# Estimating the Interferometric Vertical Wavenumber From Range Shifts 

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#### Abstract

The estimation of the interferometric vertical wavenumber over sloped terrain is an integral step for many interferometric and tomographic synthetic aperture radar (SAR) applications. The state of the art for this estimation is to calculate the angle of incidence and the local terrain slope after geometric corregistration has been performed. In this letter, an alternative approach for estimating the vertical wavenumber is proposed that requires only the estimation of the range corregistration shifts. This considerably simplifies the calculation effort without compromising the estimation performance. The proposed approach is demonstrated on Advanced Land Observing Satellite (ALOS) phased array Lband synthetic aperture radar (PALSAR) data and compared against the conventional methodology.


Index Terms-Synthetic aperture radar interferometry (InSAR), vertical wavenumber.

## I. Introduction

IN CROSS-TRACK synthetic aperture radar interferometry (InSAR), the normal baseline, $B_{\perp}$, is given by the spatial separation of two antennae on the zero-Doppler plane measured perpendicular to the line-of-sight (LOS) [1], [2], [3], [4]. The interferometric sensitivity to height is proportional to $B_{\perp}$. The parameter expressing this sensitivity is the vertical wavenumber $\kappa_{z}$, which describes the rate of change of the interferometric phase associated with a given height change [5], [6].

The conventional methodology for estimating the (slopenormal) $\kappa_{z}$ requires the calculation of incidence angle and local slope in order to define the normal vector at every range point [1], [2]. This is usually performed during (or after) the back-geocoding step, where using an external digital elevation model (DEM) and precise orbit information for each pixel in the primary (e.g., primary) image and the corresponding secondary (e.g., secondary) image coordinates are calculated [3], [5], [7], [8]. This way, the corregistration shifts required to resample the secondary image in order to match the primary one are obtained.
In this letter, an alternative approach for estimating the (slope normal) vertical wavenumber is proposed that requires only the estimation of the range corregistration shifts and can, therefore, be applied directly after the back-geocoding

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Fig. 1. Contours of $\Delta(\vec{r})$ in (left) near-field and in (right) far field. The near-field plot shows the positions of satellites $\vec{M}$ and $\vec{S}$, the LOS, and the normal baseline. The far-field plot shows the validity of both the radial and the shift of radiation center approximations.
step. For this, two scalar fields that allow representing the interferometric phase and the terrain topography are introduced in Section II. In Sections III and IV, these two fields are used to derive the vertical wavenumber as a function of range (corregistration) shifts. The demonstration on Advanced Land Observing Satellite (ALOS) phased array Lband synthetic aperture radar (PALSAR) data and comparison against the conventional methodology are performed in Section V. In Section VI, the use of the proposed approach to obtain "quick and dirty" vertical wavenumber estimates without any knowledge on orbit or topography is discussed. Finally, Section VII concludes this work.

## II. Two Scalar Fields

We consider an interferometric configuration consisting on a primary $M$ and a secondary $S$ antennae separated by a spatial baseline with a normal component $B_{\perp}$, as shown in Fig. 1. The effect of the local topography on the interferometric phase measured by the configuration can be expressed by means of two 2-D scalar fields defined on the zero-Doppler plane of the primary antenna. The first field is the difference between the distances of a given point at the end of a position vector $\vec{r}$ to the primary and secondary antennas $\vec{M}$ and $\vec{S}$

$$
\begin{equation*}
\Delta(\vec{r})=\|\vec{S}-\vec{r}\|-\|\vec{M}-\vec{r}\| \tag{1}
\end{equation*}
$$

where $\|\vec{a}\|=\left(a_{1}^{2}+a_{2}^{2}+a_{3}^{2}\right)^{1 / 2}$ is the length of a vector $\vec{a}=$ [ $\left.a_{1} a_{2} a_{3}\right]^{\mathrm{T}}$ in Euclidian 3-D space.


Fig. 2. Graphical demonstration of two scalar fields $\Delta(\vec{r})$ and $F$ and of their derivatives $\vec{\nabla} \Delta(\vec{r})$ and $\vec{C}_{t}$ in the zero-Doppler plane, respectively.

Any contour of $\Delta(\vec{r})$ forms a hyperbola with its two foci at $\vec{M}$ and $\vec{S}$. At far field, these hyperbolae are once more approximated by their asymptotes, which are rays radiating from the midpoint of two antennae $(\vec{M}+\vec{S}) / 2$. These asymptotes are approximated by rays radiated from $\vec{M}$ after its translation by $(\vec{M}-\vec{S}) / 2$. The error induced by this second approximation is negligible in the far field, especially in spaceborne SAR configurations. Each contour of $\Delta(\vec{r})$ is regarded as the LOS (see Fig. 1). Along the LOS, the interferometric phase is constant. The gradient of $\Delta(\vec{r})$, i.e., $\vec{\nabla} \Delta(\vec{r})$, is perpendicular to the LOSs and parallel to the wavefront. Scatterers on the same wavefront are imaged at the same range in the SAR image, but they would have different interferometric phases.

Fig. 1 shows the shape of the scalar potential $\Delta(\vec{r})$ indicated by the contours. On the left, the near field case is shown. Given $\|\vec{S}-\vec{M}\|=B=400 \mathrm{~m}$ as an example, the nine contours correspond to the $\Delta(\vec{r})$ values of -400 to 400 m in $100-\mathrm{m}$ steps changing color from violet to yellow. The farfield geometry for a sensor at $670-\mathrm{km}$ altitude is shown at the right. Here, the contours are plotted with a 10 times denser sampling, i.e., every 10 m . The gradient $\vec{\nabla} \Delta(\vec{r})$ is large where the contours are dense and small where they are sparse. At a range distance $R,\|\vec{\nabla} \Delta(\vec{r})\|$ takes its maximum $\left(B_{\perp} / R\right)$, where $\Delta(\vec{r})=0$ and $B=B_{\perp}$. In the orthogonal direction, where $\Delta=B$ and $B_{\perp}=0, \vec{\nabla} \Delta=0$.

The second relevant field is the potential of the surface topography $F$ defined to be $F:=0$ at the surface, positive over and negative under it, so that $\vec{\nabla} F \neq 0$ everywhere while $\|\vec{\nabla} F\|=1$ at $F=0 . F$ is smooth as it is bandpass filtered by the range resolution. The $F:=0$ contour is parameterized as $\vec{C}(t)$, where $t \in \mathbb{R}$. The parameter $t$ is defined to be the range sampling, so that for each value of $t \in \mathbb{N}, \vec{C}(t)$ becomes the $t$-th range point. In the following, its derivative $(\partial \vec{C} / \partial t)$ will be referred as $\vec{C}_{t}$. Following the definition of $F$ and $\vec{C}$, $F(\vec{C})=0$ and $\vec{\nabla} F$ become the surface normal on points on $\vec{C}$.

On the zero-Doppler plane of the primary antenna, a reference coordinate system defined by the tangent at the
intersection of Earth's ellipsoid with the LOS and its normal vector is introduced, as shown in Fig. 2. The angle between the vertical axis and the LOS is then the incidence angle $\theta$, while the angle between the wavefront and $\vec{C}$ is the local incidence angle $\alpha$. The local slope angle in the range direction is then $\theta-\alpha$.

## III. Calculation of the Vertical Wavenumber

The interferometric phase is measured by the interferometric configuration of Fig. 1

$$
\begin{equation*}
\phi_{0}=\frac{4 \pi}{\lambda} \Delta\left(\vec{r}_{0}\right) \tag{2}
\end{equation*}
$$

where $\lambda$ is the wavelength, and $\vec{r}_{0}$ is the position of a scatterer. The (slope-normal) vertical wavenumber $\kappa_{z}$ is the change rate of the interferometric phase as the scatterer ascends along the wavefront as much as a unit distance normal to the slope. We consider a point at $\vec{r}_{s}=\vec{r}_{0}+s(\vec{\nabla} \Delta(\vec{r}) /\|\vec{\nabla} \Delta(\vec{r})\|)$, where $s$ is its distance from $\vec{r}_{0}$ along the wavefront. The interferometric phase $\phi_{s}$ at $\vec{r}_{s}$ is

$$
\begin{align*}
\phi_{s} & =\frac{4 \pi}{\lambda} \Delta\left(\vec{r}_{s}\right)=\frac{4 \pi}{\lambda} \Delta\left(\vec{r}_{0}+\frac{s \vec{\nabla} \Delta(\vec{r})}{\|\vec{\nabla} \Delta(\vec{r})\|}\right) \\
& =\phi_{0}+s \frac{4 \pi}{\lambda}\|\vec{\nabla} \Delta(\vec{r})\| \tag{3}
\end{align*}
$$

As the slope-normal height difference $h$ between $\vec{r}_{0}$ and $\vec{r}_{s}$ is $h=s \sin \alpha$, the interferometric phase difference $\Delta \phi=\phi_{s}-\phi_{0}$ can be written as

$$
\begin{equation*}
\Delta \phi=\frac{4 \pi}{\lambda}\|\vec{\nabla} \Delta(\vec{r})\| \frac{h}{\sin \alpha} \tag{4}
\end{equation*}
$$

Finally, $\kappa_{z}$ becomes

$$
\begin{equation*}
\kappa_{z}=\frac{d \Delta \phi}{d h}=\frac{4 \pi}{\lambda} \frac{\|\vec{\nabla} \Delta(\vec{r})\|}{\sin \alpha} \tag{5}
\end{equation*}
$$

Substituting $\|\vec{\nabla} \Delta(\vec{r})\| \approx\left(B_{\perp} / R\right)$, where $R=\left\|\vec{r}_{0}-\vec{M}\right\|$, leads to the conventional expression of $\kappa_{z}[6]$.

## IV. Substitution of Local Incidence Angle

Although $\kappa_{z}$ is obtained in (5) from the partial derivative of the scalar field $\Delta(\vec{r})$, the contribution of the ground terrain slope is incorporated in the local incidence angle $\alpha$, which relates the wavefront to the topography that changes with range. Following the notation of Section III, the range shift $\Delta r(t)$ can be expressed as

$$
\begin{equation*}
\Delta r(t)=\Delta(\vec{C}(t)) \tag{6}
\end{equation*}
$$

The flat earth phase is merely given by $\phi_{\text {flat }}(t)=$ $(4 \pi / \lambda) \Delta(\vec{C}(t))$. The derivative of (6) with respect to $t$

$$
\Delta^{2} r=\frac{\mathrm{d}}{\mathrm{~d} t} \Delta r(t)=\frac{\mathrm{d}}{\mathrm{~d} t} \Delta(\vec{C}(t))
$$

is a composite function of $t \xrightarrow{f} \vec{r} \xrightarrow{g} \Delta$, for $f=\vec{C}(t)$ and $g=\Delta(\vec{r})$. The derivative of $f$ is $(\mathrm{d} \vec{C} / \mathrm{d} t)=\vec{C}_{t}$ and that of $g$ is $\vec{\nabla} \Delta$, so that by applying the chain rule follows:

$$
\begin{equation*}
\Delta^{2} r=\frac{\mathrm{d}}{\mathrm{~d} t} \Delta r(t)=\vec{\nabla} \Delta(\vec{r}) \cdot \vec{C}_{t}=\|\vec{\nabla} \Delta(\vec{r})\|\left\|\vec{C}_{t}\right\| \cos \alpha \tag{7}
\end{equation*}
$$



Fig. 3. Relation between wavefront and local slope in a range pixel.

As $t$ is defined to be the range sampling, (7) indicates the difference of the range shift between adjacent two range pixels. Therefore, the $\vec{C}(t)$ component parallel to $\vec{\nabla} \Delta(\vec{r})$ is $\vec{C}_{t} \cos \alpha=$ $\left(\Delta^{2} r /\|\vec{\nabla} \Delta(\vec{r})\|\right)$, as shown in Fig. 3.

For a given slant range sampling $p=(c / 2 \mathrm{RSR})$, with $c$ the speed of light and RSR the range sampling rate, a rectangle with one side along the LOS with a length of $p$ and the other side along the wavefront with a length of ( $\left.\Delta^{2} r /\|\vec{\nabla} \Delta(\vec{r})\|\right)$ is considered (see Fig. 3). The diagonal of this rectangle is $\|\vec{C}(t)\|$, and the angle between the diagonal and the $s\left(\Delta^{2} r /\|\nabla \Delta(\vec{r})\|\right)$ side is the local incidence angle $\alpha$. Therefore

$$
\begin{equation*}
\tan \alpha=\frac{p\|\vec{\nabla} \Delta(\vec{r})\|}{\Delta^{2} r} \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
\sin \alpha=\frac{\tan \alpha}{\sqrt{1+\tan ^{2} \alpha}}=\frac{p\|\vec{\nabla} \Delta(\vec{r})\|}{\sqrt{\left(\Delta^{2} r\right)^{2}+p^{2}\|\vec{\nabla} \Delta(\vec{r})\|^{2}}} \tag{9}
\end{equation*}
$$

Substituting (8) into (5) completes $\kappa_{z}$ without angle notation

$$
\begin{equation*}
\kappa_{z}=\frac{4 \pi}{\lambda p} \sqrt{\left(\Delta^{2} r\right)^{2}+p^{2}\|\vec{\nabla} \Delta(\vec{r})\|^{2}} \tag{10}
\end{equation*}
$$

Note that the vertical wavenumber appears as the (spatial) derivative of the flat earth phase as a consequence of the fact that the local slope is the derivative of the topography.

## V. Validation

In the following, the vertical wavenumber is estimated by using the approach proposed and compared to vertical wavenumber estimates obtained using the conventional approach for a set of interferometric ALOS PALSAR acquisitions. The dataset is over the southern part of the Toyama Prefecture, Japan. The two interferometric images, primary (scene ID ALPSRP071400730) and secondary (scene ID ALPSRP064690730), are acquired with a nominal incidence angle of $23.84^{\circ}$ on May 28, 2007 and April 12, 2007, respectively. The normal baseline of the interferometric pair is $B_{\perp}=638.1 \mathrm{~m}$.


Fig. 4. (First row) Pauli image of Munich test data. (Second row) Ellipsoid height, (third row) profile along the middle line, (fourth row) local incidence angle map, (fifth row) profile along the middle line, (sixth row) vertical wavenumber map, and (seventh row) vertical wavenumber profiles estimated by two methods (black: conventional and red: proposed).

Fig. 4 shows on top the Pauli RGB color composite of the site; in the second row the terrain height (obtained from


Fig. 5. Plots of (top) $\Delta r(t)=\Delta(\vec{C}(t))$ and (bottom) $\Delta^{2} r=(\mathrm{d} / \mathrm{d} t) \Delta r(t)$.
the SRTM DEM [4]); in the third row a height profile along the azimuth line indicated in the Pauli RGB image. The red vertical lines indicate lay-over ranges, where height is not defined. In the fourth row, the local incidence angle map and in the fifth row a range profile of the local incidence angle (along the indicated azimuth line) are shown.

The vertical wavenumber is estimated using the proposed approach by (9) as well as the conventional approach by (5). The terms $\Delta r(t)$ and $\Delta^{2} r$ are estimated during the backgeocoding step. However, while the conventional method requires the calculation of the incidence angle and local terrain slope, the proposed approach just needs the range shifts. Fig. 5 shows at top $\Delta r(t)=\Delta(\vec{C}(t))$ estimated for each range sample along the indicated range profile. The general decreasing trend of range shifts with increasing slant range is modulated by the local topography. The discontinuities are associated with the layover locations. The corresponding derivative of $\Delta r(t), \Delta^{2} r$, along the profile is plotted at the bottom of Fig. 5. The negative mean value reflects the monotonic decreasing trend of $\Delta r(t)$. The derivative is characterized by strongly fluctuating small-scale structures reflecting the topographic derivative (rapid changes of terrain slope).

The vertical wavenumber estimated using the proposed approach is shown in the sixth row in Fig. 4. Finally, in the
seventh row of Fig. 4, the two vertical wavenumber estimates, i.e., the proposed one (in red) and the conventional one (in black), are compared along the indicated azimuth line. The two estimates agree precisely.

## VI. DISCUSSION

Beyond this, the proposed approach can also be used to obtain "quick and dirty" vertical wavenumber estimates without requiring any knowledge on orbit or topography just by estimating the range (corregistration) shift $\Delta r$ between the two interferometric acquisitions. In the case a coherent correlation [9] is employed, the corregistration accuracy is defined by the interferometric coherence and the number of looks available for the correlation [10], [11]

$$
\sigma_{\mathrm{CR}}=\sqrt{\frac{3}{2 N}} \frac{\sqrt{1-\gamma^{2}}}{\pi \gamma}=\frac{\sigma_{\Delta r}}{p} .
$$

The estimation accuracy of the required $\kappa_{z}$ relies on the $\Delta r$ difference. Assuming uncorrelated error components, the differentiation doubles the variance, so that

$$
\sigma_{\Delta^{2} r}=\sqrt{2} p \sigma_{\mathrm{CR}}=p \sqrt{\frac{3}{N}} \frac{\sqrt{1-\gamma^{2}}}{\pi \gamma}
$$

The estimation accuracy of $\kappa_{z}$ is then

$$
\begin{aligned}
\sigma_{\kappa_{z}} & =\frac{d \kappa_{z}}{d\left(\Delta^{2} r\right)} \cdot \sigma_{\Delta^{2} r} \\
& =\frac{4 \pi}{\lambda}\left(\frac{\Delta^{2} r}{\sqrt{\left(\Delta^{2} r\right)^{2}+p^{2}\|\vec{\nabla} \Delta\|^{2}}}\right) \cdot \sqrt{\frac{3}{N}} \frac{\sqrt{1-\gamma^{2}}}{\pi \gamma}
\end{aligned}
$$

and the relative error becomes

$$
\frac{\sigma_{\kappa_{z}}}{\kappa_{z}}=\frac{p}{2} \frac{\Delta^{2} r}{\left(\Delta^{2} r\right)^{2}+p^{2}\|\vec{\nabla} \Delta\|^{2}} \cdot \sqrt{\frac{3}{N}} \frac{\sqrt{1-\gamma^{2}}}{\pi \gamma}
$$

For ALOS PALSAR where $p:=9.369 \mathrm{~m}, \Delta^{2} r: 2 \mathrm{~cm}, B_{\perp}$ : 500 m , and $R: 750 \mathrm{~km}$, the required number of looks for a $10 \%$ error even for very high coherence levels of 0.9 is on the order of 100000 associated with estimation windows on the order of $512 \times 512$ samples. Accordingly, it is only possible to obtain low-resolution estimates.

## VII. Conclusion

A formulation that relates the (slope-normal) vertical wavenumber $\kappa_{z}$ as a function of range (corregistration) shift $\Delta r$ between the primary and secondary images has been derived. This allows the estimation of $\kappa_{z}$-after backgeocoding-without the need of estimating the incidence angle nor the local terrain slope. Accordingly, the proposed approach simplifies significantly the calculational effort in the evaluation of vertical wavenumber keeping the estimation performance the same. Finally, a "quick and dirty" method for estimating vertical wavenumber was proposed and its practical applicability assessed.

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