}) \quad (6)$$

$$\triangleq \mathbb{E}_{p(\mathbf{Z}|\mathbf{X})} \left(\boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}}\right) \boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}}\right)^\top\right) \quad (7)$$

where $\boldsymbol{l}_G(\mathbf{X}, \mathbf{Y}) = \mathrm{Log}_G^\vee \left(\mathbf{X}^{-1} \mathbf{Y}\right) \;\; \forall \mathbf{X}, \mathbf{Y} \in G \times G$.

III. Intrinsic Barankin and McAulay-Seidman bounds on Lie groups

A. Background on the Euclidean Barankin and the McAulay-Seidman bounds

We consider a set of Euclidean observations $\mathbf{z} = \{\mathbf{z}_1, \ldots, \mathbf{z}_N\} \in \left(\mathbb{R}^d\right)^N$ depending of $\mathbf{x}_0 \in \mathbb{R}^p$, an unknown parameter vector, and characterized by $p(\mathbf{z}|\mathbf{x}_0)$. Let $\mathbf{g} : \mathbb{R}^p \to \mathbb{R}^s$ be a smooth function. The BB on the estimator $\widehat{\mathbf{g}(\mathbf{x}_0)}$ is given by:

$$\mathbf{P}_{BB} = \min_{\widehat{\mathbf{g}(\mathbf{x}_0)}} \mathbb{E}_{p(\mathbf{z}|\mathbf{x}_0)} \left(\left(\widehat{\mathbf{g}(\mathbf{x}_0)} - \mathbf{g}(\mathbf{x}_0)\right) \left(\widehat{\mathbf{g}(\mathbf{x}_0)} - \mathbf{g}(\mathbf{x}_0)\right)^\top\right)$$
$$\text{w.r.t. } \mathbb{E}_{p(\mathbf{z}|\mathbf{x})} \left(\widehat{\mathbf{g}(\mathbf{x}_0)} - \mathbf{g}(\mathbf{x}_0)\right) = \mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}_0) \quad \forall \mathbf{x} \in \mathbb{R}^p \quad (8)$$

This uniform unbiasedness constraint can be seen as a continuum of constraints, consequently, solving the minimization problem (8) is difficult. To overcome this issue, $\mathbf{P}_{BB}$ can be approached by using a set of test points, $\mathbf{x}^{(1:L)} = \{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(L)}\}$, verifying the unbiasedness condition in (8). Then, the latter can be written as:

$$\mathbb{E}_{p(\mathbf{z}|\mathbf{x}_0)} \left(\mathbf{v}_{\mathbf{x}_0}(\mathbf{x}^{(1:L)}) \left(\mathbf{g}(\mathbf{x}_0) - \widehat{g\left(\mathbf{x}_0\right)}\right)^\top\right) = \begin{bmatrix} \left(\mathbf{g}(\mathbf{x}^{(1)}) - \mathbf{g}(\mathbf{x}_0)\right)^\top \\ \vdots \\ \left(\mathbf{g}(\mathbf{x}^{(L)}) - \mathbf{g}(\mathbf{x}_0)\right)^\top \end{bmatrix} \quad (9)$$

with $\mathbf{v}_{\mathbf{x}_0}(\mathbf{x}^{(1:L)}) = \left[\frac{p(\mathbf{z}|\mathbf{x}^{(1)})}{p(\mathbf{z}|\mathbf{x}_0)}, \ldots, \frac{p(\mathbf{z}|\mathbf{x}^{(L)})}{p(\mathbf{z}|\mathbf{x}_0)}\right]^\top$. This new matrix condition allows to obtain [19, Lemma 1]

$$\mathbb{E}_{p(\mathbf{z}|\mathbf{x}_0)} \left(\left(\widehat{\mathbf{g}(\mathbf{x}_0)} - \mathbf{g}(\mathbf{x}_0)\right) \left(\widehat{\mathbf{g}(\mathbf{x}_0)} - \mathbf{g}(\mathbf{x}_0)\right)^\top\right) \succeq \boldsymbol{\Delta} \mathbf{R}_{\mathbf{v}_{\mathbf{x}_0}}^{-1} \boldsymbol{\Delta}^\top, \quad (10)$$

where $\succeq$ is defined such that $\forall \mathbf{A}, \mathbf{B}$, $\mathbf{A} \succeq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is a positive definite matrix.
The right-hand term of the inequality is the MSB,

$$\begin{cases} \boldsymbol{\Delta} = \left[\mathbf{g}(\mathbf{x}^{(1)}) - \mathbf{g}(\mathbf{x}_0), \ldots, \mathbf{g}(\mathbf{x}^{(L)}) - \mathbf{g}(\mathbf{x}_0)\right], & (11) \\ \mathbf{R}_{\mathbf{v}_{\mathbf{x}_0}} = \mathbb{E}_{p(\mathbf{z}|\mathbf{x}_0)} \left(\mathbf{v}_{\mathbf{x}_0}(\mathbf{x}^{(1:L)}) \, \mathbf{v}_{\mathbf{x}_0}(\mathbf{x}^{(1:L)})^\top\right). & (12) \end{cases}$$

If the test points are written in the following form:

$$\mathbf{x}^{(1)} = \mathbf{x}_0 \quad (13)$$
$$\mathbf{x}^{(l)} = \mathbf{x}_0 + \mathbf{i}_l \delta_l \;\; \forall l \in \{1, \ldots, L-1\} \quad (14)$$

with

$$\mathbf{i}_l = \Big[0, \ldots, \underbrace{1}_{l^{\text{th}}\text{component}} \ldots, 0\Big]^\top \in \mathbb{R}^p, \quad (15)$$

B. Development of the intrinsic McAulay-Seidman bound

Now, let be G, G' and G'' three matrix Lie groups. We assume a set of random measurements $\mathbf{Z} \in G''$ depending on a smooth application $\mathbf{H}(\cdot) : G \to G'$ function of an unknown parameter $\mathbf{X}_0 \in G$. If the statistical relation between $\mathbf{Z}$ and $\mathbf{X}_0$ is modelled by the likelihood $p(\mathbf{Z}|\mathbf{X}_0)$, then the IBB $\mathbf{P}_{\text{IBB}}$ on $\mathbf{X}_0$ can be defined as the minimum of the intrinsic covariance error estimation:

$$\mathbf{P}_{\text{IBB}} = \min_{\widehat{\mathbf{H}(\mathbf{X}_0)}} \mathbf{C}_{\mathbf{Z}|\mathbf{X}_0}(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)})$$
$$\text{s.t. } \mathbf{b}_{\mathbf{Z}|\mathbf{X}} \left(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)}\right) = \boldsymbol{l}_{G''} \left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X})\right)$$
$$\forall \, \mathbf{X} \in G \quad (16)$$

In order to approximate $\mathbf{P}_{\text{IBB}}$, we can define an intrinsic unbiasedness condition under the following form,

$$\mathbf{b}_{\mathbf{Z}|\mathbf{X}} \left(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)}\right) = \boldsymbol{l}_{G''} \left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}^{(l)})\right), \quad \forall \, l \in \{1, \ldots, L\} \quad (17)$$

where $\{\mathbf{X}_{(l)}\}_{l=1}^L$ are a set of test points belonging to G. According to [19], we can show that this condition provides the following approximation of $\mathbf{P}_{\text{IBB}}$, called IMSB,

$$\mathbf{P}_{\text{IMSB}} = \boldsymbol{\Delta}_G \, \mathbf{R}_{\mathbf{v}_{\mathbf{X}_0}}^{-1} \boldsymbol{\Delta}_G^\top \quad (18)$$

$$\boldsymbol{\Delta}_G = \left[\boldsymbol{l}_{G''} \left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}^{(1)})\right), \ldots, \boldsymbol{l}_{G''} \left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}^{(L)})\right)\right] \quad (19)$$

$$\mathbf{R}_{\mathbf{v}_{\mathbf{X}_0}} = \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)} \left(\mathbf{v}_{\mathbf{X}_0} \left(\mathbf{Z}; \mathbf{X}^{(1:L)}\right), \mathbf{v}_{\mathbf{X}_0} \left(\mathbf{Z}; \mathbf{X}^{(1:L)}\right)^\top\right), \quad (20)$$

C. Closed-form expression

In order to obtain a tractable expression of $\mathbf{P}_{\text{IMSB}}$, we place ourselves in the case where $G = G' = SO(3)$, $G'' = \mathbb{R}^3$ and $\mathbf{H} = \mathbf{I}_3$. Then, we consider the following observation model, also known as the Wahba's problem:

$$\mathbf{z}_i = \mathbf{X}_0 \, \mathbf{p}_i + \mathbf{n}_i \quad \forall i \in [\![1, N]\!] \quad (21)$$

where $\mathbf{X}_0 \in SO(3)$, $\{\mathbf{z}_i\}_{i=1}^N$ and $\{\mathbf{p}_i\}_{i=1}^N$ are a set of 3D known points, and $\mathbf{n}_i$ is a white Gaussian noise with covariance $\mathbf{\Sigma}$. Thus, we can establish the analytical expression of the IMSB for the unknown parameter $\mathbf{X}_0$.

- First, we consider a set of L test points $\{\mathbf{X}_{(l)}\}_{l=1}^L \in SO(3)$ and $\mathbf{p} = \left[\mathbf{p}_1^\top, \ldots, \mathbf{p}_N^\top\right]^\top$. In this way, the term $[\mathbf{P}]_{i,j}$ can be written with the following form $\forall (i,j) \in [\![1, L]\!]^2$,

$$[\mathbf{P}]_{i,j} = \exp\left(0.5\left(\mathbf{m}_{ij}^\top\left(\mathbf{I}_3 \otimes \mathbf{\Sigma}\right)\mathbf{m}_{ij} - \delta_{ij}\right)\right) \tag{22}$$

$$\mathbf{m}_{ij} = \left(\mathbf{I}_3 \otimes \mathbf{\Sigma}\right)^{-1}\left(\mathbf{I}_3 \otimes \mathbf{X}_{(i)}\right)\mathbf{p} + \left(\mathbf{I}_3 \otimes \mathbf{\Sigma}\right)^{-1}\left(\mathbf{I}_3 \otimes \mathbf{X}_{(j)}\right)\mathbf{p} - \left(\mathbf{I}_3 \otimes \mathbf{\Sigma}\right)\left(\mathbf{I}_3 \otimes \mathbf{X}_0\right)\mathbf{p} \tag{23}$$

$$\begin{aligned}\delta_{ij} =& \left(\left(\mathbf{I}_3 \otimes \mathbf{X}_{(i)}\right)\mathbf{p}\right)^\top\left(\mathbf{I}_3 \otimes \mathbf{\Sigma}\right)^{-1}\left(\left(\mathbf{I}_3 \otimes \mathbf{X}_{(i)}\right)\mathbf{p}\right) \\ &+ \left(\left(\mathbf{I}_3 \otimes \mathbf{X}_{(j)}\right)\mathbf{p}\right)^\top\left(\mathbf{I}_3 \otimes \mathbf{\Sigma}\right)^{-1}\left(\left(\mathbf{I}_3 \otimes \mathbf{X}_{(j)}\right)\mathbf{p}\right) \\ &- \left(\left(\mathbf{I}_3 \otimes \mathbf{X}_0\right)\mathbf{p}\right)^\top\left(\mathbf{I}_3 \otimes \mathbf{\Sigma}\right)^{-1}\left(\left(\mathbf{I}_3 \otimes \mathbf{X}_0\right)\mathbf{p}\right)\end{aligned} \tag{24}$$

  with $\otimes$ the Kronecker product.

- $\mathbf{\Delta}$ can be easily computed because $\mathrm{Log}_{SO(3)}^\vee(.)$ corresponds directly to the logarithm matrix.

IV. Numerical simulations

In this section, we propose to validate numerically the IMSB by studying the influence of the noise of the covariance observation matrix and the number of observations on its behaviour.

A. Simulation parameters

- The test points are defined according to the following relation,

$$\mathbf{X}_{(l)} = \mathbf{X}_0\,\mathrm{Exp}_{SO(3)}^\wedge\left(\boldsymbol{\delta}_l\right) \quad \forall l \in \{1, \ldots L\} \tag{25}$$

  with $L = 100$ and $\boldsymbol{\delta}_l$ a zero-mean Gaussian random vector with covariance $\sigma_l^2\,\mathbf{I}_3$. This generative model allows to browse the whole group, especially when σ_l is sufficiently high.
- The estimator is gathered by searching the likelihood maximum of $p(\mathbf{z}|\mathbf{X}_0)$. It amounts to find the minima of the criterion $\sum_{i=1}^N \|\mathbf{z}_i - \mathbf{X}_0\,\mathbf{p}_i\|_{\mathbf{\Sigma}}^2$. To obtain a sufficiently accurate estimator, a Gauss-Newton algorithm on LGs is used [20].

- The IMSE is given by the trace of the covariance estimation error:

$$\mathbb{E}_{p(\mathbf{z}|\mathbf{X}_0)}\left(\left\|\mathrm{Log}_{SO(3)}^\vee\left(\mathbf{X}_0^{-1}\,\widehat{\mathbf{X}}_0\right)\right\|^2\right)$$

  where $\widehat{\mathbf{X}}_0$ is an estimator of $\mathbf{X}_0$ in the maximum likelihood sense. Such estimator is built with a Gauss-Newton algorithm dedicated to LGs [20]. In order to evaluate it, we approximate it by Monte Carlo,

$$\frac{1}{N_{mc}}\sum_{i=1}^{N_{mc}}\left\|\mathrm{Log}_{SO(3)}^\vee\left(\mathbf{X}_0^{-1}\left(\widehat{\mathbf{X}}_0\right)_i\right)\right\|^2,$$

  where N_{mc} is the number of realizations of the algorithm, fixed to 500, and $\left(\widehat{\mathbf{X}}_0\right)_i$ the estimator for the i-th realization.

B. Influence of the observation noise

First, we implement the IMSB by considering $N = 3$ with with $\mathbf{p}_1 = [1, 2, 2]^\top$, $\mathbf{p}_2 = [3, 4, 5]^\top$ and $\mathbf{p}_3 = [0.1, 0.2, 2]^\top$ and $\mathbf{\Sigma} = \sigma^2\,\mathbf{I}_3$. In Fig. 5, we show the IMSB evolution as function of the noise variance σ^2. Particularly, we observe that the IMSB admits a stable behaviour with respect to the IMSE. When the noise becomes high, the IMSE and the IMSB. It is consistent because the covariance error of the estimator depends of $\mathbf{\Sigma}$. We can also notice that more the variance is low, the closer the two are. Such convergence of the IMSE to the IMSB in the high SNR regime (asymptotic efficiency) validates the proposed bound.

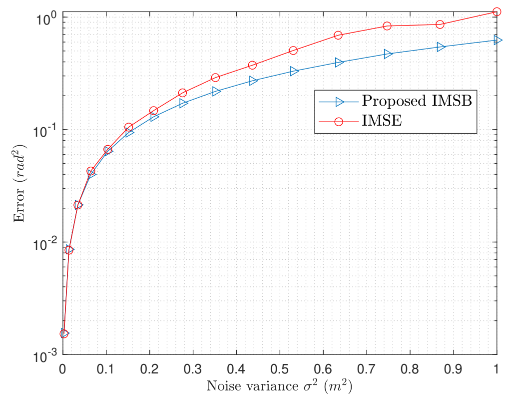

Fig. 4. Evolution of the proposed IMSB and IMSE for different σ^2 values.

C. Influence of the number of observations

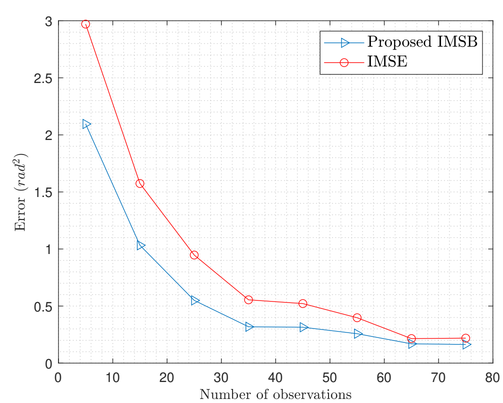

Fig. 5. Evolution of the proposed IMSB and IMSE for different number of observations with $\sigma^2 = 0.5$

In order to evaluate the impact of the number of observations, we simulate N random points $\{\mathbf{p}_i\}_{i=1}^N$ in the following way:

$$\mathbf{p}_i \sim \mathcal{N}_{\mathbb{R}^3}(\mathbf{p}_m, \sigma_m^2\,\mathbf{I}_3)\ \forall i \in \{1, \ldots N\} \tag{26}$$

where $\mathbf{p}_m = [1,1,1]^\top$ and $\sigma_m = 0.5\,\text{m}$. Then, $\{\mathbf{z}_i\}_{i=1}^N$ are generated with the model (21)

As in the previous case, we observe in Fig. 5 the consistency of the bound for different number of observations. Indeed, when this number becomes high ($\simeq 65$), so when the maximum likelihood estimator becomes more accurate, the IMSE decreases and gets closer and closer to the bound. This allows to confirm that the proposed bound behaves as theoretically expected.

V. Conclusions

In this communication, we proposed a new intrinsic bound on Lie groups called intrinsic McAulay-Seidman bound. This new bound is inspired by the Euclidean Barankin and McAulay-Seidman bounds derivation. The bound was computed with a closed-form expression in the case of the Gaussian Euclidean observation model. The latter was tested and validated by numerical simulations. The perspectives of this work are multiple. Two of them would be to compute the proposed bound for other Lie groups of interest (for instance $SE(3)$) and to derive an intrinsic Cramér-Rao bound from the intrinsic McAulay-Seidman bound in a similar fashion as in the Euclidean case.

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