MEAN-FIELD APPROXIMATE OPTIMIZATION ALGORITHM

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Motivation: Quantum Approximate Optimization Algorithm

Goal: Find ground state of

$$H_P = \sum_{i=1}^{N} \left[h_i + \sum_{j>i} J_{ij} \sigma_j^z \right] \sigma_i^z \, .$$

Adiabatic Quantum Computation (AQC):

- Initialize qubits in the ground state of $H_D = \sum_{i=1}^{N} \sigma_i^x$
- Evolve the system adiabatically to H_P and measure the output.

Problem: Tiny gaps enforce (exponentially) small sweep velocities!

Diabatic Alternative: QAOA:

- Discretize the time evolution of a linear AQC schedule by the standard Suzuki-Trotter decomposition, such that it becomes an alternating sequence of parameterized unitary gates of some given length *p*, applied to $|+\rangle^{\otimes N}$.
- Optimize the unitary‘s parameters in a closed loop to minimize the energy.

Problem: The adiabatic theorem does not hold. And what about quantum advantage?

Mean-Field AOA: Derivation

In the mean-field approximation, the total Hamiltonian becomes

$$\mathrm{H}(t) = -\gamma(t)\sum_{i=1}^{N}\left[h_i + \sum_{j>i} J_{ij} n_j^z(t)\right] n_i^z(t) - \beta(t)\sum_{i=1}^{N} n_i^x(t),$$

where $\beta(t)$ and $\gamma(t)$ are piecwise constant functions of time, and the classical spin vectors

$$\boldsymbol{n}_i(t) = \left(n_i^x(t), n_i^y(t), n_i^z(t)\right)^T.$$

The dynamics can be found exactly:

$$\boldsymbol{n}_i(t) = \prod_{k=1}^{p} V_i^D(k)\, V_i^P(k)\ \ \boldsymbol{n}_i(0),$$

$$V_i^D(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\beta_k) & \sin(2\beta_k) \\ 0 & -\sin(2\beta_k) & \cos(2\beta_k) \end{pmatrix},$$

$$V_i^P(k) = \begin{pmatrix} \cos(2m_i(t_{k-1})\gamma_k) & \sin(2m_i(t_{k-1})\gamma_k) & 0 \\ -\sin(2m_i(t_{k-1})\gamma_k) & \cos(2m_i(t_{k-1})\gamma_k) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Mean-Field AOA: Algorithm

1. Initialization: $\boldsymbol{n}_i(0) = (1,0,0)^T \quad \forall i$
2. Apply a mean-field evolution sequence of length *p*

$$\boldsymbol{n}_i(0) = \prod_{k=1}^{p} V_i^D(k)\ V_i^P(k)\ \boldsymbol{n}_i(0).$$

Observe that $V_i^P(k)$ depends on $\boldsymbol{n}_i(k-1)$.

3. Compute the cost function

$$H_P[p,\tau] = -\sum_{i=1}^{N}\left[h_i + \sum_{j>i} J_{ij} n_j^z(p)\right] n_i^z(p)$$

where τ is the step size such that $\gamma_k = \tau k/p$ and $\beta_k = \tau\left(1 - \frac{k-1}{p}\right)$.

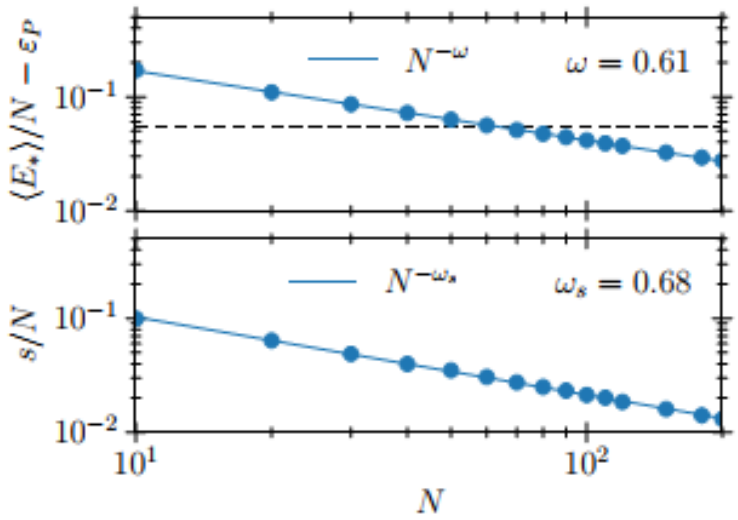
4. Adjust the number of steps *p* and the step size τ to minimize $H_P[p,\tau]$.
5. Repeat steps 2 to 4 until a convergence threshold is reached.
6. Round the *z*-components of the spin vektors $\boldsymbol{n}_i(p)$ to obtain the result

$$\sigma_* = (\mathrm{sign}(n_1^z), \ldots, \mathrm{sign}\,(n_N^z)).$$

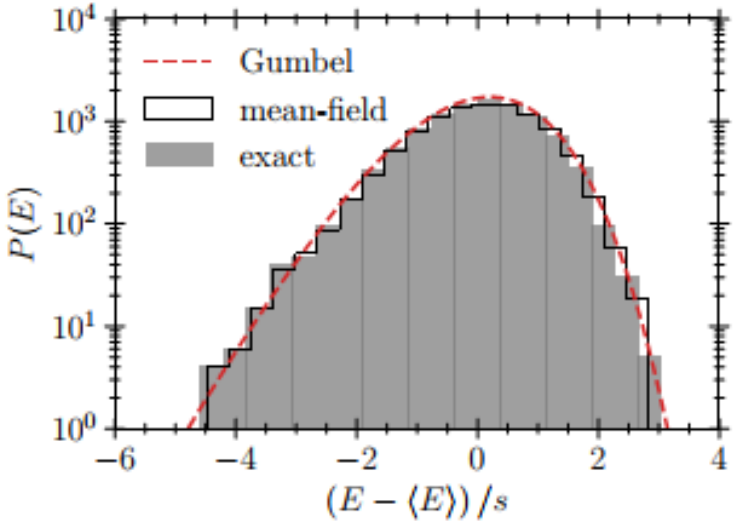
Performance Benchmark: Sherrington-Kirkpatrick

Model: $H_P = -\frac{1}{\sqrt{N}} \sum_{i<j\leq N} J_{ij} \sigma_i^z \sigma_j^z$, with $\langle J_{ij} \rangle = 0, \langle J_{ij}^2 \rangle = J^2$; Parisi 1979: $\epsilon_P = \lim_{N\to\infty} \left\langle \frac{E_0}{N} \right\rangle_J = -0.763 \ldots$

Scaling of the energy E_* resulting from mean-field AOA with N

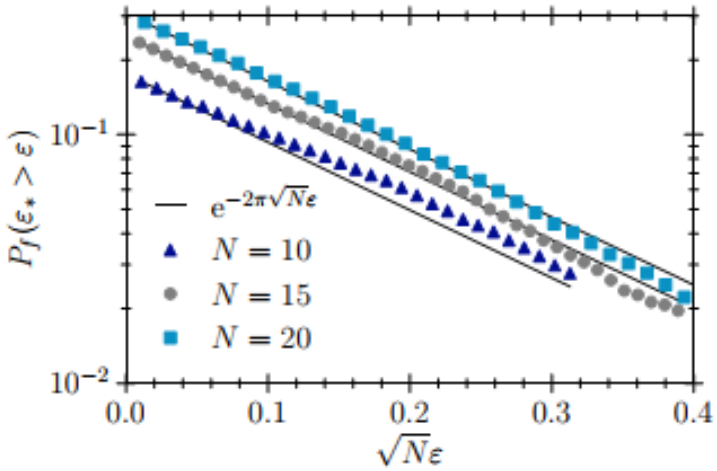


Comparison of the resulting to the exact distribution with $N = 20$



Tail distribution

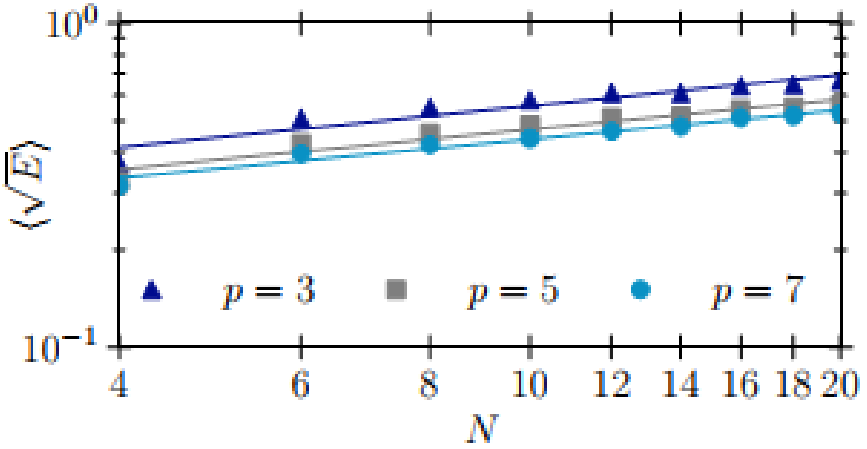
$P_f(\epsilon^* > \epsilon) \sim \exp(-2\pi\sqrt{N}\epsilon)$

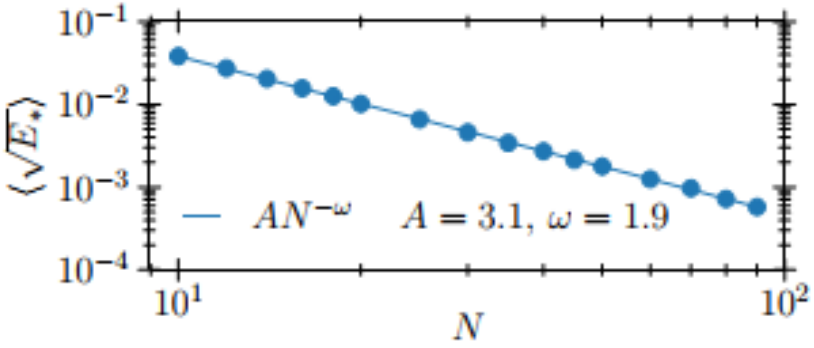


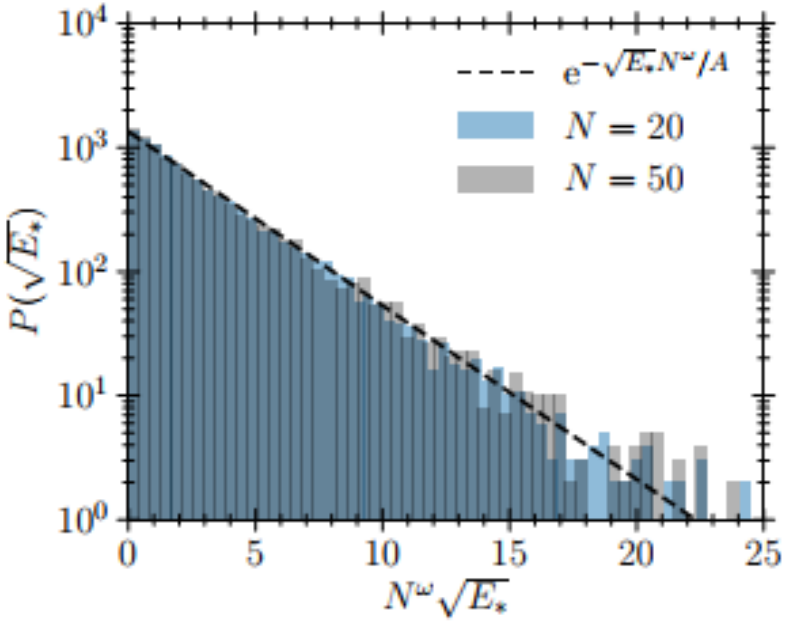
With a probability of at least $1 - O(e^{-2\pi\sqrt[4]{N}})$ over possible realizations of the SK Hamiltonian, the mean-field AOA delivers an approximate optimum E_* with a relative accuracy bounded by $\sqrt[4]{N}$ from above.

Performance: Number Partitioning

- $C(S) = \sum_{a_i \in S} a_i \sigma_i^z$
- $H_P = \sum_{i=1}^{N} a_i^2 - \sum_{i<j\leq N} J_{ij} \sigma_i^z \sigma_j^z$
- $P(\sqrt{E_*}) = \left(\frac{N^\omega}{A}\right) \exp\left(-\sqrt{E_*}\frac{N^\omega}{A}\right)$
- $P_f(c) = \exp\left(-c\frac{N^\omega}{A}\right)$

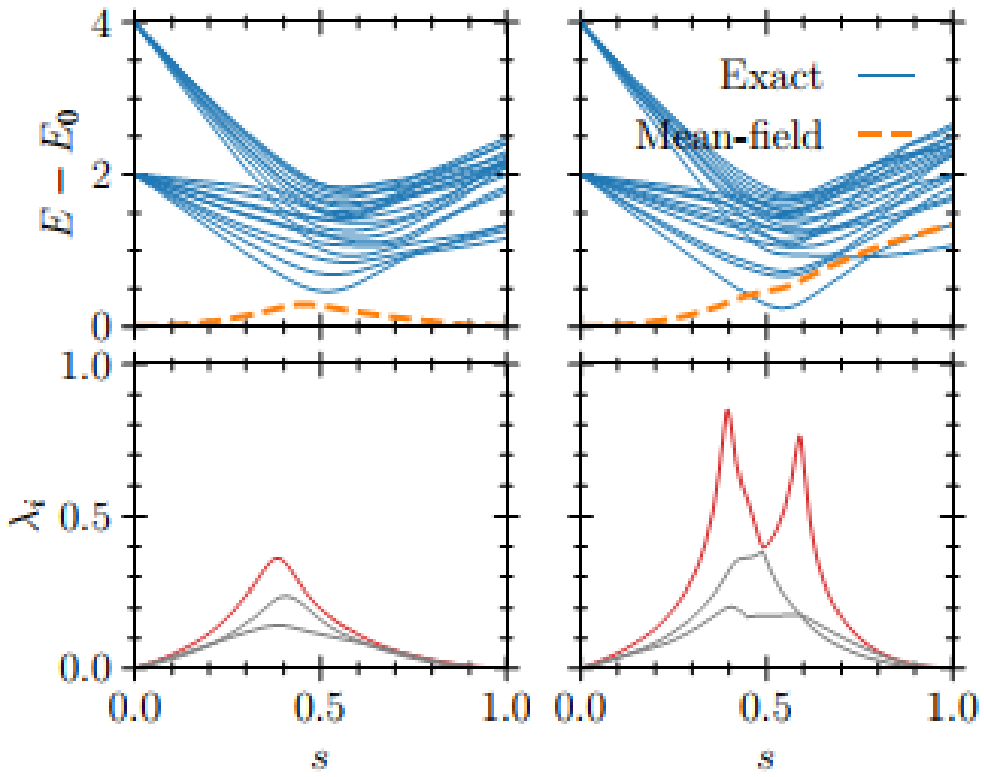


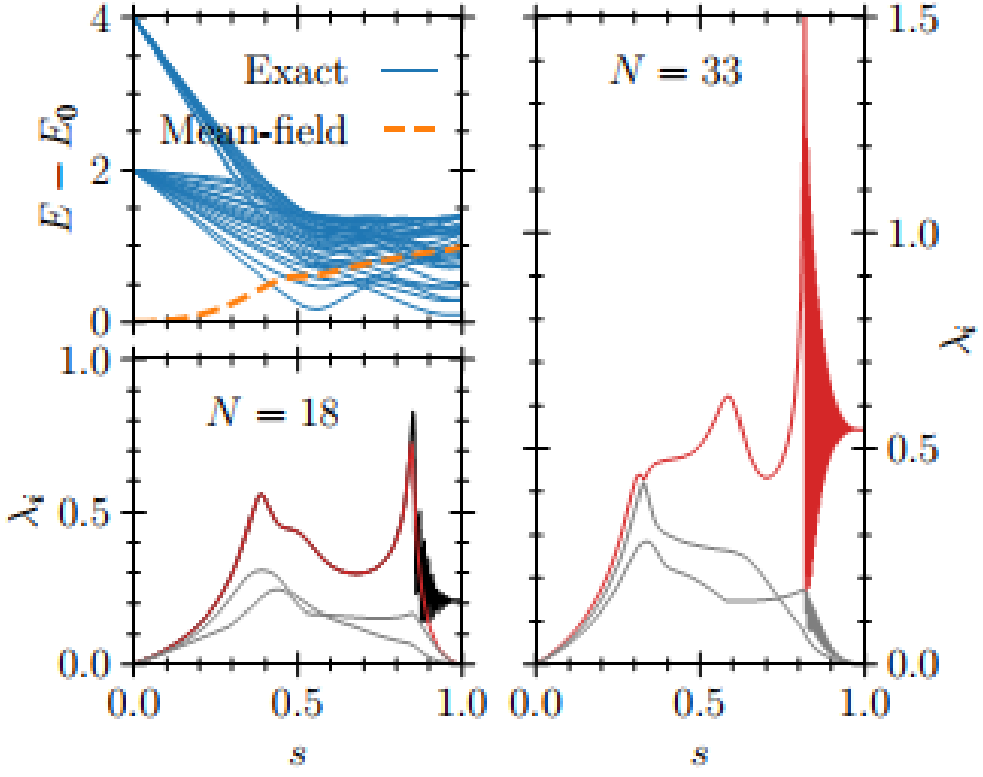




With a probability of at least $1 - O(e^{-N^\delta/A})$ over possible realizations of random sets in the partition problem, the mean-field AOA delivers an approximate optimum $c \leq N^{-\delta}$, where the exponent $\delta = 0.95$.

Beyond Mean-Field: Lyaponov Exponents





To quantify quantum fluctuations around the mean-field spin trajectories, we solve an effective scattering problem in time. It describes the propagation of collective ‘paramagnon’ modes above the instantaneous ground state of the adiabatic Hamiltonian, and is characterized by a spectrum of positive Lyapunov exponents. The largest Lyapunov exponent shows a number of maxima, which are pinned to level crossings or minimal gaps in the lowest part of the Hamiltonian spectrum. The case where all Lyapunov exponents are relatively small indicates an ‘easy’ instance, where the optimum can be found classically, i.e. without invoking quantum algorithms. For hard instances, there occur some large maxima in the spectrum, and the mean-field AOA typically fails to deliver the exact solution.