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RESEARCH ARTICLE

# A FEM-PD coupling based on Arlequin approach to impose boundary conditions in peridynamics

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**Abstract**

The peridynamic approach (PD) is a continuous theory that is well suited for solving damage problems. Because of the nonlocal formulation, PD can predict the response of a material and fracture patterns with high probability in high dynamic processes. In PD, some parameters differ from the continuum formulation and have some deviation in discretized PD systems, such as a horizon. A material constant becomes a parameter dependent on the mesh size. A sticking point, which has to be considered, is that an incomplete horizon at the boundaries results in an unphysical variation of the material's stiffness in these regions. Material points at the boundaries do not have an entire nonlocal neighborhood, meaning the points have fewer bonds and are softer than points within the domain. This leads to the so-called surface effect. The difficulties in applying the classical local initial and boundary conditions happen because of the nonlocal character of the PD. To overcome this problem, several correction techniques have been developed. Nevertheless, a standard method to describe them is not available yet. An alternative approach is the application of the earlier proposed FEM-PD coupling, which can be seen as a local-nonlocal coupling method. The damage-free zones are analyzed by the FEM as classical local theory, while the domain where the fracture is expected is modeled with the PD as a nonlocal theory. Consequently, the reduction of the computational effort as well as the imposing of the conventional local boundary conditions, is achieved. The coupling method is based on the Arlequin method—an energy-based procedure where the energy of a system is found as a weighted average of both systems. The mechanical compatibility in the overlapping zone of both domains is reached by implementing constraints with the help of the penalty method. In the paper at hand, the focus is on imposing BCs. The proposed method is applied to both static and dynamic applications. The accuracy and convergence behavior is evaluated by analyzing test examples.

# 1 | INTRODUCTION

The field of damage mechanics deals with understanding and predicting the behavior of materials and structures under various loading conditions. It plays a crucial role in engineering, as it helps design safe and reliable structures, machines, and components. The main task is to identify potential failure mechanisms and understand how damage initiates and propagates within a material or structure. There are plenty of theories, local and nonlocal, that already deal with such problems. Nevertheless, nonlocal damage models are beneficial in fracture mechanics, where cracks' behavior and propagation are analyzed. Traditional local fracture mechanics approaches, such as linear elastic fracture mechanics (LEFM) [1], have limitations in capturing crack-tip plasticity or crack branching. Nonlocal models can provide a more comprehensive understanding of crack propagation behavior by considering the influence of nonlocal effects on the crack-tip stress and strain fields [2]. Nonlocal damage behavior in materials refers to the phenomenon where the initiation and propagation of damage, such as cracks or fractures, are influenced by conditions or factors beyond the immediate vicinity of the damage itself. In traditional damage mechanics, the fracture is typically described using local models, where the state of a material at a given point is only dependent on the local stresses and strains [3]. Nonlocal damage models can handle material heterogeneity more effectively. In heterogeneous materials, the damage may not be localized at a single point but can spread over a larger region due to different material properties or microstructural features [4].

The peridynamic theory has been developed to overcome such restrictions [5] and has some significant advantages as lack of partial differential questions that fail by any discontinuity and implemented in governing equation damage concept with the prediction of crack bifurcation and delamination with high accuracy. Despite all, imposing boundary conditions is quite a challenging task. They do not appear naturally, as, for example, in the case of CE as a result of solving PDE, but should be defined additionally. The nonlocal theory requires the application of nonlocal BCs. Nevertheless, there is no general rule for peridynamics, and the proposed methods are either complicated or not applicable to complex geometries.

FEM is well established for handling a wide range of boundary conditions, including prescribed displacements, forces, and constraints. It provides a versatile framework for enforcing boundary conditions on a fixed mesh. By coupling peridynamics with FEM, the advantages of FEM in handling complex boundary conditions can be utilized.

Another motivation for coupling peridynamics and FEM arises when dealing with multiscale problems. Peridynamics can capture the behavior at multiple scales due to its mesh-free nature, whereas FEM is limited to the scale of the finite elements. Coupling these two methods makes it possible to bridge the gap between different length scales and simulate phenomena occurring at various scales within the same framework.

In regions where the material is intact and behaves linearly, FEM can be employed to model the behavior and enforce appropriate boundary conditions accurately. As the material undergoes damage or discontinuities, peridynamics can be invoked to handle the evolving behavior and capture the effects of cracks or fractures. This smooth transition between the two methods allows for a comprehensive analysis while respecting the different requirements of boundary conditions in intact and damaged regions.

# 2 | GOVERNING EQUATIONS

## 2.1 | Continuum mechanics

The strong form of the boundary value problem [6] can be given as:

$$\rho\ddot{\mathbf{u}} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{b}. \tag{1}$$

The kinematics of continuum mechanics describes the motion and deformation of materials. Key concepts include displacement, velocity, and deformation gradient. Deformation gradient quantifies the local change in shape and size of infinitesimal material elements: $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$, for $\mathbf{x}$ as a position vector in a current configuration, and $\mathbf{X}$ in a reference one.

The strain tensor, $\epsilon$, provides a measure of the deformation at each point in the material. It characterizes the local changes in length, angles, and shape due to deformation. The strain tensor is related to the deformation gradient tensor through the formula:

$$\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}), \tag{2}$$

with $\mathbf{I}$ as the identity tensor.

In linear elasticity, the relationship between stress and strain is assumed to be linear and reversible, meaning that the material's response to loading is instantaneous and utterly recoverable upon unloading. Hooke's law is the fundamental constitutive equation in linear elasticity. It states that the stress experienced by a material is directly proportional to the corresponding strain: $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon}$. The elasticity tensor, $\mathbf{C}$, depends on the material's symmetry and isotropy:

$$\boldsymbol{\sigma} = E\boldsymbol{\epsilon} - \nu E(Tr(\boldsymbol{\epsilon}))\mathbf{I}. \tag{3}$$

## 2.2 | Bond-based peridynamics

Discussion of the concept of bonds and their role in capturing long-range interactions, the governing equations, and highlight of the unique features that distinguish bond-based peridynamics from traditional models are already widely presented in [7–9].

In bond-based peridynamics, the material is discretized into a set of material points $\mathbf{x}_{(k)}, k = (1, 2, ..\infty)$, each representing a small region within the continuum. These material points are connected through bonds:

$$\boldsymbol{\xi}_{(k)(j)} = \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, \tag{4}$$

which encode the interactions between neighboring points. The relative displacement vector is found as:

$$\boldsymbol{\eta}_{(\boldsymbol{k})(\boldsymbol{j})} = \mathbf{u}_{(j)} - \mathbf{u}_{(k)}. \tag{5}$$

The displacement and deformation of material points are governed by the forces exerted through these bonds, allowing for the modeling of long-range interactions and the propagation of damage. The governing equation of motion in bond-based peridynamics is derived based on the principle of virtual work.

$$\rho\ddot{\mathbf{u}}(\mathbf{x}, t) = \int_{H_{\mathrm{x}}} \mathbf{f}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t)\mathrm{d}V_{\mathrm{x}'} + \mathbf{b}(\mathbf{x}, t). \tag{6}$$

Where the bond force relates the forces between material points to their displacements.

$$\mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \frac{\partial w}{\partial \boldsymbol{\eta}}(\boldsymbol{\eta}, \boldsymbol{\xi}) \quad \forall \boldsymbol{\eta}, \boldsymbol{\xi}. \tag{7}$$

with $w$ as a micropotential. If we assume a small deformations, the bond force vector and his components for 2D case have the form:

$$\mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \frac{\boldsymbol{\xi} + \boldsymbol{\eta}}{|\boldsymbol{\xi} + \boldsymbol{\eta}|} \cdot cs; \qquad s = \frac{|\boldsymbol{\xi} + \boldsymbol{\eta}| - |\boldsymbol{\xi}|}{\boldsymbol{\xi}}; \qquad c = \frac{9E}{\pi h \delta^3}. \tag{8}$$

### 2.2.1 | Boundary conditions in peridynamics

As mentioned earlier, the definition of boundary conditions is problematic because of the nonlocal character of the theory. Using classical local boundary conditions, material points do not have an entire nonlocal neighborhood, meaning the points have fewer bonds and are softer than points within the domain. This leads to stiffness fluctuations, the so-called surface effect [10]. Few methods have been developed to reduce or eliminate when possible the surface effect. For instance, the boundary layer can be expanded on the horizon size value, so the displacement difference between PD and CE is much lower [11]. Beyond the horizon, the PD surface effect depends on the micromodule function (in bond-based PD formulation). So such methods as the volume correction method aim to increase the micromodulus function by incomplete family nodes close to the boundary edge so that the nodes have the same strain energy density as inner nodes [12]. A similar purpose has the force density method. It is based on prescribing uniaxial tension BC [13] in all directions. The multiplication factor is calculated to achieve the same force density vector for boundary points. However, both correction schemes are not exact. The energy-based method [13] is quite familiar to the last approach, but instead of force density, the energy density of a PD point is recalculated with compensating constant. Several methods were suggested to circumvent the lack of data. For instance, the extended domain method [14, 15] constructs an extended solution of displacement to the real and

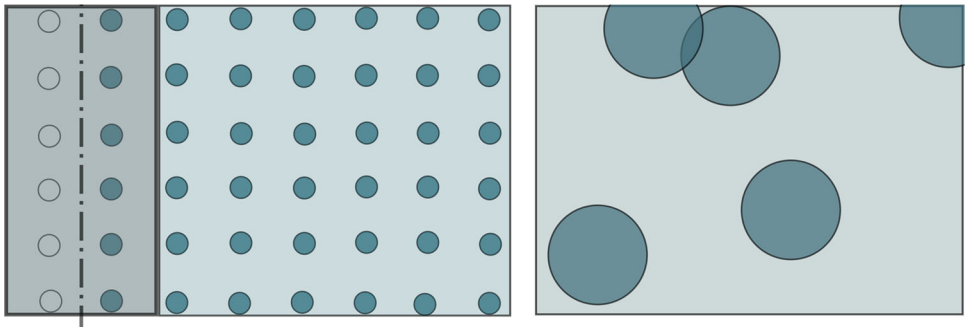

**FIGURE 1** Correction methods in peridynamics: left—extended domain method, right—strain energy correction method.

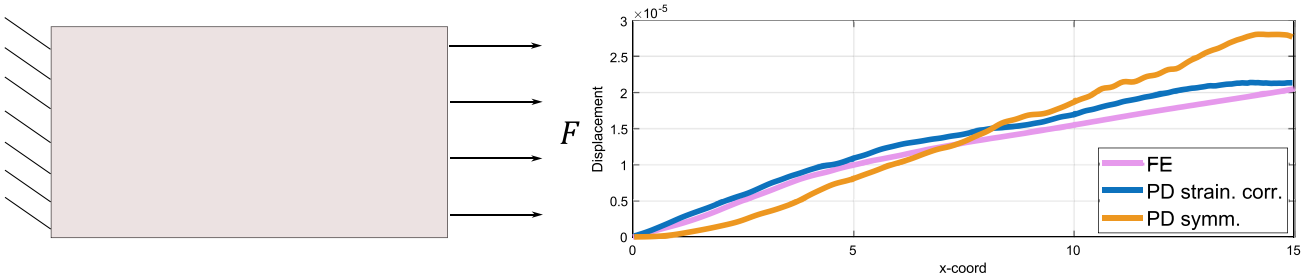


**FIGURE 2** Left: test configuration for validation diverse BC correction methods. Right: displacement of the middle line of the plate over the $x$-coordinate.

fictitious domain with width as double horizon size, where the BCs are applied. One more alternative was proposed in the same article, the variable horizon method. The second consists then in decreasing the horizon from a constant value in the domain's interior to zero at the boundary so that one can directly apply the classical boundary conditions. A series of works [16–18] introduce the concept of state-based PD, where the displacements of the fictitious nodes are expressed as functions of the displacements of the closest real nodes employing multiple Taylor series expansions.

We chose two methods to display the problem of applying diverse strategies and show the working principle: the extended domain method and surface correction factor strategy. The schematic explanation is seen in Figure 1. The extended domain method, for example, suggests filling the boundary with fictitious points that complete this truncated neighborhood. In this way, there is no need to demand displacement to be zero for the application of clamped BCs. However, instead of this, the points close to the boundary in the real domain that has the width of the horizon should fulfill the antisymmetric rule with the points in the fictitious region. Another method is a surface correction factor strategy, which calculates the strain energy density of a peridynamic point close to the boundary with a truncated nonlocal neighborhood and defines the multiplication factor to compensate surface effect.

To demonstrate the ambiguity of both methods, the test simulation is provided. The quasi-static analysis of a rectangular 2D plate with dimensions $15 \times 2$ m is shown in Figure 2. The elastic and isotropic plate stretches under constant load $F = 200 \times 10^3$ Pa in $X$ direction. The material is assumed to be isotropic and elastic. The numerical parameters are $\Delta x = 0.006$ m, horizon $\delta = 0.018$ m, and total time $t = 0.004$ s. The displacement of the middle line of a plate at the time $t$ over the $x$-coordinate is illustrated in Figure 2. As a reference solution, the FE result is taken. Both methods do not converge to the FE solution and have a deviation. In this context, coupling PD and FEM can be supposed as an alternative method to impose boundary conditions for damage problems, where fracture-free zone together with boundaries (if they are also fracture free) are modeled with FE and the rest with peridynamics (Figure 3).

## 3 | COUPLING STRATEGY

Arlequin Method [19] is an energy based approach. The detailed information about this application and validation for 1D case is given in [20]. Tho whole model is divided to the pure peridynamic (PD), continuum mechanic (CM) domains, and

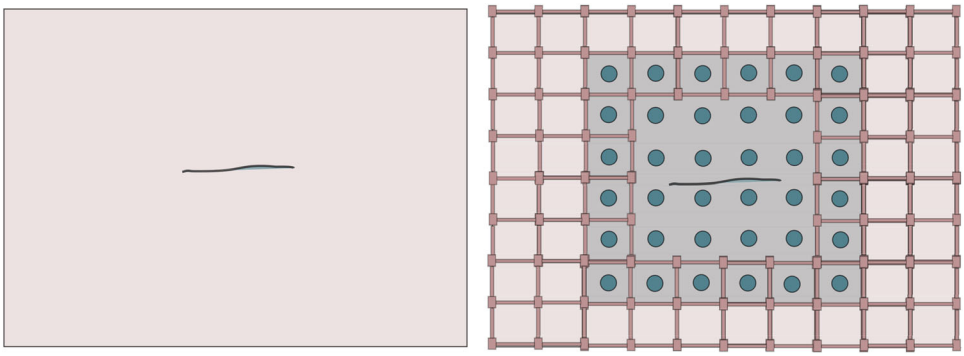

**FIGURE 3** The schematic model of the coupled system: the PD domain with crack, damaged-free zone with the FE .

the area, where they both overlap. The total Hamiltonian of the system is written as follows:

$$\mathcal{H}_{tot}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{d}, \dot{\mathbf{d}}) = \alpha(\mathbf{x})\mathcal{H}_{CM}(\mathbf{u}, \dot{\mathbf{u}}) + (1 - \alpha(\mathbf{x}))\mathcal{H}_{PD}(\mathbf{d}, \dot{\mathbf{d}}). \tag{9}$$

with a weighting function $\alpha(\mathbf{x})$, which is either 1 for pure CM or 0 for pure PD. For overlapping zone $\alpha(\mathbf{x})$ performs as a blending function that enables to reach a seamless transition between both regions. If it remains constant, for example, as half of the total energy, the rapid drop between the energies happens. The choice of linear or cubic functions that gradually increase or decrease could significantly reduce the error.

The total Hamiltonian in both theories is set to equal to the total energy system:

$$\mathcal{H} = \mathcal{W}^{kin} + \mathcal{W}^{int} - \mathcal{W}^{ext}. \tag{10}$$

The equation of motion for both systems obtained by taking the derivative of Equation (10) by the displacement. And after that, the additional constrains in the overlapping zone should be included.

Let the equation of constraints in the general case $\mathbf{z}$ consist of displacement-dependent term $\mathbf{Zu}$ and some constant term $\mathbf{z}_0$, as $\mathbf{z}(\mathbf{u}) = \mathbf{z}_0 + \mathbf{Zu}$. These constraints can be included in the equation of motion using the Penalty Method. The Penalty method provides a straightforward approach to handling constraints in coupling-type problems. For this purpose, the functional $\Psi$ should be constructed concerning the introduced degrees of freedom. Considering constraints, the functional is demanded to reach a minimum value [21].

$$\Psi = \mathcal{L}(\mathbf{x}, \mathbf{u}, t) + \frac{1}{2}\kappa \int_{\Omega_o} \mathbf{z}^T \mathbf{z} d\Omega_o \rightarrow \text{Min}, \tag{11}$$

where $\mathcal{L}(\mathbf{x}, \mathbf{u}, t)$ is Lagrangian of the dynamic system and $\kappa$ the penalty number. The above-mentioned functional should be minimized, that means that the partial derivative of the functional over the displacement is calculated. These lead to the equation of motion of the coupled system in discretized form [20]:

$$\mathbf{M\ddot{u}} + (\mathbf{K} + \mathbf{K}_z)\mathbf{u} = \mathbf{f} - \mathbf{f}_z; \tag{12}$$

with

$$\mathbf{K}_z = \kappa \mathbf{Z}^T \mathbf{Z}; \qquad \mathbf{f}_z = \kappa \mathbf{Z}^T \mathbf{z}_0.$$

## 4 | VALIDATION AND DISCUSSION

It was previously shown that Arlequin coupling performs suitable for static cases, so we focused on the dynamic tests, especially on the wave propagation problem under the impulse load. The wave propagation test shows the coupling quality and accuracy. Traveling through the coupling area if no interference is happening means that media is considered to have the same properties and give the same response. The plate with dimensions 10

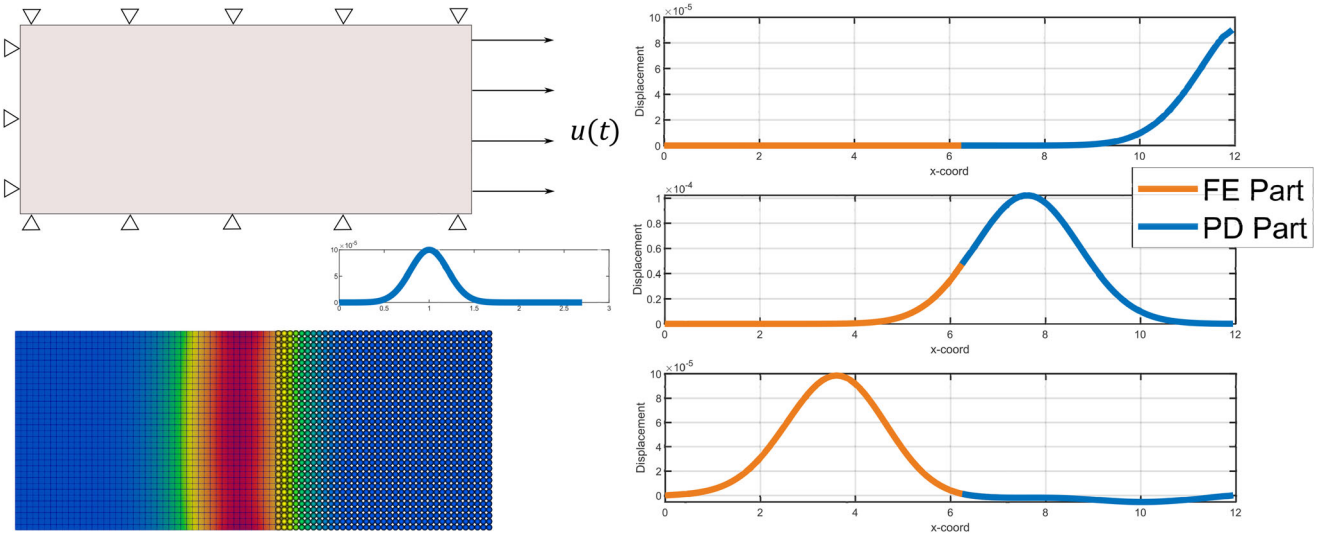


**FIGURE 4** Test configuration for wave propagation analysis and the result of the displacement at the time: top: $t = 0.0011$ s, middle: $t = 0.0018$ s, bottom: $t = 0.0024$ s.

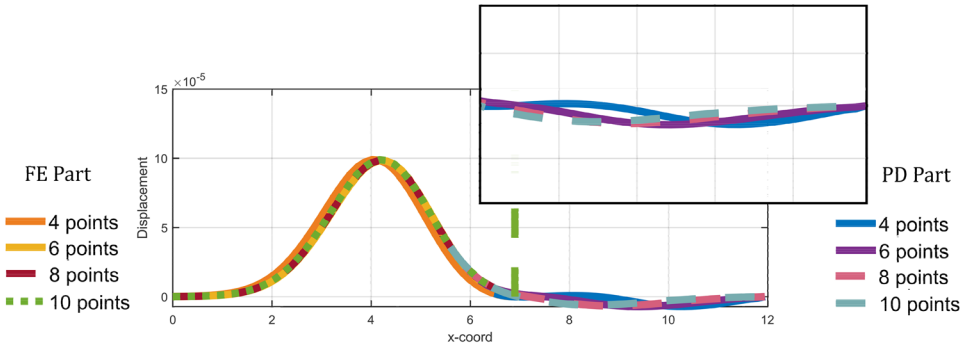


**FIGURE 5** The parameter study: variation of the size of the overlapping area.

5 m is under the impulse load of a form $u(t) = ae^{-\frac{(t-\beta)^2}{2c^2}}$, $a = 10^{-5}$ m, $\beta = 10^{-4}$ s, and $c = 210^{-6}$ s. The $y$ displacements are constrained at the top and bottom edges, whereas the $x$ displacements are constrained at the left edge of the plate (Figure 4). The material is assumed to be isotropic and elastic. Numerical parameter was chosen as $\Delta x = 0.1$ m, $\delta = 0.301$ m, $\Delta t = 10^{-8}$ s. The left side of the plate is modeled with finite elements, whereas the right side, where the impulse load is applied, is discretized mesh-free and is solved with bond-based peridynamics. Both meshes are uniform.

As expected, the left upper figure shows that the propagation wave is constant over the plate width. The graphs in Figure 4 illustrate a displacement of the middle line of the plate at a particular time. Spreading through the coupling area is smooth, so the PD curve does not create a nonphysical angle with the FE ones. Nevertheless, it can be seen on the third graph that the small reflection wave arises and goes in the opposite direction. The coupling domain study is also provided in Figure 5. It can be seen that 4 points in the overlapping area give the small shifted amplitude to the left in the FE part, but after the choice of 6 points and more, it is not the case. The difference is visible by analyzing the reflected wave in the PD domain. The result with 10 points gives a slightly lower amplitude than in other cases. The reflections were not avoided entirely, but it is possible to minimize them by carefully selecting the model parameters.

## 5 | CONCLUSION

In conclusion, the article discusses the peridynamic approach (PD) and its application in solving damage problems, particularly in the context of wave propagation. The study highlights the challenges associated with boundary conditions, the surface effect in PD, and the limitations of existing correction techniques.

The article proposes an alternative approach, the FEM-PD coupling, which combines the Finite Element Method (FEM) as a local and PD as a nonlocal theory. Based on the Arlequin method, the coupling method aims to achieve mechanical compatibility and reduce computational effort by dividing the analysis between the FEM and PD domains.

Through various examples and simulations, the article demonstrates the performance of the FEM-PD coupling method in imposing boundary conditions and analyzing dynamic wave propagation. While the method shows promising results regarding computational efficiency and accurate representation of wave traveling, certain deviations and unphysical effects are observed, indicating the need for further investigation and refinement.

Overall, the FEM-PD coupling approach offers a potential solution to address the challenges associated with boundary conditions in damage problems. However, further research and development are required to enhance the method's accuracy, convergence behavior, and physical consistency. By bridging the gap between local and nonlocal theories, the coupling approach holds promise for advancing the understanding and analysis of wave propagation and damage phenomena.

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