# Symmetries in non-relativistic quantum electrodynamics 

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January 23, 2023


#### Abstract

We define symmetries in non-relativistic quantum electrodynamics, which have the physical interpretation of rotation, parity and time reversal symmetry. We collect transformation properties related to these symmetries in Fock space representation as well as in the Schrödinger representation. As an application, we generalize and improve theorems about Kramer's degeneracy in non-relativistic quantum electrodynamics.


## 1 Introduction

Symmetries are often used to analyze various properties of physical systems. In particular in quantum mechanics symmetries are used to determine spectral properties of the Hamiltonian. In this paper we study symmetries of non-relativistic quantum electrodynamics (qed), which have the physical interpretation of rotation, parity and time reversal symmetry. We give explicit formulas for these symmetries both in Fock space representation as well as in the so called Schrödinger representation and apply these symmetries to prove multiplicities of eigenvalues.

The transformation properties described in the present paper are of general interest in non-relativistic qed. In particular, in the Fock representation these symmetries are helpful for operator theoretic renormalization analysis of non-relativistic qed. On the one hand,

[^0]symmetries can be used to control marginal terms [12,13, 14, 25]. On the other hand symmetries allow the treatment of degenerate eigenvalues in the frame work of renormalization, provided the symmetries act irreducibly on the eigenspace [15]. In fact, the latter is the main interest, which we had in mind, for collecting the transformation properties of the aforementioned symmetries.

In physics literature continuous symmetries are often described by means of their infinitesimal generator. That is, as a representation of the Lie-algebra. For non-relativistic qed the generators of the Lie-algebra of $S U(2)$ are readily available in textbooks about non-relativistic aspects of quantum electrodynamics [4, 27]. In this paper we express the $S U(2)$-symmetry directly as a representation of the Lie-group.

As already mentioned, symmetries are helpful in the spectral analysis of Hamiltonian operators of quantum mechanics. For example the classical Kramers degeneracy theorem states, that the eigenvalues of a time-reversal symmetric Hamiltonian describing an odd number of spin $1 / 2$-particles have even multiplicity. Using a theorem of this type it was shown in $[20,21]$ that Hamiltonians of non-relativistic qed, which describe odd number of spin $1 / 2$-particles have a doubly degenerate ground state, provided the external potential is symmetric with respect to parity. In this paper we improve that result and show that parity symmetry is not necessary. This is of physical relevance, since potentials describing molecules with static nuclei, are not necessarily symmetric with respect to parity. Furthermore, we include external magnetic fields in the mathematical model. Finally, we consider translation invariant systems and generalize degeneracy results for a single spin $1 / 2$-particle $[16,17]$ to atoms and molecules.

Let us give a short outline of the paper. In the next section we review the notion of a symmetry in quantum mechanics and state an abstract version of Kramers degeneracy theorem. In Section 3 we introduce non-relativistic qed. In Section 4 we define rotation, parity, and time-reversal symmetry. Moreover we collect various transformation properties. In Section 5 we study symmetry properties of Hamiltonians of non-relativistic qed. In particular we show the aforementioned degeneracy theorems. In Section 6 we study symmetry properties of fibers of translationally invariant Hamiltonians of non-relativistic qed. In Section 7 we define rotation, parity, and time-reversal symmetry in the so called Schrödinger representation. We show that the definitions in Schrödinger representation agree with the definitions in Fock space representation. To show this, we use the canonical unitary transformation mapping the Fock space representation to the Schrödinger representation.

## 2 Symmetries in Quantum Mechanical Systems

In this section we collect some well-known definitions and properties.
Definition 2.1. Let $V$ be a complex vector space. A mapping $A: V \rightarrow V$ is called antilinear (or conjugate linear) if
(i) $A(x+y)=A x+A y$ for all $x, y \in V$.
(ii) $A(\alpha x)=\bar{\alpha} A x$, for all $x \in V$ and $\alpha \in \mathbb{C}$.

If $\mathcal{H}$ is a complex Hilbert space and $T: \mathcal{H} \rightarrow \mathcal{H}$ anti-linear, then the adjoint $T^{*}: \mathcal{H} \rightarrow \mathcal{H}$ is defined by

$$
\left\langle T^{*} x, y\right\rangle=\langle T y, x\rangle \quad, \quad \forall x, y \in \mathcal{H} .
$$

If $\mathcal{H}$ is a complex Hilbert space and $S: \mathcal{H} \rightarrow \mathcal{H}$ anti-linear, then $S$ is called anti-unitary if it is surjective and satisfies

$$
\langle S x, S y\rangle=\langle y, x\rangle \quad, \quad \forall x, y \in \mathcal{H} .
$$

The assertions of the following Lemma are straightforward to verify.
Lemma 2.2. The following holds.
(a) Let $C_{i}$ be anti-linear (anti-unitary) transformations on complex vector spaces $V_{i}$ (Hilbert spaces), $i=1,2$. Then $C_{1} \otimes C_{2}: V_{1} \otimes V_{2} \rightarrow V_{1} \otimes V_{2}$ is also anti-linear (anti-unitary).
(b) If $T: \mathcal{H} \rightarrow \mathcal{H}$ is an anti-linear mapping on a Hilbert space $\mathcal{H}$, then also $T^{*}$ is anti-linear.
(c) If $S$ is anti-unitary, then $S$ is bijective and $S^{*} S=1$ and $S S^{*}=1$.

Definition 2.3. Let $S$ be a unitary or anti-unitary operator. Let $H$ be a densely defined operator in $\mathcal{H}$. We call $S$ a symmetry of $H$, if

$$
S H=H S
$$

when $S$ is unitary, and

$$
S H=H^{*} S
$$

when $S$ is anti-unitary.
The following theorem, whose formulation is from [20], can be viewed as an abstract version of Kramer's degeneracy theorem, [18, 29].

Theorem 2.4 (Abstract Kramers Degeneracy). Let $\theta$ be a an anti-unitary symmetry of a self-adjoint operator $H$ and $\theta^{2}=-1$. Then each eigenvalue of $H$ is at least doubly degenerate. Any eigenvalue of $H$ with finite multiplicity has even multiplicity.

The proof follows from the following lemma.
Lemma 2.5. Let $J$ be an anti-unitary operator on a complex Hilbert space $V$ with $J^{2}=-1$. Then the following holds.
(a) For any nonzero $v \in V$, also $J v$ is nonzero and $v \perp J v$.
(b) The Hilbert space $V$ cannot have finite odd dimension.

Proof. (a) Since $J(J v)=J^{2} v=-v$, the vector $J v$ is nonzero. Since $J$ is anti-unitary

$$
\langle v, J v\rangle=\langle J J v, J v\rangle=-\langle v, J v\rangle .
$$

So $\langle v, J v\rangle=0$.
(b) We show by induction that $V$ cannot have dimension $2 n-1$ for $n \in \mathbb{N}$. Clearly, the induction hypothesis holds true for $n=1$ by (a). Suppose the induction hypothesis holds for $n$, and suppose $V$ has dimension $2 n+1$. Pick a nonzero $v \in V$. Then $J v \in V$ and $J v \perp v$ by (a). Thus $W:=\{v, J v\}^{\perp}$ is a complex vector space, which has dimension $2 n-1$. Since $J^{2}=-1$, it follows that $J$ leaves the complex linear span $\operatorname{lin}_{\mathbb{C}}\{v, J v\}$ invariant. Since $J$ is anti-unitary, it leaves also $W$ invariant. But the complex vector space $W$ together with $\left.J\right|_{W}$ contradict the induction hypothesis.

Proof of Theorem 2.4. Let $E$ be an eigenvalue of $H$. Since $H$ is self-adjoint $E$ is real. So $\theta$ leaves the space $V=\operatorname{ker}(H-E)$ invariant, since $(H-E) \theta \psi=\theta(H-E) \psi$. Thus the first and second statement follow from (b) of Lemma 2.5 with $J=\theta$.

## 3 Non-relativistic qed

For a complex Hilbert space $\mathcal{H}$ we denote the $n$-fold tensor product by

$$
\mathcal{H}^{\otimes n}:=\bigotimes_{j=1}^{n} \mathcal{H}
$$

and we set $\mathcal{H}^{\otimes 0}:=\mathbb{C}$. Let $\mathfrak{S}_{\{1, \ldots, n\}}$ be the permutation group of the set $\{1, \ldots, n\}$. For each $\sigma \in \mathfrak{S}_{\{1, \ldots, n\}}$ we define an operator $\mathfrak{U}(\sigma)$ on $\mathcal{H}^{\otimes n}$ by

$$
\begin{equation*}
\mathfrak{U}(\sigma)\left(\varphi_{1} \otimes \varphi_{2} \otimes \cdots \otimes \varphi_{n}\right)=\varphi_{\sigma(1)} \otimes \varphi_{\sigma(2)} \otimes \cdots \otimes \varphi_{\sigma(n)} \tag{3.1}
\end{equation*}
$$

for any $\varphi_{j} \in \mathcal{H}, j=1, \ldots, n$, and extending it linearly. This yields a bounded operator (of norm one) on $\mathcal{H}^{\otimes n}$ so we can define $S_{n}=\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_{\{1, \ldots, n\}}} \mathfrak{U}(\sigma)$. We define the symmetric $n$-fold tensor product of $\mathcal{H}$ by

$$
\mathcal{H}^{\otimes_{s} n}:=S_{n}\left(\mathcal{H}^{\otimes n}\right) .
$$

Let $\mathcal{D}_{s}$ denote the representation space of $S U(2)$ with dimension $2 s+1$. In this paper we shall only consider the case $s=0$, describing spinless particles, and the case $s=\frac{1}{2}$, describing particles with spin $1 / 2$.

The model consists of $N$ particles with spins $s_{j} \in\{0,1 / 2\}$, masses $m_{j}>0$, charges $q_{j} \in \mathbb{R}$, values of the spin magnetic moments $\mu_{j} \in \mathbb{R}, j=1, \ldots, N$. By $x_{j} \in \mathbb{R}^{3}$ we shall denote the position of the $j$-the particle. The Hilbert space describing the non-relativistic quantum mechanical matter is

$$
\mathcal{H}_{\mathrm{mat}}=\bigotimes_{j=1}^{N} L^{2}\left(\mathbb{R}^{3} ; \mathcal{D}_{s_{j}}\right)
$$

We note that the description of physical systems usually requires the restriction to a subspace determined by the particle statistics of identical particles. This will be considered below.

If $s=0$, let $\hat{s}_{l}=0$ for $l=1,2,3$, and if $s=1 / 2$, let $\hat{s}_{l}=\frac{1}{2} \sigma_{l}$ for $l=1,2,3$, where $\sigma_{l}$ denotes the $l$-th Pauli-matrix.

Remark 3.1. Note that $\hat{s}_{1}, \hat{s}_{2}$ and $\hat{s}_{3}$ are representations of the generators of $s u(2)$ in the representation $\mathcal{D}_{s}=\mathbb{C}^{2 s+1}, s \in\{0,1 / 2\}$. They are linear maps in $\mathcal{D}_{s}$ satisfying

$$
\begin{align*}
& {\left[\hat{s}_{j}, \hat{s}_{k}\right]=\sum_{l=1}^{3} i \epsilon_{j, k, l} \hat{s}_{l}, \quad \hat{s}_{l}^{*}=\hat{s}_{l}, l=1,2,3,} \\
& \overline{\hat{s}}_{1}=\hat{s}_{1}, \quad \overline{\hat{s}}_{2}=-\hat{s}_{2}, \quad \overline{\hat{s}}_{3}=\hat{s}_{3}, \tag{3.2}
\end{align*}
$$

where $\epsilon_{j, k, l}$ denotes the totally antisymmetric tensor in three dimensions.
For $j=1, \ldots, N$ and $l=1,2,3$ we define

$$
\left(\widehat{S}_{j}\right)_{l}=\left(\bigotimes_{k=1}^{j-1} \mathbb{I}_{\mathcal{D}_{s_{k}}}\right) \otimes \hat{s}_{l} \otimes\left(\bigotimes_{k=j+1}^{N} \mathbb{1}_{\mathcal{D}_{s_{k}}}\right)
$$

For a Hilbert space $\mathfrak{h}$ define the symmetric Fock space over $\mathfrak{h}$ by

$$
\mathcal{F}_{s}(\mathfrak{h}):=\bigoplus_{n=0}^{\infty} \mathfrak{h}^{\otimes_{s} n} .
$$

Thus we can identify $\psi \in \mathcal{F}_{s}(\mathfrak{h})$ with a sequence of functions $\psi=\left(\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots.\right)$ such that $\psi^{(n)} \in \mathfrak{h}^{\otimes_{s} n}$. We introduce the set $F_{0}(\mathfrak{h}):=\left\{\psi \in \mathcal{F}_{s}(\mathfrak{h}): \exists N, \forall n \geq N, \psi^{(n)}=0\right\}$ of finite particle vectors. For $f \in \mathfrak{h}$ let $a^{*}(f)$ denote the usual creation operator, which is a densely defined closed linear operator which satisfies for $\eta \in \mathfrak{h}^{\otimes_{s} n}$

$$
\begin{equation*}
a^{*}(f) \eta=\sqrt{n+1} S_{n+1}(f \otimes \eta) \tag{3.3}
\end{equation*}
$$

Let $a(f)$ denote the adjoint of the creation operator. If $T$ be a symmetry in $\mathfrak{h}$, then $\Gamma(T)$ denotes the unique operator on $\mathcal{F}(\mathfrak{h})$ such that on $\mathfrak{h}^{\otimes_{s} n}$

$$
\left.\Gamma(T)\right|_{h^{\otimes s n}}=\bigotimes_{j=1}^{n} T
$$

It is straight forward to see that also $\Gamma(T)$ is a symmetry. Let $A$ be any self-adjoint operator on $\mathcal{H}$ with domain of essential self-adjointness $D$. Let $D_{A}=\left\{\psi \in F_{0}(\mathfrak{h}): \psi^{(n)} \in\right.$ $\otimes_{k=1}^{n} D$ for each $\left.n\right\}$ and define $d \Gamma(A)$ on $D_{A} \cap \mathfrak{h}^{\otimes_{s} n}$ as

$$
A \otimes 1 \otimes \cdots \otimes 1+1 \otimes A \otimes \cdots \otimes 1+\cdots+1 \otimes \cdots \otimes 1 \otimes A
$$

In [24, Section VIII.10] it is shown that $d \Gamma(A)$ is essentially self-adjoint on $D_{A}$, and we shall denote this self-adjoint extension again by $d \Gamma(A)$. It follows from the definitions that for a symmetry $T$ on $\mathfrak{h}$ and $f \in \mathfrak{h}$

$$
\begin{align*}
& \Gamma(T) a^{\#}(f) \Gamma(T)^{*}=a^{\#}(T f),  \tag{3.4}\\
& \Gamma(T) d \Gamma(A) \Gamma(T)^{*}=d \Gamma\left(T A T^{*}\right) \tag{3.5}
\end{align*}
$$

where $a^{\#}$ stands for $a$ or $a^{*}$. Let us now define operators acting on the composite Hilbert space

$$
\mathcal{H} \otimes \mathcal{F}_{s}(\mathfrak{h})
$$

where $\mathcal{H}$ denotes a Hilbert space, which is used to describe the matter. For a bounded linear operator $G \in \mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathfrak{h})$ we define for $\varphi \in \mathcal{H}$ and $\eta \in \mathfrak{h}^{\otimes_{s} n}$

$$
\begin{equation*}
a^{*}(G)(\varphi \otimes \eta)=\sqrt{n+1}\left(1 \otimes S_{n+1}\right)((G \varphi) \otimes \eta) \tag{3.6}
\end{equation*}
$$

This extends by linearity to a closable operator in $\mathcal{H} \otimes \mathcal{F}_{s}(\mathfrak{h})$, which we shall again denote by $a^{*}(G)$. We define $a(G)=\left[a^{*}(G)\right]^{*}$.

In non-relativistic qed one consider the Fock space over $\mathfrak{g}:=L^{2}\left(\mathbb{R}^{3} \times \mathbb{Z}_{2}\right)$. In that case we can identify $\psi \in \mathcal{F}_{s}(\mathfrak{g})$ with a sequence of functions $\psi=\left(\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots\right)$ such that $\psi^{(n)} \in L_{s}^{2}\left(\left(\mathbb{R}^{3} \times \mathbb{Z}_{2}\right)^{n}\right)$, where the subscript $s$ stands for wave functions which are symmetric with respect to interchange of components of the $n$-fold Cartesian product. Let $M_{f}$ denote the operator of multiplication by the function $f$. We define

$$
H_{\mathrm{f}}=d \Gamma\left(M_{\omega}\right)
$$

where the so-called dispersion relation $\omega: \mathbb{R}^{3} \rightarrow[0, \infty)$ is defined such that $\omega(k)=\omega\left(k^{\prime}\right)$ whenever $\left|k^{\prime}\right|=|k|$. Moreover define

$$
P_{\mathrm{f}}=d \Gamma\left(M_{\pi_{j}}\right),
$$

where $\pi_{j}: \mathbb{R}^{3} \rightarrow \mathbb{R}$ with $\pi_{j}(k)=k_{j}$. Next we introduce creation and annihilation operators in terms of operator valued distributions. We define

$$
\mathcal{D}_{\mathcal{S}}:=\left\{\psi \in F_{0}(\mathfrak{g}): \psi^{(n)} \in \mathcal{S}\left(\left(\mathbb{R}^{3} \times \mathbb{Z}_{2}\right)^{n}\right)\right\} .
$$

where $\mathcal{S}\left(\left(\mathbb{R}^{3} \times \mathbb{Z}_{2}\right)^{n}\right)$ denotes the space of smooth rapidly decaying functions. For each $(k, \lambda) \in \mathbb{R}^{3} \times \mathbb{Z}_{2}$ we define an operator $a(k, \lambda)$ on $\mathcal{F}_{s}(\mathfrak{g})$ with domain $\mathcal{D}_{\mathcal{S}}$ by

$$
(a(k, \lambda) \psi)_{n}\left(k_{1}, \lambda_{1}, \ldots, k_{n}, \lambda_{n}\right)=\sqrt{n+1} \psi_{n+1}\left(k, \lambda, k_{1}, \lambda_{1}, \ldots, k_{n}, \lambda_{n}\right)
$$

We define $a^{*}(k, \lambda)$ in the sense of quadratic forms on $\mathcal{D}_{\mathcal{S}} \times \mathcal{D}_{\mathcal{S}}$ by

$$
\left\langle\psi_{1}, a^{*}(k, \lambda) \psi_{2}\right\rangle=\left\langle a(k, \lambda) \psi_{1}, \psi_{2}\right\rangle
$$

Then it is straight forward to see that

$$
a^{*}(f)=\sum_{\lambda=1,2} \int_{\mathbb{R}^{3}} f(k, \lambda) a^{*}(k, \lambda) d k, \quad a(f)=\sum_{\lambda=1,2} \int_{\mathbb{R}^{3}} \overline{f(k, \lambda)} a(k, \lambda) d k
$$

where the equalities are understood in the sense of quadratic forms and the integrals are understood as weak integrals. Let us now relate the definition given in (3.6) to integrals of operator valued distributions. To this end we use the natural embedding

$$
\begin{aligned}
I: L^{2}\left(\mathbb{R}^{3} \times \mathbb{Z}_{2} ; \mathcal{L}\left(\mathcal{H}_{\mathrm{mat}}\right)\right) & \rightarrow \mathcal{L}\left(\mathcal{H}_{\mathrm{mat}} ; L^{2}\left(\mathbb{R}^{3} \times \mathbb{Z}_{2} ; \mathcal{H}_{\mathrm{mat}}\right)\right) \cong \mathcal{L}\left(\mathcal{H}_{\mathrm{mat}} ; \mathcal{H}_{\mathrm{mat}} \otimes \mathfrak{g}\right) \\
g & \mapsto(\varphi \mapsto[(k, \lambda) \mapsto g(k, \lambda) \varphi])
\end{aligned}
$$

which is a bounded injection, cf. [24, Theorem II.10]. Then for $g \in L^{2}\left(\mathbb{R}^{3} \times \mathbb{Z}_{2} ; \mathcal{L}\left(\mathcal{H}_{\text {mat }}\right)\right)$ it is straight forward to show that

$$
\begin{equation*}
a^{*}(I(g))=\sum_{\lambda=1,2} \int_{\mathbb{R}^{3}} g(k) \otimes a^{*}(k, \lambda) d k, \quad a(I(g))=\sum_{\lambda=1,2} \int_{\mathbb{R}^{3}} g(k)^{*} \otimes a(k, \lambda) d k \tag{3.7}
\end{equation*}
$$

in the sense of quadratic forms on $\mathcal{H}_{\text {mat }} \otimes \mathcal{D}_{\mathcal{S}}$, where the integral is a weak integral. Henceforth we shall drop the tensor sign in (3.7) if it is clear on which factor the operator acts. The definition of the vector potential involves the so called polarization vectors. For $\lambda=1,2$ we choose a measurable function

$$
\begin{equation*}
\varepsilon(\cdot, \lambda): S_{2} \rightarrow \mathbb{R}^{3} \tag{3.8}
\end{equation*}
$$

on the 3-dimensional sphere $\mathrm{S}_{2}$ with the following properties. For each $k \in \mathrm{~S}_{2}$ the vectors $(\varepsilon(k, 1), \varepsilon(k, 2), k)$ form an orthonormal basis of $\mathbb{R}^{3}$. We extend $\varepsilon(\cdot, \lambda)$ to $\mathbb{R}^{3} \backslash\{0\}$ by setting $\varepsilon(k, \lambda):=\varepsilon(k /|k|, \lambda)$ for all nonzero $k$. We assume that we are given a measurable coupling function $\kappa: \mathbb{R}^{3} \rightarrow \mathbb{C}$. We note that the Fourier transform of $\kappa$ is real, if and only if

$$
\begin{equation*}
\overline{\kappa(k)}=\kappa(-k) . \tag{3.9}
\end{equation*}
$$

We define the coupling functions for $l=1,2,3$ and $x \in \mathbb{R}^{3}$

$$
g_{x, l}^{(\varepsilon)}(k, \lambda)=\frac{\varepsilon_{l}(k, \lambda)}{\sqrt{2 \omega(k)}} \overline{\kappa(k)} e^{-i k \cdot x} .
$$

We can now define the field operators. If $\omega^{-1 / 2} \kappa \in L^{2}\left(\mathbb{R}^{3}\right)$, we define the magnetic vector potential

$$
\begin{aligned}
A_{l}(x) & :=a\left(g_{x, l}^{(\varepsilon)}\right)+a^{*}\left(g_{x, l}^{(\varepsilon)}\right) \\
& =\sum_{\lambda=1,2} \int_{\mathbb{R}^{3}} \frac{\varepsilon_{l}(k, \lambda)}{\sqrt{2 \omega(k)}}\left(\kappa(k) e^{i k \cdot x} a(k, \lambda)+\overline{\kappa(k)} e^{-i k \cdot x} a^{*}(k, \lambda)\right) d k, \quad l=1,2,3,
\end{aligned}
$$

where in the second line we made use of (3.7). If $|\cdot| \omega^{-1 / 2} \kappa \in L^{2}\left(\mathbb{R}^{3}\right)$, we define the quantized magnetic field

$$
\begin{aligned}
B_{l}(x) & :=[\nabla \times A(x)]_{l} \\
& =\sum_{\lambda=1,2} \int_{\mathbb{R}^{3}} \frac{i[k \times \varepsilon(k, \lambda)]_{l}}{\sqrt{2 \omega(k)}}\left(\kappa(k) e^{i k \cdot x} a(k, \lambda)-\overline{\kappa(k)} e^{-i k \cdot x} a^{*}(k, \lambda)\right) d k, \quad l=1,2,3 .
\end{aligned}
$$

If $\omega^{1 / 2} \kappa \in L^{2}\left(\mathbb{R}^{3}\right)$, we define the quantized electric field

$$
\begin{aligned}
E_{l}^{\perp}(x) & :=a\left(-i \omega g_{x, j}^{(\varepsilon)}\right)+a^{*}\left(-i \omega g_{x, j}^{(\varepsilon)}\right) \\
& =\sum_{\lambda=1,2} \int_{\mathbb{R}^{3}} i \varepsilon_{l}(k, \lambda) \sqrt{\frac{\omega(k)}{2}}\left(\kappa(k) e^{i k \cdot x} a(k, \lambda)-\overline{\kappa(k)} e^{-i k \cdot x} a^{*}(k, \lambda)\right) d k, \quad l=1,2,3 .
\end{aligned}
$$

The Hamiltonian acting in the Hilbert space

$$
\mathcal{H}_{\mathrm{mat}} \otimes \mathcal{F}_{s}(\mathfrak{g})
$$

is given by

$$
\begin{align*}
H= & \sum_{j=1}^{N}\left\{\left(p_{j} \otimes 1+q_{j}\left(A\left(\hat{x}_{j}\right)+A_{\mathrm{ext}}\left(\hat{x}_{j}\right)\right)\right)^{2}+\mu_{j} \widehat{S}_{j} \cdot\left(B\left(\hat{x}_{j}\right)+B_{\mathrm{ext}}\left(\hat{x}_{j}\right)\right)\right\} \\
& +1 \otimes H_{\mathrm{f}}+V\left(\hat{x}_{1}, \ldots, \hat{x}_{N}\right) \otimes 1 \tag{3.10}
\end{align*}
$$

where $\hat{x}_{j}$ denotes the operator of multiplication with $x_{j}$, the coordinates of the $j$-th particle, and $p_{j}=-i \nabla_{x_{j}}$. We assume that $V: \mathbb{R}^{3 N} \rightarrow \mathbb{R}$ is a function and that $B_{\text {ext }}: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ is a function. Furthermore, we defined

$$
\begin{equation*}
A_{\mathrm{ext}}(x):=-\int \frac{(x-y) \times B_{\mathrm{ext}}(y)}{4 \pi|x-y|^{3}} d y \tag{3.11}
\end{equation*}
$$

cf. Remark 3.2.
Remark 3.2. Provided $B_{\text {ext }}$ is sufficiently regular and has sufficient decay, we can write

$$
\begin{equation*}
A_{\mathrm{ext}}(x):=-\int \frac{(x-y) \times B_{\mathrm{ext}}(y)}{4 \pi|x-y|^{3}} d y=\nabla_{x} \times \int \frac{B_{\mathrm{ext}}(y)}{4 \pi|x-y|} d y=\int \frac{\nabla_{y} \times B_{\mathrm{ext}}(y)}{4 \pi|x-y|} d y \tag{3.12}
\end{equation*}
$$

by calculating the derivative and using integration by parts, respectively. In particular, if $\nabla \cdot B_{\text {ext }}=0$, it follows that $\nabla \times A_{\text {ext }}=B_{\text {ext }}$.

Physically, $V$ is called the external potential, $B_{\text {ext }}$ the external magnetic field, $A_{\text {ext }}$ the external magnetic vector potential. We assume that $B_{\text {ext }}$ is such that $A_{\text {ext }}$ in (3.11) is well defined for almost all $x \in \mathbb{R}^{3}$. Moreover, we assume that $\kappa$ and $\omega$ are such that the fields occurring in the Hamiltonian exist. Furthermore, we assume that $\kappa, \omega, V$, and $B_{\text {ext }}$ are such that the Hamiltonian is essentially self-adjoint on $\left(\otimes_{j=1}^{N} C_{c}^{\infty}\left(\mathbb{R}^{3} ; \mathcal{D}_{s_{j}}\right)\right) \otimes F_{0}(\mathfrak{g})$, for details we refer the reader to [23, Theorem X.35, Theorem X.34] and [11].

## 4 Symmetries

In this subsection we define symmetries associated to rotations, space inversion and time inversion. To define these symmetries on Fock space it is convenient to identity $\mathfrak{h}$ with the space of so called divergence free vector fields. In this section we shall denote by $F$ the Fourier transform and by $F^{-1}$ its inverse, i.e. for $f \in L^{1}\left(\mathbb{R}^{3}\right)$

$$
\begin{aligned}
(F f)(k) & =(2 \pi)^{-3 / 2} \int_{\mathbb{R}^{3}} e^{-i k \cdot x} f(x) d x \\
\left(F^{-1} f\right)(x) & =(2 \pi)^{-3 / 2} \int_{\mathbb{R}^{3}} e^{i k \cdot x} f(k) d k
\end{aligned}
$$

where both transformations are canonically extended to $L^{2}\left(\mathbb{R}^{3}\right)$ by Plancherel's theorem.

### 4.1 Space of divergent free vector fields

We introduce the space of divergence free vector fields

$$
\mathfrak{v}:=\left\{v \in L^{2}\left(\mathbb{R}^{3} ; \mathbb{C}^{3}\right): \sum_{j=1}^{3} k_{j} \widehat{v}_{j}(k)=0 \text {, a.e. } k \in \mathbb{R}^{3}\right\} .
$$

Given a specific measurable choice for the polarization vectors (3.8) we obtain a canonical identification with the one photon Hilbert space $\mathfrak{g}=L^{2}\left(\mathbb{R}^{3} \times \mathbb{Z}_{2}\right)$. This is the content of the following lemma.

Lemma 4.1. For the polarization vector $\varepsilon: S_{2} \times \mathbb{Z}_{2} \rightarrow \mathbb{R}^{3}$, as in (3.8), the map

$$
\tau_{\varepsilon}: \mathfrak{g} \rightarrow \mathfrak{v}, \quad h \mapsto\left(F^{-1} \sum_{\lambda=1,2} \varepsilon_{j}(\cdot, \lambda) h(\cdot, \lambda)\right)_{j=1,2,3}
$$

is unitary and its inverse acting on $v \in \mathfrak{v}$ is determined by $\left(\tau_{\varepsilon}^{-1} v\right)(k, \lambda)=\varepsilon(k, \lambda) \cdot(F v)(k)$ for almost all $(k, \lambda) \in \mathbb{R}^{3} \times\{1,2\}$.

For the proof we first note the following. For $k \in \mathbb{R}^{3} \backslash\{0\}$ define

$$
\begin{equation*}
P(k)_{a, b}:=\delta_{a b}-\frac{k_{a} k_{b}}{|k|^{2}}, \quad a, b=1,2,3, \quad k \neq 0 \tag{4.1}
\end{equation*}
$$

From the definition it follows that $P(k)_{a, b}=P(k)_{b, a}$, and that $P(k)$ is equal to the projection operator in $\mathbb{C}^{3}$ onto the subspace in $\mathbb{C}^{3}$, which is perpendicular to $k$. Thus from the definition of the polarization vectors, (3.8), we infer that for $k \in \mathbb{R}^{3} \backslash\{0\}$

$$
\begin{equation*}
P(k)_{a, b}=\sum_{\lambda=1,2} \varepsilon_{a}(k, \lambda) \varepsilon_{b}(k, \lambda) . \tag{4.2}
\end{equation*}
$$

Proof of Lemma 4.1. The lemma follows from a straight forward calculation using the properties of the polarization vectors. Let $h \in \mathfrak{g}$. Clearly, $\tau_{\varepsilon}$ is well defined, since $k \cdot F\left(\tau_{\varepsilon}(h)\right)(k)=k \cdot \sum_{\lambda=1,2} \varepsilon(k, \lambda) h(k, \lambda)=0$. The map is an isometry, since

$$
\begin{aligned}
\left\|\tau_{\varepsilon} h\right\|^{2} & =\int_{\mathbb{R}^{3}} \sum_{j=1}^{3} \sum_{\lambda, \lambda^{\prime}=1,2} \overline{\varepsilon_{j}(k, \lambda) h(k, \lambda)} \varepsilon_{j}\left(k, \lambda^{\prime}\right) h\left(k, \lambda^{\prime}\right) d^{3} k \\
& =\int_{\mathbb{R}^{3}} \sum_{\lambda, \lambda^{\prime}=1,2} \delta_{\lambda, \lambda^{\prime}} \overline{h(k, \lambda)} h\left(k, \lambda^{\prime}\right) d^{3} k=\|h\|^{2}
\end{aligned}
$$

Furthermore for $v \in \mathfrak{v}$ let $\left(\beta_{\epsilon} v\right)(k, \lambda)=\varepsilon(k, \lambda) \cdot(F v)(k)$. Then

$$
\begin{aligned}
F\left(\tau_{\varepsilon}\left(\beta_{\varepsilon} v\right)_{j}\right)(k) & =\sum_{\lambda=1,2} \varepsilon_{j}(k, \lambda)\left(\beta_{\varepsilon} v\right)(k, \lambda) \\
& =\sum_{\lambda=1,2} \varepsilon_{j}(k, \lambda) \sum_{l=1}^{3} \varepsilon_{l}(k, \lambda) \cdot\left(F v_{l}\right)(k) \\
& =F v_{j}(k)
\end{aligned}
$$

where we used that (4.2) and that $v$ is divergence free. This shows the surjectivity of $\tau_{\varepsilon}$ and that its inverse is given by $\beta_{\varepsilon}$.

Define for $x \in \mathbb{R}^{3}$ and $a=1,2,3$ the function $v_{x, a}: \mathbb{R}^{3} \rightarrow \mathbb{C}^{3}$ by

$$
\begin{equation*}
\left[v_{x, b}(y)\right]_{a}:=\frac{1}{(2 \pi)^{3 / 2}} \int_{\mathbb{R}^{3}} e^{-i k \cdot(x-y)} \frac{\overline{\kappa(k)}}{\sqrt{2 \omega(k)}} P(k)_{a, b} d k, \quad y \in \mathbb{R}^{3} . \tag{4.3}
\end{equation*}
$$

The properties collected in the following lemma are straight forward to verify using the definitions.

Lemma 4.2. We have the following properties for $x \in \mathbb{R}^{3}$ and $b=1,2,3$
(a) $v_{x, b} \in \mathfrak{v}$,
(b) $v_{x, b}=\tau_{\varepsilon} g_{x, b}^{(\varepsilon)}, \quad \tau_{\varepsilon}^{-1} v_{x, b}=g_{x, b}^{(\varepsilon)}$.

The next lemma will be needed to determine transformation properties of the field energy and field momentum with respect to rotations, parity transformations, and time reversal symmetry.

Lemma 4.3. Let $f: \mathbb{R}^{3} \rightarrow \mathbb{C}$ be a measurable function. Then we have the following properties.
(a) $\tau_{\varepsilon} M_{f} \tau_{\varepsilon}^{-1}=F^{-1} M_{f} F$.
(b) For $\varphi \in L^{2}\left(\mathbb{R}^{3}\right)$ and $S \in O(3)$ we define the transformation $T_{S} \varphi=\varphi \circ S^{-1}$. Then

$$
\begin{equation*}
T_{S} F=F T_{S}, \quad T_{S} F^{-1}=F^{-1} T_{S} . \tag{4.4}
\end{equation*}
$$

(c) Let $T_{S}$ be defined as in (b). Then $T_{S}^{-1}=T_{S^{-1}}$ and

$$
\begin{equation*}
T_{S} M_{f} T_{S}^{-1}=M_{f \circ S^{-1}} \tag{4.5}
\end{equation*}
$$

Proof. Part (a) follows from

$$
\begin{align*}
\left(\tau_{\varepsilon} M_{f} \tau_{\varepsilon}^{-1} v\right)_{j} & =F^{-1} \sum_{\lambda=1,2} \varepsilon_{j}(\cdot, \lambda) M_{f}\left(\tau_{\varepsilon}^{-1} v\right)(\cdot, \lambda) \\
& =F^{-1} \sum_{\lambda=1,2} \varepsilon_{j}(\cdot, \lambda) f(\cdot) \varepsilon(\cdot, \lambda) \cdot(F v)(\cdot) \\
& =F^{-1}\left(M_{f} F v_{j}\right) \tag{4.6}
\end{align*}
$$

(b) If $\varphi \in \mathcal{S}\left(\mathbb{R}^{3}\right)$, we find by the transformation formula for integrals for arbitrary $S \in O(3)$

$$
\begin{equation*}
\left(T_{S} F \varphi\right)(k)=(2 \pi)^{-3 / 2} \int_{\mathbb{R}^{3}} e^{-i\left(S^{-1} k\right) \cdot x} \varphi(x) d x=(2 \pi)^{-3 / 2} \int_{\mathbb{R}^{3}} e^{-i k \cdot x} \varphi\left(S^{-1} x\right) d x=\left(F T_{S} \varphi\right)(k) \tag{4.7}
\end{equation*}
$$

So (b) follows by density and continuity. Part (c) is straight forward to verify.
The following lemma will be needed to determine transformation properties of the interaction with respect to rotations and parity transformations.

Lemma 4.4. Let $S \in O(3)$. Then the following holds.
(a) For all $k \in \mathbb{R}^{3}$ we have $P(S k)=S P(k) S^{T}$.
(b) For all $x \in \mathbb{R}^{3}$ and $b=1,2,3$

$$
\begin{align*}
& \sum_{c^{\prime}=1}^{3} S_{c, c^{\prime}} \int_{\mathbb{R}^{3}} e^{-i k \cdot\left(x-S^{-1} y\right)} \frac{\overline{\kappa(k)}}{\sqrt{2 \omega(k)}} P_{b, c^{\prime}}(k) d k  \tag{4.8}\\
& \quad=\sum_{b^{\prime}=1}^{3} S_{b^{\prime}, b} \int_{\mathbb{R}^{3}} e^{-i k \cdot(S x-y)} \frac{\overline{\kappa\left(S^{-1} k\right)}}{\sqrt{2 \omega(k)}} P_{b^{\prime}, c}(k) d k \tag{4.9}
\end{align*}
$$

(c) If $\kappa(S \cdot)=\kappa(\cdot)$, then for all $x \in \mathbb{R}^{3}$ and $b=1,2,3$

$$
\begin{equation*}
S v_{x, b}\left(S^{-1} y\right)=\sum_{b^{\prime}=1}^{3} S_{b^{\prime}, b} v_{S x, b^{\prime}}(y) \tag{4.10}
\end{equation*}
$$

Proof. Part (a) is straight forward to verify using the definition (4.1). For $x \in \mathbb{C}^{3}$ and $k \in \mathbb{R}^{3} \backslash\{0\}$ we find for $\widehat{k}=k /|k|$

$$
P(S k) x=x-S \widehat{k}(S \widehat{k} \cdot x)=S S^{T} x-S \widehat{k}\left(\widehat{k} \cdot S^{T} x\right)=S P(k) S^{T} x
$$

(b) follows from a change of variables and (a)

$$
\begin{aligned}
(4.8) & =\sum_{c^{\prime}=1}^{3} \int e^{-i(S k) \cdot(S x-y)} \frac{\overline{\kappa(k)}}{\sqrt{2 \omega(k)}} P_{b, c^{\prime}}(k) S_{c, c^{\prime}} d k \\
& =\sum_{c^{\prime}=1}^{3} \int e^{-i k \cdot(S x-y)} \frac{\overline{\kappa\left(S^{-1} k\right)}}{\sqrt{2 \omega(k)}} P_{b, c^{\prime}}\left(S^{-1} k\right) S_{c, c^{\prime}} d k=(4.9) .
\end{aligned}
$$

(c) Now (4.10) follows from (4.9) and the definition (4.3).

### 4.2 Rotation Invariance

We introduce the so called canonical double covering homomorphism

$$
\pi: S U(2) \rightarrow S O(3), \quad U \mapsto \pi(U)
$$

where $\pi(U)$ is the unique element of $S O(3)$ such that

$$
U \sigma_{m} U^{*}=\sum_{l=1}^{3} \pi(U)_{l, m} \sigma_{l}, \quad m=1,2,3
$$

with $\sigma_{1}, \sigma_{2}, \sigma_{3}$ denoting the Pauli matrices. On the one electron Hilbert space $L^{2}\left(\mathbb{R}^{3} ; \mathcal{D}_{s}\right)$ we define

$$
\left(\mathcal{U}_{\mathrm{p}, s}(U) \psi\right)(x)=D_{s}(U) \psi\left(\pi(U)^{-1} x\right),
$$

where $D_{s}$ denotes the representation of $S U(2)$ with spin $s$. Similarly we define for $v \in \mathfrak{v}$ the transformation for $R \in S O(3)$

$$
\left(\mathcal{U}_{\mathfrak{v}}(R) v\right)(x)=R v\left(R^{-1} x\right) .
$$

Moreover, we define

$$
\mathcal{U}_{\mathfrak{g}}(R)=\tau_{\epsilon}^{-1} \mathcal{U}_{\mathfrak{v}}(R) \tau_{\epsilon},
$$

which depends on the choice of the polarization vectors. For $R \in S O(3)$ we define the unitary mapping

$$
\mathcal{U}_{\mathfrak{f}}(R)=\Gamma\left(\mathcal{U}_{\mathfrak{g}}(R)\right),
$$

and for $U \in S U(2)$ we define the unitary mappings

$$
\begin{aligned}
\mathcal{U}_{\mathrm{mat}}(U) & =\bigotimes_{j=1}^{N} \mathcal{U}_{\mathrm{p}, s_{j}}(U) \\
\mathcal{U}(U) & =\mathcal{U}_{\mathrm{mat}}(U) \otimes \mathcal{U}_{\mathrm{f}}(\pi(U))
\end{aligned}
$$

on the Hilbert spaces $\mathcal{H}_{\text {mat }}$ and $\mathcal{H}_{\text {mat }} \otimes \mathcal{F}_{s}(\mathfrak{g})$, respectively. This defines a representation of $S U(2)$ on these Hilbert spaces. The next proposition collects elementary properties, which follow directly from the definitions.

Proposition 4.5. The map $\mathcal{U}_{\mathrm{f}}$ is a unitary representation of $R \in S O(3)$, and the maps $\mathcal{U}_{\mathrm{mat}}$, and $\mathcal{U}$ are unitary representations of $\operatorname{SU}(2)$.

Remark 4.6. By abuse of notation we denote the unitary representation $\mathcal{U}_{\mathrm{f}} \circ \pi$ on $S U(2)$ also by $\mathcal{U}_{\mathrm{f}}$.

Lemma 4.7. Let $R \in S O(3)$ and $\kappa(R \cdot)=\kappa(\cdot)$. Then
(a) $\quad \mathcal{U}_{\mathrm{f}}(R) A(x) \mathcal{U}_{\mathrm{f}}^{*}(R)=R^{-1} A(R x)$,
(b) $\quad \mathcal{U}_{\mathrm{f}}(R) B(x) \mathcal{U}_{\mathrm{f}}^{*}(R)=R^{-1} B(R x)$,
(c) $\quad \mathcal{U}_{\mathrm{f}}(R) E^{\perp}(x) \mathcal{U}_{\mathrm{f}}^{*}(R)=R^{-1} E^{\perp}(R x)$.

Proof. We observe that for $R \in S O$ (3) we find

$$
\begin{equation*}
\left(\mathcal{U}_{\mathfrak{v}}(R) v_{x, b}\right)(y)=R v_{x, b}\left(R^{-1} y\right)=\sum_{b^{\prime}=1}^{3} R_{b^{\prime}, b} v_{R x, b^{\prime}}(y) \tag{4.11}
\end{equation*}
$$

where we used Lemma 4.4 (c). Using Eqs. (3.4) and (4.11) as well as Lemma 4.2 we obtain

$$
\begin{aligned}
\mathcal{U}_{\mathfrak{f}}(R) a^{\#}\left(g_{x, b}^{(\varepsilon)}\right) \mathcal{U}_{\mathrm{f}}^{*}(R) & =a^{\#}\left(\mathcal{U}_{\mathfrak{g}}(R) g_{x, b}^{(\varepsilon)}\right)=a^{\#}\left(\tau_{\varepsilon}^{-1} \mathcal{U}_{\mathfrak{v}}(R) \tau_{\varepsilon} g_{x, b}^{(\varepsilon)}\right) \\
& =a^{\#}\left(\tau_{\varepsilon}^{-1} \mathcal{U}_{\mathfrak{v}}(R) v_{x, b}\right)=\sum_{b^{\prime}=1}^{3} R_{b^{\prime}, b} a^{\#}\left(\tau_{\varepsilon}^{-1} v_{R x, b^{\prime}}\right) \\
& =\sum_{b^{\prime}=1}^{3} R_{b^{\prime}, b} a^{\#}\left(g_{R x, b^{\prime}}^{(\varepsilon)}\right)
\end{aligned}
$$

This implies

$$
\mathcal{U}_{\mathfrak{f}}(R) A_{b}(x) \mathcal{U}_{\mathrm{f}}^{*}(R)=\sum_{b^{\prime}=1}^{3} R_{b^{\prime}, b} A_{b^{\prime}}(R x)
$$

Thus (a) follows. Now (b) follows from (a) and by calculating the rotation. (c) Follows similarly as (a) observing that $\omega$ is invariant under rotations.

Proposition 4.8. Let $U \in S U(2)$ and $R=\pi(U)$. Then the following holds
(a) $\mathcal{U}(U) \hat{x}_{j} \mathcal{U}(U)^{*}=R^{-1} \hat{x}_{j}$,
(b) $\mathcal{U}(U) p_{j} \mathcal{U}(U)^{*}=R^{-1} p_{j}$,
(c) $\mathcal{U}(U) \widehat{S}_{j} \mathcal{U}(U)^{*}=R^{-1} \widehat{S}_{j}$,
(d) $\mathcal{U}(U) A\left(\hat{x}_{j}\right) \mathcal{U}(U)^{*}=R^{-1} A\left(\hat{x}_{j}\right), \quad$ if $\kappa(R \cdot)=\kappa(\cdot)$,
(e) $\quad \mathcal{U}(U) B\left(\hat{x}_{j}\right) \mathcal{U}(U)^{*}=R^{-1} B\left(\hat{x}_{j}\right), \quad$ if $\kappa(R \cdot)=\kappa(\cdot)$,
(f) $\quad \mathcal{U}(U) E^{\perp}\left(\hat{x}_{j}\right) \mathcal{U}(U)^{*}=R^{-1} E^{\perp}\left(\hat{x}_{j}\right), \quad$ if $\kappa(R \cdot)=\kappa(\cdot)$,
(g) $\mathcal{U}(U) H_{\mathrm{f}} \mathcal{U}(U)^{*}=H_{\mathrm{f}}$,
(h) $\mathcal{U}(U) P_{\mathrm{f}} \mathcal{U}(U)^{*}=R^{-1} P_{\mathrm{f}}$.

Proof. Parts (a), (b), and (c) are straight forward to verify. Parts (d)-(f) follow from (a) and Lemma 4.7. Next we show (g) and (h). Using Lemma 4.3 and the identity (3.5) we find for any measurable $f: \mathbb{R}^{3} \rightarrow \mathbb{R}$ and $U \in S U(2)$ with $R=\pi(U)$

$$
\begin{aligned}
\mathcal{U}(U) d \Gamma\left(M_{f}\right) \mathcal{U}(U)^{*} & =d \Gamma\left(\tau_{\varepsilon}^{-1} \mathcal{U}_{\mathfrak{v}}(\pi(U)) \tau_{\varepsilon} M_{f} \tau_{\varepsilon}^{-1} \mathcal{U}_{\mathfrak{v}}^{*}(\pi(U)) \tau_{\varepsilon}\right) \\
& =d \Gamma\left(\tau_{\varepsilon}^{-1} \mathcal{U}_{\mathfrak{v}}(R) F^{-1} M_{f} F \mathcal{U}_{\mathfrak{v}}^{*}(R) \tau_{\varepsilon}\right) \\
& =d \Gamma\left(\tau_{\varepsilon}^{-1} F^{-1} M_{f \circ R^{-1}} F \tau_{\varepsilon}\right) \\
& =d \Gamma\left(M_{f \circ R^{-1}}\right)
\end{aligned}
$$

Now choosing $f=\omega$ or $f: k \mapsto k_{j}$ Parts (g) and (h) follow.
In the following proposition we give a formula for the action of the rotation transformation in $\mathfrak{g}$.

Proposition 4.9. For $R \in S O(3)$ define

$$
\mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{u}}(R ; k):=\left(R^{-1} \varepsilon(k, \lambda)\right) \cdot \varepsilon\left(R^{-1} k, \lambda^{\prime}\right)
$$

Then for $R \in S O(3)$

$$
\begin{equation*}
\mathcal{D}_{\lambda, \lambda^{\prime}}^{u}\left(R^{-1} ; k\right)=\mathcal{D}_{\lambda^{\prime}, \lambda}^{\mathcal{U}}(R ; R k) \tag{4.12}
\end{equation*}
$$

and the following holds.
(a) For any $h \in \mathfrak{g}$

$$
\begin{equation*}
\left(\mathcal{U}_{\mathfrak{g}}(R) h\right)(k, \lambda)=\sum_{\lambda^{\prime}=1,2} \mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{U}}(R ; k) h\left(R^{-1} k, \lambda^{\prime}\right) . \tag{4.13}
\end{equation*}
$$

(b) In the sense of operator valued distributions for all $(k, \lambda) \in \mathbb{R}^{3} \times \mathbb{Z}_{2}$

$$
\mathcal{U}_{\mathrm{f}}(R) a^{\#}(k, \lambda) \mathcal{U}_{\mathrm{f}}(R)^{*}=\sum_{\lambda^{\prime}=1,2} \mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{U}}\left(R^{-1} ; k\right) a^{\#}\left(R k, \lambda^{\prime}\right)
$$

Proof. Equation (4.12) follows from a straight forward calculation using that the elements of $S O(3)$ preserve the inner product. Now we prove (a). Using the property (4.4) of the Fourier transform, we find

$$
\begin{aligned}
\left(\mathcal{U}_{\mathfrak{g}}(R) h\right)(k, \lambda) & =\varepsilon(k, \lambda) \cdot F\left(F^{-1} \sum_{\lambda^{\prime}=1,2} R \varepsilon\left(\cdot, \lambda^{\prime}\right) h\left(\cdot, \lambda^{\prime}\right)\right)\left(R^{-1} k\right) \\
& =\sum_{\lambda^{\prime}=1,2} \varepsilon(k, \lambda) \cdot \operatorname{Re}\left(R^{-1} k, \lambda^{\prime}\right) h\left(R^{-1} k, \lambda^{\prime}\right) .
\end{aligned}
$$

(b) We have by linearity and (a)

$$
\begin{aligned}
& \sum_{\lambda=1,2} \int_{\mathbb{R}^{3}} h(k, \lambda) \mathcal{U}_{\mathrm{f}}(R) a^{*}(k, \lambda) \mathcal{U}_{\mathrm{f}}^{*}(R) d k=\mathcal{U}_{\mathrm{f}}(R) a^{*}(h) \mathcal{U}_{\mathfrak{f}}^{*}(R)=a^{*}\left(\mathcal{U}_{\mathfrak{g}}(R) h\right) \\
& =\sum_{\lambda=1,2} \int_{\mathbb{R}^{3}}\left(\mathcal{U}_{\mathfrak{g}}(R) h\right)(k, \lambda) a^{*}(k, \lambda) d k \\
& =\sum_{\lambda, \lambda^{\prime}=1,2} \int_{\mathbb{R}^{3}} \mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{U}}(R ; k) h\left(R^{-1} k, \lambda^{\prime}\right) a^{*}(k, \lambda) d k \\
& =\sum_{\lambda, \lambda^{\prime}=1,2} \int_{\mathbb{R}^{3}} \mathcal{D}_{\lambda^{\prime}, \lambda}^{\mathcal{U}}(R ; R k) h(k, \lambda) a^{*}\left(R k, \lambda^{\prime}\right) d k .
\end{aligned}
$$

Since $h \in \mathfrak{g}$ is arbitrary the claim follows for $a^{*}(k, \lambda)$ in view of (4.12). Taking adjoints the claim then follows also for $a(k, \lambda)$.

### 4.3 Parity Symmetry

Parity is the operation $x \mapsto-x$. On the particle space we define

$$
\mathcal{P}_{\mathrm{p}, s}: L^{2}\left(\mathbb{R}^{3} ; \mathcal{D}_{s}\right) \rightarrow L^{2}\left(\mathbb{R}^{3} ; \mathcal{D}_{s}\right), \quad \psi \mapsto(x \mapsto \psi(-x))
$$

for $s=0,1 / 2$. On the photon space we define

$$
\mathcal{P}_{\mathfrak{v}}: \mathfrak{v} \rightarrow \mathfrak{v}, \quad v \mapsto(x \mapsto-v(-x)),
$$

and

$$
\mathcal{P}_{\mathfrak{g}}=\tau_{\varepsilon}^{-1} \mathcal{P}_{\mathfrak{v}} \tau_{\varepsilon}
$$

We define

$$
\begin{aligned}
\mathcal{P}_{\mathrm{mat}} & =\bigotimes_{j=1}^{N} \mathcal{P}_{\mathrm{p}, s_{j}}, \\
\mathcal{P}_{\mathrm{f}} & =\Gamma\left(\mathcal{P}_{\mathfrak{g}}\right), \\
\mathcal{P} & =\mathcal{P}_{\mathrm{mat}} \otimes \mathcal{P}_{\mathrm{f}} .
\end{aligned}
$$

Proposition 4.10. The maps $\mathcal{P}_{\mathrm{mat}}, \mathcal{P}_{\mathrm{f}}$ and $\mathcal{P}$ are unitary and commute with the representations $\mathcal{U}_{\text {mat }}, \mathcal{U}_{\mathrm{f}}$ and $\mathcal{U}$, respectively.

Proof. The unitarity property is straight forward to verify. The commutativity follows from the commutativity of $\mathcal{P}_{\mathrm{p}}$ with $\mathcal{U}_{\mathrm{p}}$ and $\mathcal{P}_{\mathfrak{v}}$ with $\mathcal{U}_{\mathfrak{v}}$, which are straight forward to verify.

Lemma 4.11. Suppose $\kappa(-\cdot)=\kappa(\cdot)$. Then
(a) $\quad \mathcal{P}_{\mathrm{f}} A(x) \mathcal{P}_{\mathrm{f}}^{*}=-A(-x)$,
(b) $\quad \mathcal{P}_{\mathrm{f}} B(x) \mathcal{P}_{\mathrm{f}}^{*}=B(-x)$,
(c) $\quad \mathcal{P}_{\mathrm{f}} E^{\perp}(x) \mathcal{P}_{\mathrm{f}}^{*}=-E(-x)$.

Proof. We observe that for $S=-\mathbb{I}_{3 \times 3}$ we find from (4.10)

$$
\begin{equation*}
\left(\mathcal{P}_{\mathfrak{v}} v_{x, b}\right)(y)=-v_{x, b}(-y)=-v_{-x, b}(y) \tag{4.14}
\end{equation*}
$$

Now we find similar as in the proof of Lemma 4.7 using Lemma 4.2 and (4.14)

$$
\begin{aligned}
\left.\mathcal{P}_{\mathfrak{f}} a^{\#}\left(g_{x, b}^{(\varepsilon)}\right)\right) \mathcal{P}_{\mathrm{f}}^{*} & =a^{\#}\left(\mathcal{P}_{\mathfrak{g}} g_{x, b}^{(\varepsilon)}\right)=a^{\#}\left(\tau_{\varepsilon}^{-1} \mathcal{P}_{\mathfrak{v}} \tau_{\varepsilon} g_{x, b}^{(\varepsilon)}\right) \\
& =a^{\#}\left(\tau_{\varepsilon}^{-1} \mathcal{P}_{\mathfrak{v}} v_{x, b}\right)=a^{\#}\left(-\tau_{\varepsilon}^{-1} v_{-x, b}\right) \\
& =-a^{\#}\left(g_{-x, b}^{(\varepsilon)} .\right.
\end{aligned}
$$

This implies

$$
\mathcal{P}_{\mathrm{f}} A_{b}(x) \mathcal{P}_{\mathrm{f}}^{*}=-A_{b}(-x) .
$$

Thus (a) follows. Now (b) follows from (a) and by calculating the rotation. (c) Follows similarly as in (a) observing that $\omega(-\cdot)=\omega$.

In view of the following proposition we see that $\mathcal{P}$ has the physical interpretation of parity inversion.

Proposition 4.12. $\mathcal{P}$ has satisfies the following properties.

$$
\begin{array}{ll}
\text { (a) } & \mathcal{P} \hat{x}_{j} \mathcal{P}=-\hat{x}_{j}, \\
\text { (b) } & \mathcal{P} p_{j} \mathcal{P}^{*}=-p_{j}, \\
\text { (c) } & \mathcal{P} \widehat{S}_{j} \mathcal{P}^{*}=\widehat{S}_{j}, \\
\text { (d) } & \mathcal{P} A\left(\hat{x}_{j}\right) \mathcal{P}^{*}=-A\left(\hat{x}_{j}\right), \quad \text { if } \kappa(-\cdot)=\kappa(\cdot), \\
\text { (e) } & \mathcal{P} B\left(\hat{x}_{j}\right) \mathcal{P}^{*}=B\left(\hat{x}_{j}\right), \quad \text { if } \kappa(-\cdot)=\kappa(\cdot), \\
\text { (f) } & \mathcal{P} E^{\perp}\left(\hat{x}_{j}\right) \mathcal{P}^{*}=-E^{\perp}\left(\hat{x}_{j}\right), \quad \text { if } \kappa(-\cdot)=\kappa(\cdot), \\
\text { (g) } & \mathcal{P} H_{\mathrm{f}} \mathcal{P}^{*}=H_{\mathrm{f}}, \\
\text { (h) } & \mathcal{P} P_{\mathrm{f}} \mathcal{P}^{*}=-P_{\mathrm{f}} .
\end{array}
$$

Proof. The proof is analogous to that of Proposition 4.8.
In the following proposition we give a formula for the action of the parity in $\mathfrak{g}$.
Proposition 4.13. The map $\mathcal{P}_{\mathfrak{g}}$ has the following properties. Define

$$
\mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{P}}(k):=-\varepsilon(k, \lambda) \cdot \varepsilon\left(-k, \lambda^{\prime}\right) .
$$

Then $\mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{P}}(k)=\mathcal{D}_{\lambda^{\prime}, \lambda}^{\mathcal{P}}(-k)$.
(a) For any $h \in \mathfrak{g}$ we have for almost all $(k, \lambda) \in \mathbb{R}^{3} \times\{1,2\}$

$$
\left(\mathcal{P}_{\mathfrak{g}} h\right)(k, \lambda)=\sum_{\lambda^{\prime}=1,2} \mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{P}}(k) h\left(-k, \lambda^{\prime}\right) .
$$

(b) We have in the sense of operator valued distributions for all $(k, \lambda) \in \mathbb{R}^{3} \times \mathbb{Z}_{2}$

$$
\mathcal{P}_{\mathrm{f}} a^{\#}(k, \lambda) \mathcal{P}_{\mathrm{f}}^{*}=\sum_{\lambda^{\prime}=1,2} \mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{P}}(k) a^{\#}\left(-k, \lambda^{\prime}\right)
$$

Proof. The first statement follows from the symmetry of the scalar product. (a) Using (4.4), we find

$$
\begin{aligned}
\left(\mathcal{P}_{\mathfrak{g}} h\right)(k, \lambda) & =\varepsilon(k, \lambda) \cdot F\left(F^{-1} \sum_{\lambda^{\prime}=1,2}\left(-\varepsilon\left(\cdot, \lambda^{\prime}\right)\right) h\left(\cdot, \lambda^{\prime}\right)\right)(-k) \\
& =\sum_{\lambda^{\prime}=1,2}(-\varepsilon(k, \lambda)) \cdot \varepsilon\left(-k, \lambda^{\prime}\right) h\left(-k, \lambda^{\prime}\right)
\end{aligned}
$$

(b) We have by linearity and (a)

$$
\begin{aligned}
& \sum_{\lambda=1,2} \int h(k, \lambda) \mathcal{P}_{\mathrm{f}} a^{*}(k, \lambda) \mathcal{P}_{\mathrm{f}}^{*} d k=\mathcal{P}_{\mathrm{f}} a^{*}(h) \mathcal{P}_{\mathrm{f}}^{*}=a^{*}\left(\mathcal{P}_{\mathfrak{g}} h\right) \\
& =\sum_{\lambda=1,2} \int_{\mathbb{R}^{3}}\left(\mathcal{P}_{\mathfrak{g}} h\right)(\lambda, k) a^{*}(k, \lambda) d k \\
& =\sum_{\lambda, \lambda^{\prime}=1,2} \int_{\mathbb{R}^{3}} \mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{P}}(k) a^{*}(k, \lambda) h\left(-k, \lambda^{\prime}\right) d k \\
& =\sum_{\lambda, \lambda^{\prime}=1,2} \int_{\mathbb{R}^{3}} \mathcal{D}_{\lambda^{\prime}, \lambda}^{\mathcal{P}}(-k) a^{*}\left(-k, \lambda^{\prime}\right) h\left(k, \lambda^{\prime}\right) d k
\end{aligned}
$$

Since $h \in \mathfrak{g}$ is arbitrary the claim follows for $a^{*}(k, \lambda)$. Taking adjoints the claim then follows also for $a(k, \lambda)$.

### 4.4 Time reversal symmetry

We define time reversal symmetry. Let $K$ denote complex conjugation on $L^{2}\left(\mathbb{R}^{3} ; \mathcal{D}_{s}\right)$. Define the operators

$$
\mathcal{T}_{\mathrm{p}, s}:=\left\{\begin{array}{lll}
K & , & \text { if } \\
\left(K \sigma_{2}\right), & \text { if } & s=1 / 2
\end{array}\right.
$$

and

$$
\mathcal{T}_{\mathrm{mat}}:=\bigotimes_{j=1}^{N} \mathcal{T}_{\mathrm{p}, s_{j}}
$$

Let $\mathcal{K}_{\mathfrak{v}}$ denote complex conjugation in $\mathfrak{v}$, and let

$$
\begin{equation*}
\mathcal{K}_{\mathfrak{g}}=\tau_{\varepsilon}^{-1} \mathcal{K}_{\mathfrak{v}} \tau_{\varepsilon} \tag{4.15}
\end{equation*}
$$

denote its action on $\mathfrak{g}$. Next we define operator of time reversal on the quantum field

$$
\begin{equation*}
\mathcal{T}_{\mathrm{f}}:=\Gamma\left(-\mathcal{K}_{\mathfrak{g}}\right) . \tag{4.16}
\end{equation*}
$$

We define the operator of time reversal in the full Hilbert space by

$$
\begin{equation*}
\mathcal{T}=\mathcal{T}_{\mathrm{mat}} \otimes \mathcal{T}_{\mathrm{f}} \tag{4.17}
\end{equation*}
$$

Proposition 4.14. The maps $\mathcal{T}_{\text {mat }}, \mathcal{T}_{\mathrm{f}}$, and $\mathcal{T}$ are anti-unitary operators, which commute with the representations $\mathcal{U}_{\text {mat }}, \mathcal{U}_{\mathrm{f}}$, and $\mathcal{U}$ and the operators $\mathcal{P}_{\text {mat }}$, $\mathcal{P}_{\mathrm{f}}$, and $\mathcal{P}$, respectively. We have $\mathcal{T}_{\mathrm{f}}^{2}=1$, and

$$
\mathcal{T}_{\text {mat }}^{2}=(-1)^{\sum_{j=1}^{N} 2 s_{j}}, \quad \mathcal{T}^{2}=(-1)^{\sum_{j=1}^{N} 2 s_{j}}
$$

Proof. The anti-unitarity is straight forward to verify on the one particle spaces. On the tensor product it then follows by Lemma 2.2. The commutativity can be seen by verifying it on the one particle spaces. The last statement follows from

$$
\mathcal{T}_{\mathrm{mat}}^{2}=\bigotimes_{j=1}^{N}\left(\mathcal{T}_{\mathrm{mat}, s_{j}}\right)^{2}
$$

with $\left(\mathcal{T}_{\text {mat }, 0}\right)^{2}=1$ and $\left(\mathcal{T}_{\text {mat }, 1 / 2}\right)^{2}=\left(K \sigma_{2}\right)\left(K \sigma_{2}\right)=K^{2} \sigma_{2}\left(-\sigma_{2}\right)=-1$,
Lemma 4.15. Suppose $\overline{\kappa(\cdot)}=\kappa(-\cdot)$. Then the following holds
(d) $\quad \mathcal{T}_{\mathrm{f}} A(x) \mathcal{T}_{\mathrm{f}}{ }^{*}=-A(x)$,
(e) $\quad \mathcal{T}_{\mathrm{f}} B(x) \mathcal{T}_{\mathrm{f}}^{*}=-B(x)$,
(e) $\quad \mathcal{T}_{\mathrm{f}} E^{\perp}(x) \mathcal{T}_{\mathrm{f}}{ }^{*}=E^{\perp}(x)$.

Proof. It follows directly from the definition, a trivial change of variables, and the assumption about $\kappa$ that

$$
\begin{equation*}
\left(\mathcal{K}_{\mathfrak{v}} v_{x, b}\right)(y)=v_{x, b}(y) . \tag{4.18}
\end{equation*}
$$

Now we find using Lemma 4.2

$$
\begin{aligned}
\Gamma\left(-\mathcal{K}_{\mathfrak{g}}\right) a^{*}\left(g_{x, b}^{(\varepsilon)}\right) \Gamma\left(-\mathcal{K}_{\mathfrak{g}}\right)^{*} & =a^{*}\left(-\mathcal{K}_{\mathfrak{g}} g_{x, b}^{(\varepsilon)}\right)=-a^{*}\left(\tau_{\varepsilon}^{-1} \mathcal{K}_{\mathfrak{v}} \tau_{\varepsilon} g_{x, b}^{(\varepsilon)}\right) \\
& =-a^{*}\left(\tau_{\varepsilon}^{-1} \mathcal{K}_{\mathfrak{v}} v_{x, b}\right)=-a^{*}\left(\tau_{\varepsilon}^{-1} v_{x, b}\right) \\
& =-a^{*}\left(g_{x, b}^{(\varepsilon)}\right)
\end{aligned}
$$

This implies $\mathcal{T}_{\mathrm{f}} a^{*}\left(g_{x, b}^{(\varepsilon)}\right) \mathcal{T}_{\mathrm{f}}^{*}=-a^{*}\left(g_{x, b}^{(\varepsilon)}\right)$ and by taking adjoints $\mathcal{T}_{\mathrm{f}} a\left(g_{x, b}^{(\varepsilon)}\right) \mathcal{T}_{\mathrm{f}}{ }^{*}=-a\left(g_{x, b}^{(\varepsilon)}\right)$. Hence

$$
\mathcal{T}_{\mathrm{f}} A_{b}(x) \mathcal{T}_{\mathrm{f}}^{*}=-A_{b}(x) .
$$

This shows (a). Now (b) follows from (a) and by calculating the rotation. (c) Follows similarly as in (a) observing that $i \omega$ changes sign when complex conjugating.

In view of the following proposition we see that $\mathcal{T}$ has the physical interpretation of time reversal.

Proposition 4.16. Suppose $\overline{\kappa(\cdot)}=\kappa(-\cdot)$. Then $\mathcal{T}$ is anti-unitary and satisfies the following properties
(a) $\mathcal{T} \hat{x}_{j} \mathcal{T}^{*}=\hat{x}_{j}$,
(b) $\mathcal{T} p_{j} \mathcal{T}^{*}=-p_{j}$,
(c) $\mathcal{T} \widehat{S}_{j} \mathcal{T}^{*}=-\widehat{S}_{j}$,
(d) $\mathcal{T} A\left(\hat{x}_{j}\right) \mathcal{T}^{*}=-A\left(\hat{x}_{j}\right), \quad$ if $\overline{\kappa(-\cdot)}=\kappa(\cdot)$,
(e) $\mathcal{T} B\left(\hat{x}_{j}\right) \mathcal{T}^{*}=-B\left(\hat{x}_{j}\right), \quad$ if $\overline{\kappa(-\cdot)}=\kappa(\cdot)$,
(f) $\quad \mathcal{T} E^{\perp}\left(\hat{x}_{j}\right) \mathcal{T}^{*}=E^{\perp}\left(\hat{x}_{j}\right), \quad$ if $\overline{\kappa(-\cdot)}=\kappa(\cdot)$,
(g) $\mathcal{T} H_{\mathrm{f}} \mathcal{T}^{*}=H_{\mathrm{f}}$,
(h) $\mathcal{T} P_{\mathrm{f}} \mathcal{T}^{*}=-P_{\mathrm{f}}$.

Proof. Parts (a), (b), and (c) are straight forward to verify. Parts (d), (e), and (f) follow from Lemma 4.15. Using Lemma 4.3 we find for any measurable $f: \mathbb{R}^{3} \rightarrow \mathbb{R}$

$$
\begin{aligned}
\mathcal{T}_{\mathrm{f}} d \Gamma\left(M_{f}\right) \mathcal{T}_{\mathrm{f}}^{*} & =d \Gamma\left(\tau_{\varepsilon}^{-1} \mathcal{K}_{\mathfrak{v}} \tau_{\varepsilon} M_{f} \tau_{\varepsilon}^{-1} \mathcal{K}_{\mathfrak{v}}^{*} \tau_{\varepsilon}\right) \\
& =d \Gamma\left(\tau_{\varepsilon}^{-1} \mathcal{K}_{\mathfrak{v}} F^{-1} M_{f} F \mathcal{K}_{\mathfrak{v}}^{*} \tau_{\varepsilon}\right) \\
& =d \Gamma\left(\tau_{\varepsilon}^{-1} F^{-1} M_{f(-)} F \tau_{\varepsilon}\right) \\
& =d \Gamma\left(M_{f(-\cdot)}\right),
\end{aligned}
$$

where in the third equality we used that the Fourier transform satisfies the following properties $F \bar{\varphi}=\overline{F \varphi}(-\cdot)$ and $\overline{F^{-1} \varphi}=F^{-1} \bar{\varphi}(-\cdot)$ for $\varphi \in L^{2}\left(\mathbb{R}^{3}\right)$. Now choosing $f=\omega$ or $f: k \mapsto k_{j}$ Parts (g) and (h) follow.

In the following proposition we give a formula for the action of the time reversal symmetry in $\mathfrak{g}$.

Proposition 4.17. For $h \in \mathfrak{g}$ we have for almost all $(k, \lambda) \in \mathbb{R}^{3} \times\{1,2\}$

$$
\left(\mathcal{K}_{\mathfrak{g}} h\right)(k, \lambda)=\sum_{\lambda^{\prime}=1,2} \mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{T}}(k) \overline{h\left(-k, \lambda^{\prime}\right)},
$$

where $\mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{T}}(k):=\varepsilon(k, \lambda) \cdot \varepsilon\left(-k, \lambda^{\prime}\right)$. Then $\mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{T}}(k)=\mathcal{D}_{\lambda^{\prime}, \lambda}^{\mathcal{T}}(-k)$ and in the sense of operator valued distributions for all $(k, \lambda) \in \mathbb{R}^{3} \times \mathbb{Z}_{2}$

$$
\mathcal{T}_{\mathrm{f}} a^{\#}(k, \lambda) \mathcal{T}_{\mathrm{f}}^{*}=-\sum_{\lambda^{\prime}=1,2} \mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{T}}(k) a^{\#}\left(-k, \lambda^{\prime}\right) .
$$

Proof. Using for $\varphi \in L^{2}\left(\mathbb{R}^{3}\right)$ the following property of the Fourier transform $\overline{F\left(F^{-1} \varphi\right)}(k)=$ $\bar{\varphi}(-k)$, we find

$$
\begin{aligned}
\left(\mathcal{K}_{\mathfrak{g}} h\right)(k, \lambda) & =\epsilon(k, \lambda) \cdot F \overline{\left(F^{-1} \sum_{\lambda^{\prime}=1,2} \varepsilon\left(\cdot, \lambda^{\prime}\right) h\left(\cdot, \lambda^{\prime}\right)\right)}(k) \\
& =\sum_{\lambda^{\prime}=1,2} \varepsilon(k, \lambda) \cdot \varepsilon\left(-k, \lambda^{\prime}\right) \overline{h\left(-k, \lambda^{\prime}\right)} .
\end{aligned}
$$

This shows the first identity. Using this, we find by anti-linearity

$$
\begin{aligned}
& \sum_{\lambda=1,2} \int \overline{h(k, \lambda)} \mathcal{T}_{\mathrm{f}} a^{*}(k, \lambda) \mathcal{T}_{\mathrm{f}}^{*} d k=\mathcal{T}_{\mathrm{f}} a^{*}(h) \mathcal{T}_{\mathrm{f}}^{*}=a^{*}\left(-\mathcal{K}_{\mathfrak{g}} h\right) \\
& =-\sum_{\lambda=1,2} \int_{\mathbb{R}^{3}}\left(\mathcal{K}_{\mathfrak{g}} h\right)(\lambda, k) a^{*}(k, \lambda) d k \\
& =-\sum_{\lambda, \lambda^{\prime}=1,2} \int_{\mathbb{R}^{3}} \overline{h\left(-k, \lambda^{\prime}\right)} \mathcal{D}_{\lambda, \lambda^{\prime}}^{\mathcal{T}}(k) a^{*}(k, \lambda) d k \\
& =-\sum_{\lambda, \lambda^{\prime}=1,2} \int_{\mathbb{R}^{3}} \overline{h(k, \lambda)} \mathcal{D}_{\lambda^{\prime}, \lambda}^{\mathcal{T}}(-k) a^{*}\left(-k, \lambda^{\prime}\right) d k
\end{aligned}
$$

Since $h \in \mathfrak{g}$ is arbitrary the second identity follows for $a^{*}(k, \lambda)$. Taking adjoints the claim then follows also for $a(k, \lambda)$.

## 5 Hamiltonians with Symmetries

In this section we consider Hamiltonians of non-relativistic qed, and discuss their symmetry properties.
Theorem 5.1. Suppose $U \in S U(2), R=\pi(U), V\left(x_{1}, \ldots, x_{N}\right)=V\left(R x_{1}, \ldots, R x_{N}\right)$ for all $x_{1}, \ldots, x_{N} \in \mathbb{R}^{3}$, $B_{\text {ext }}(x)=R B_{\text {ext }}\left(R^{-1} x\right)$ for all $x \in \mathbb{R}^{3}$, and $\kappa(R \cdot)=\kappa(\cdot)$. Then

$$
\mathcal{U}(U) H \mathcal{U}(U)^{*}=H
$$

Proof. Using (3.11), properties of the cross product, a change of variables, and the symmetry properties of $B_{\text {ext }}$ we find

$$
R A_{\mathrm{ext}}\left(R^{-1} x\right)=-\int \frac{(x-R y) \times R B_{\mathrm{ext}}(y)}{4 \pi|x-R y|^{3}} d y=A_{\mathrm{ext}}(x)
$$

Thus using Proposition 4.8

$$
\begin{aligned}
& \mathcal{U}(U) H \mathcal{U}(U)^{*} \\
& =\sum_{j=1}^{N}\left\{\frac{1}{2 m_{j}}\left(R^{-1} p_{j}+q_{j}\left(R^{-1} A\left(\hat{x}_{j}\right)+A_{\mathrm{ext}}\left(R^{-1} \hat{x}_{j}\right)\right)\right)^{2}+\mu_{j} R^{-1} \widehat{S}_{j} \cdot\left(R^{-1} B\left(\hat{x}_{j}\right)+B_{\mathrm{ext}}\left(R^{-1} \hat{x}_{j}\right)\right)\right\} \\
& \quad+H_{\mathrm{f}}+V\left(R^{-1} \hat{x}_{1}, \ldots, R^{-1} \hat{x}_{N}\right) \\
& =H
\end{aligned}
$$

where in the last line we used the assumed properties of $B_{\text {ext }}$ and $V$.
Theorem 5.2. Suppose $V\left(x_{1}, \ldots, x_{N}\right)=V\left(-x_{1}, \ldots,-x_{N}\right)$ for all $x_{1}, \ldots, x_{N} \in \mathbb{R}^{3}$, $B_{\text {ext }}(\cdot)=$ $B_{\text {ext }}(-\cdot)$, and $\kappa(-\cdot)=\kappa(\cdot)$. Then

$$
\mathcal{P} H \mathcal{P}^{*}=H .
$$

Proof. Using (3.11), the properties of the cross product, a change of variables, and the symmetry properties of $B_{\text {ext }}$ we find

$$
A_{\mathrm{ext}}(-x)=-\int \frac{(-(x-y)) \times B_{\mathrm{ext}}(-y)}{4 \pi|x-y|^{3}} d y=-A_{\mathrm{ext}}(x) .
$$

Thus we find from Proposition 4.12

$$
\begin{aligned}
& \mathcal{P} H \mathcal{P}^{*} \\
& \left.=\sum_{j=1}^{N}\left\{\frac{1}{2 m_{j}}\left(-p_{j}-q_{j} A\left(\hat{x}_{j}\right)+q_{j} A_{\mathrm{ext}}\left(-\hat{x}_{j}\right)\right)\right)^{2}+\mu_{j} \widehat{S}_{j} \cdot\left(B\left(\hat{x}_{j}\right)+B_{\mathrm{ext}}\left(-\hat{x}_{j}\right)\right)\right\} \\
& +H_{\mathrm{f}}+V\left(-\hat{x}_{1}, \ldots,-\hat{x}_{N}\right) \\
& =H
\end{aligned}
$$

where in the last line we used the assumed properties of $B_{\text {ext }}$ and $V$.
Theorem 5.3. Suppose $B_{\mathrm{ext}}=0$ and $\overline{\kappa(\cdot)}=\kappa(-\cdot)$. Then

$$
\mathcal{T} H \mathcal{T}^{*}=H
$$

Proof. We find from Proposition 4.16

$$
\mathcal{T} H \mathcal{T}^{*}=\sum_{j=1}^{N}\left\{\frac{1}{2 m_{j}}\left(-p_{j}-q_{j} A\left(\hat{x}_{j}\right)\right)^{2}+\mu_{j} \widehat{S}_{j} \cdot B\left(\hat{x}_{j}\right)\right\}+H_{\mathrm{f}}+V\left(\hat{x}_{1}, \ldots, \hat{x}_{N}\right)=H
$$

Theorem 5.4. If $V\left(x_{1}, \ldots, x_{N}\right)=V\left(-x_{1}, \ldots,-x_{N}\right)$ for all $x_{1}, \ldots, x_{N} \in \mathbb{R}^{3}$, $B_{\text {ext }}(\cdot)=$ $-B_{\text {ext }}(-\cdot)$, and $\overline{\kappa(\cdot)}=\kappa(-\cdot)=\kappa(\cdot)$. Then

$$
\mathcal{T} \mathcal{P} H(\mathcal{T P})^{*}=H
$$

Proof. Using (3.11), the properties of the cross product, a change of variables, and the symmetry properties of $B_{\text {ext }}$ we find

$$
A_{\mathrm{ext}}(-x)=-\int \frac{(-(x-y)) \times B_{\mathrm{ext}}(-y)}{4 \pi|x-y|^{3}} d y=A_{\mathrm{ext}}(x) .
$$

Thus we find from Propositions 4.12 and 4.16

$$
\begin{aligned}
& \mathcal{T P} \mathcal{P} \mathcal{P}^{*} \mathcal{T}^{*} \\
& =\mathcal{T}\left(\sum_{j=1}^{N}\left\{\frac{1}{2 m_{j}}\left(-p_{j}-q_{j} A\left(\hat{x}_{j}\right)+q_{j} A_{\mathrm{ext}}\left(-\hat{x}_{j}\right)\right)^{2}+\mu_{j} \widehat{S}_{j} \cdot\left(B\left(\hat{x}_{j}\right)+B_{\mathrm{ext}}\left(-\hat{x}_{j}\right)\right)\right\}\right. \\
& \left.+H_{\mathrm{f}}+V\left(-\hat{x}_{1}, \ldots,-\hat{x}_{N}\right)\right) \mathcal{T}^{*} \\
& \left.=\sum_{j=1}^{N}\left\{\frac{1}{2 m_{j}}\left(p_{j}+q_{j} A\left(\hat{x}_{j}\right)+q_{j} A_{\mathrm{ext}}\left(\hat{x}_{j}\right)\right)\right)^{2}+\mu_{j} \widehat{S}_{j}\left(B\left(\hat{x}_{j}\right)-B_{\mathrm{ext}}\left(-\hat{x}_{j}\right)\right)\right\} \\
& +H_{\mathrm{f}}+V\left(-\hat{x}_{1}, \ldots,-\hat{x}_{N}\right) \\
& =H
\end{aligned}
$$

where we used the assumed properties of $B_{\text {ext }}$ and $V$.
As an application of the abstract Kramer theorem, we now show the following degeneracy result.
Theorem 5.5. Suppose $\sum_{j=1}^{N} 2 s_{j}$ is odd, and that at least one of the following two assumptions hold.
(i) $B_{\text {ext }}=0$ and $\overline{\kappa(\cdot)}=\kappa(-\cdot)$
(ii) $V\left(-x_{1}, \ldots,-x_{N}\right)=V\left(x_{1}, \ldots, x_{N}\right)$ and $B_{\text {ext }}(-x)=-B_{\text {ext }}(x)$, and $\overline{\kappa(\cdot)}=\kappa(-\cdot)=\kappa(\cdot)$.

Then, any eigenvalue of $H$ is at least two fold degenerate. If the multiplicity of an eigenvalue is finite, it is even.

Proof. In case (i) the assertion follows from Kramers degeneracy theorem 2.4 for $\theta=$ $\mathcal{T}$, Proposition 4.14, and Theorem 5.3. In case (ii) the assertion follows from Kramers degeneracy theorem 2.4 for $\theta=\mathcal{T} \mathcal{P}$, Proposition 4.14, and Theorem 5.4.

## Remark 5.6.

(a) We note that Theorem 5.5 for the case $N=1, s_{1}=1 / 2$, and (i) with the additional assumption $V(-x)=V(x)$ was shown in $[20,21]$. Thus Theorem 5.5 relaxes the unnecessary parity-symmetry assumption for the external potential $V$. In fact, the proof given in [20] uses the symmetry $\mathcal{P} \mathcal{T}$, while the proof in [21] uses the symmetry $\mathcal{T}$ in the so called Schrödinger representation, cf. Section 7 of this paper.
(b) Since the classical Kramer theorem uses time inversion symmetry it cannot be applied to situations with external magnetic fields. However if one considers the anti-linear symmetry $\mathcal{P} \mathcal{T}$ one can include external magnetic fields, which satisfy a symmetry condition. We note that the result (ii) also holds for an ordinary Schrödinger operator without any quantized electromagnetic field, as the proof also applies to such a situation with a straight forward (trivial) modification of the proof.

Next we consider the restriction to symmetric subspaces. To this end we introduce notation satisfying the following hypothesis.

Hypothesis A. The set $\mathfrak{P}=\left\{p_{1}, \ldots, p_{L}\right\}, L \in \mathbb{N} \cap\{1, \ldots, N\}$, is a partition of $\{1, \ldots, N\}$ such that on each element $p \in \mathfrak{P}$ of the partition the numbers $m_{j}, s_{j}, q_{j}$, and $\mu_{j}$ are equal (cf. (3.10)). The function $\tau$ maps $\mathfrak{P}$ to $\{0,1\}$. The potential $V$ is symmetric with respect to interchange of particle coordinates of particles which belong to the same element $p \in \mathfrak{P}$.

Remark 5.7. The function $\tau$ in Hypothesis A is used to specify the statistics of identical particles. The value 0 will be used to describe bosons while the value 1 will be used describe fermions. By physical laws, spin zero particles are bosons while spin $1 / 2$ particles are fermions.

For a finite set $S$ we shall denote by $\mathfrak{S}_{S}$ the set of all permutations of the set $S$. For a subset $S \subset\{1, \ldots, N\}$ and $\sigma \in \mathfrak{S}_{S}$ we denote by $\underline{\sigma}$ its extension to $\{1, \ldots, N\}$ by setting it equal to the identity on $\{1, \ldots, N\} \backslash S$. Suppose the partition $\mathfrak{P}$ satisfies Hypothesis A. Then for any $p \in \mathfrak{P}$ and $\sigma \in \mathfrak{S}_{p}$ it follows that $\mathfrak{U}(\underline{\sigma})$, defined in (3.1), leaves $\mathcal{H}_{\text {mat }}$ invariant, and we can define the subspace

$$
\begin{equation*}
\mathcal{H}_{\mathrm{mat}, \mathfrak{P}, \tau}=\left\{\psi \in \mathcal{H}_{\mathrm{mat}}: \forall p \in \mathfrak{P}, \forall \sigma \in \mathfrak{S}_{p}, \mathfrak{U}(\underline{\sigma}) \psi=\operatorname{sgn}(\sigma)^{\tau(p)} \psi\right\} \tag{5.1}
\end{equation*}
$$

where $\operatorname{sgn}(\sigma)$ defines the signum of the permutation $\sigma$. Furthermore, it follows from the definitions that $\mathfrak{U}(\underline{\sigma})$ commutes with the symmetries $\mathcal{U}_{\text {mat }}, \mathcal{P}_{\text {mat }}, \mathcal{T}_{\text {mat }}$ as well as the Hamiltonian $H$. In particular, $\mathcal{H}_{\text {mat, } \mathfrak{\beta}, \tau} \otimes \mathcal{F}_{s}(\mathfrak{g})$ is an invariant subspace of $H$.

Theorem 5.8. Suppose that the partition $\mathfrak{P}$, the function $\tau$ and the potential $V$, satisfy Hypothesis A. Suppose $\sum_{j=1}^{N} 2 s_{j}$ is odd, and (i) or (ii) of Theorem 5.5 holds. Then, any eigenvalue of $\left.H\right|_{\mathcal{H}_{\text {mat }, \mathfrak{\beta}, \tau} \otimes \mathcal{F}_{s}(\mathfrak{g})}$ has even or infinite multiplicity.

Proof. Follows from the same proof as Theorem 5.5, by observing in addition that $\mathcal{T}$ and $\mathcal{P}$ commute with $\mathfrak{U}(\underline{\sigma})$ for any $\sigma \in \mathfrak{S}_{p}$ and $p \in \mathfrak{P}$, and thus leave $\mathcal{H}_{\text {mat }, \mathfrak{P}, \tau}$ invariant.

Remark 5.9. We note that Theorem 5.8 for the special case $\mathfrak{P}=\{p\}$ with $p=\{1, \ldots, N\}$, $s_{j}=1 / 2$ for all $j \in p$, and $\tau(p)=1$, and with the additional assumption that $V$ is given by the Coulomb potential of $N$ electrons in the presence of the electric field of a nucleus was shown in [20].

## 6 Translationally invariant Hamiltonians

We write the Hamiltonian (3.10) acting in the Hilbert space $\mathcal{H}_{\text {mat }} \otimes \mathcal{F}_{s}(\mathfrak{g})$ in the following notation

$$
H=\sum_{j=1}^{N} T_{j}+H_{\mathrm{f}}+V\left(\widehat{x}_{1}, \ldots, \widehat{x}_{N}\right), \quad T_{j}:=\frac{1}{2 m_{j}}\left(p_{j}+q_{j} A\left(\widehat{x}_{j}\right)\right)^{2}+\mu_{j} \widehat{S}_{j} \cdot B\left(\widehat{x}_{j}\right),
$$

and we assume that there is no external magnetic field. Furthermore, we assume that the potential $V$ in the definition of the Hamiltonian (3.10) is translationally invariant, i.e., that for all $a \in \mathbb{R}^{3}$

$$
\begin{equation*}
V\left(x_{1}+a, \ldots, x_{N}+a\right)=V\left(x_{1}, \ldots, x_{N}\right) . \tag{6.1}
\end{equation*}
$$

Using the unitary transformation

$$
U=\exp \left(i x_{N} \cdot\left(P_{\mathrm{f}}+\sum_{j=1}^{N-1} p_{j}\right)\right)
$$

and a Fourier transform in the variable $x_{N}$ we can write

$$
H=\int_{\mathbb{R}^{3}}^{\oplus} H(\xi) d \xi,
$$

where
$H(\xi):=\frac{1}{2 m_{N}}\left(\xi-\sum_{j=1}^{N-1} p_{j}-P_{\mathrm{f}}+q_{N} A(0)\right)^{2}+\mu_{N} \widehat{S}_{N} \cdot B(0)+\sum_{j=1}^{N-1} T_{j}+H_{\mathrm{f}}+V\left(\widehat{x}_{1}, \ldots, \widehat{x}_{N-1}, 0\right)$
acts in

$$
\begin{equation*}
\mathcal{H}_{\mathrm{mat}}^{\prime} \otimes \mathcal{D}_{s_{N}} \otimes \mathcal{F}_{s}(\mathfrak{g}) \tag{6.2}
\end{equation*}
$$

where

$$
\mathcal{H}_{\mathrm{mat}}^{\prime}:=\bigotimes_{j=1}^{N-1} L^{2}\left(\mathbb{R}^{3} ; \mathcal{D}_{s_{j}}\right),
$$

cf. $[10,19]$. We define $\mathcal{U}_{\text {mat }}^{\prime}, \mathcal{P}_{\text {mat }}^{\prime}$, and $\mathcal{T}_{\text {mat }}^{\prime}$ on $\mathcal{H}_{\text {mat }}^{\prime}$ as in Section 4. On (6.2) we define the symmetries

$$
\begin{aligned}
\mathcal{U}^{\prime}(U) & :=\mathcal{U}_{\text {mat }}^{\prime}(U) \otimes D_{s_{N}}(U) \otimes \mathcal{U}_{\mathrm{f}}(\pi(U)), \quad U \in S U(2) \\
\mathcal{P}^{\prime} & :=\mathcal{P}_{\text {mat }}^{\prime} \otimes \mathbb{I}_{\mathcal{D}_{s_{N}}} \otimes \mathcal{P}_{\mathrm{f}} \\
\mathcal{T}^{\prime} & :=\mathcal{T}_{\text {mat }}^{\prime} \otimes \mathcal{T}_{\mathrm{p}, s}^{\prime} \otimes \mathcal{T}_{\mathrm{f}},
\end{aligned}
$$

where we defined

$$
\mathcal{T}_{\mathrm{p}, s}^{\prime}:=\left\{\begin{array}{lll}
K_{s}, & \text { if } \quad s=0, \\
\left(K_{s} \sigma_{2}\right), & \text { if } \quad s=1 / 2
\end{array}\right.
$$

where $K_{s}$ denotes complex conjugation on $\mathcal{D}_{s}=\mathbb{C}^{2 s+1}$.
Lemma 6.1. Suppose $V$ is translationally invariant, cf. (6.1).
(a) Let $U \in S U(2), R=\pi(U), V\left(R x_{1}, \ldots, R x_{N}, 0\right)=V\left(x_{1}, \ldots, x_{N}, 0\right)$ for all $x_{j} \in \mathbb{R}^{3}$, and $\kappa(\cdot)=\kappa(R \cdot)$. Then for all $\xi \in \mathbb{R}^{3}$

$$
\mathcal{U}^{\prime}(U) H(\xi) \mathcal{U}^{\prime}(U)^{*}=H(R \xi) .
$$

(b) Let $V\left(x_{1}, \ldots, x_{N-1}, 0\right)=V\left(-x_{1}, \ldots,-x_{N-1}, 0\right)$ for all $x_{j} \in \mathbb{R}^{3}$ and $\kappa(\cdot)=\kappa(-\cdot)$. Then for all $\xi \in \mathbb{R}^{3}$

$$
\mathcal{P}^{\prime} H(\xi) \mathcal{P}^{\prime *}=H(-\xi) .
$$

(c) If $\overline{\kappa(\cdot)}=\kappa(-\cdot)$, then for all $\xi \in \mathbb{R}^{3}$

$$
\mathcal{T}^{\prime} H(\xi) \mathcal{T}^{\prime *}=H(-\xi)
$$

Proof. The Lemma follows as a consequence of Lemmas 4.7, 4.11, and 4.15 and Propositions 4.8, 4.12, and 4.16, respectively, and their trivial adaption to (6.2).

Theorem 6.2. Suppose $V$ is translationally invariant and $\sum_{j=1}^{N} 2 s_{j}$ is odd. If $\overline{\kappa(\cdot)}=\kappa(-\cdot)$ each eigenvalue of $H(0)$ has even or infinite multiplicity. If in addition $V\left(x_{1}, \ldots, x_{N-1}, 0\right)=$ $V\left(-x_{1}, \ldots,-x_{N-1}, 0\right)$ for all $x_{j} \in \mathbb{R}^{3}$ and $\kappa(-\cdot)=\kappa(\cdot)$, then for all $\xi \in \mathbb{R}^{3}$ each eigenvalue of $H(\xi)$ has even or infinite multiplicity.

Proof. The theorem follows as a consequence of Parts (c) and (b) of Lemma 6.1, Theorem 2.4. The first statement follows using the anti-linear symmetry $\mathcal{T}^{\prime}$. The second statement follows using the anti-linear symmetry $\mathcal{P}^{\prime} \mathcal{T}^{\prime}$ and their commutativity property, cf. Proposition 4.14 and its trivial adaption to (6.2).

Next we consider quantum systems with identical particles. For notational simplicity, we shall assume that there is a single particle which is distinguishable from the rest. This is satisfied for atoms, ions and many molecules. Otherwise, a further restriction to subspaces would be necessary.

Theorem 6.3. Suppose $V$ is translationally invariant and $\sum_{j=1}^{N} 2 s_{j}$ is odd. Suppose that the partition $\mathfrak{P}$, the function $\tau$ and the potential $V$, satisfy Hypothesis A. Furthermore, assume $\{N\} \in \mathfrak{P}$ and let $\mathfrak{P}^{\prime}=\mathfrak{P} \backslash\{\{N\}\}$ and $\tau^{\prime}=\left.\tau\right|_{\mathfrak{P}^{\prime}}$. If $\overline{\kappa(\cdot)}=\kappa(-\cdot)$ each eigenvalue of $H(0)$ when restricted to $\mathcal{H}_{\text {mat }, \mathfrak{F}^{\prime}, \tau^{\prime}}^{\prime} \otimes \mathcal{D}_{s_{N}} \otimes \mathcal{F}_{s}(\mathfrak{g})$ has even or infinite multiplicity. If in addition $V\left(x_{1}, \ldots, x_{N-1}, 0\right)=V\left(-x_{1}, \ldots,-x_{N-1}, 0\right)$ for all $x_{j} \in \mathbb{R}^{3}$ and $\kappa(-\cdot)=\kappa(\cdot)$, then each eigenvalue of $H(\xi)$ when restricted to $\mathcal{H}_{\text {mat }, \mathfrak{F}^{\prime}, \tau^{\prime}}^{\prime} \otimes \mathcal{D}_{s_{N}} \otimes \mathcal{F}_{s}(\mathfrak{g})$ has even or infinite multiplicity.

Proof. Follows from the same proof as Theorem 6.2, by observing in addition that $\mathcal{T}^{\prime}$ and $\mathcal{P}^{\prime}$ commute with $\mathfrak{U}(\underline{\sigma})$ for any $\sigma \in \mathfrak{S}_{p}$ and $p \in \mathfrak{P}^{\prime}$.

Remark 6.4. We note that the statement of Theorem 6.2 was proven for the special case where $N=1$ and $V=0$ for small coupling in [16] and for general coupling in [17]. Clearly, Theorem 6.3 covers the special case of $N-1$ electrons with spin $1 / 2$ and a spinless nucleus with pairwise Coulomb interactions $(\mathcal{P}=\{\{1, \ldots, N-1\},\{N\}\})$, cf. Remark 5.2 in [20]. We note that whereas ground states of fiber Hamiltonians describing electrons do not exist for nonzero momentum [10], they are shown to exist for atoms and small absolute values of the momentum [19].

## 7 Schrödinger Representation

In this section we define rotation, parity and time reversal symmetry in the so called Schrödinger representation of non-relativistic qed. To this end, we recall the Schwartz space of smooth functions of rapid decrease $\mathcal{S}\left(\mathbb{R}^{d} ; \mathbb{F}\right)$, with $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$, which is the set of infinitely differentiable $\mathbb{F}$-valued functions $f(x)$ on $\mathbb{R}^{d}$ for which

$$
\begin{equation*}
\|f\|_{\alpha, \beta}=\sup _{x \in \mathbb{R}^{d}}\left|x^{\alpha} \partial^{\beta} f(x)\right|<\infty \tag{7.1}
\end{equation*}
$$

for all $\alpha, \beta \in \mathbb{N}_{0}^{d}$. Let $\underline{\mathcal{S}}=\mathcal{S}\left(\mathbb{R}^{3} ; \mathbb{R}\right)^{3}$ equipped with the product topology. The topological dual space $\underline{\mathcal{S}}^{\prime}$ can be identified with the set of all $T \in \mathcal{S}^{\prime}\left(\mathbb{R}^{3} ; \mathbb{R}\right)^{3}$, with $T(f)=T_{1}\left(f_{1}\right)+$ $T_{2}\left(f_{2}\right)+T_{3}\left(f_{3}\right)$.

On $\underline{\mathcal{S}}$ we define the symmetric positive semi-definite form

$$
\begin{equation*}
B(v, w)=\sum_{i, j} \int \frac{1}{|k|} \overline{\hat{v}_{i}(k)} P_{i, j}(k) \hat{w}_{j}(k) d^{3} k, \tag{7.2}
\end{equation*}
$$

where we recall

$$
\begin{equation*}
P(k)_{a, b}:=\delta_{a b}-\frac{k_{a} k_{b}}{|k|^{2}}, \quad a, b=1,2,3, \quad k \neq 0 \tag{7.3}
\end{equation*}
$$

Let

$$
c(f)=e^{-\frac{1}{4} B(f, f)}
$$

for $f \in \underline{\mathcal{S}}$.
By definition a cylinder set in $\underline{\mathcal{S}}^{\prime}$ is a set

$$
\left\{T \in \underline{\mathcal{S}}^{\prime}:\left(T\left(f_{1}\right), \ldots ., T\left(f_{n}\right)\right) \in \Omega\right\}
$$

where $f_{1}, \ldots, f_{n}$ are $n$ fixed elements in $\underline{\mathcal{S}}$ and $\Omega$ is a fixed Borel set in $\mathbb{R}^{n}$. A cylinder set measure on $\underline{\mathcal{S}}^{\prime}$ is a measure, $\mu$, on the $\sigma$-algebra, generated by the cylinder sets, with $\mu\left(\underline{\mathcal{S}}^{\prime}\right)=1$. By construction, each $f \in \underline{\mathcal{S}}$ defines a measurable function $\varphi(f)$ on $\underline{\mathcal{S}}^{\prime}$ by

$$
\begin{equation*}
\varphi(f)(T)=T(f) \tag{7.4}
\end{equation*}
$$

In particular it follows that for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in \underline{\mathcal{S}}$

$$
\begin{equation*}
\varphi(\alpha f+\beta g)=\alpha \varphi(f)+\beta \varphi(g) \tag{7.5}
\end{equation*}
$$

We shall use the following theorem, see $[5,6,7,8,9]$.
Theorem 7.1. There exists a unique cylinder set measure $\nu$ on $\underline{\mathcal{S}}^{\prime}$ such that for all $f \in \underline{\mathcal{S}}$

$$
\begin{equation*}
\exp \left(-\frac{1}{4} B(f, f)\right)=\int \exp (i \varphi(f)) d \nu \tag{7.6}
\end{equation*}
$$

Furthermore, $\nu$ has the following properties.
(a) For each $f \in \underline{\mathcal{S}}$ the function $\varphi(f)$ is a Gaussian random variable with mean zero and variance $\frac{1}{2} B(f, f)$.
(b) For $f_{1}, \ldots f_{n} \in \underline{\mathcal{S}}$ the random variables $\varphi\left(f_{1}\right), \ldots, \varphi\left(f_{n}\right)$ are jointly Gaussian random variables.
(c) Let $\mathcal{U}=\left\{F\left(\varphi\left(f_{1}\right), \ldots, \varphi\left(f_{n}\right)\right): F \in \mathcal{S}\left(\mathbb{R}^{n} ; \mathbb{C}\right), f_{1}, \ldots, f_{n} \in \underline{\mathcal{S}}\right\}$. Then $\mathcal{U}$ is dense in $L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$.
(d) If $f \in \underline{\mathcal{S}}$ and $P \widehat{f}=0$, then $\varphi(f)=0$ almost surely, cf. (7.3). In particular, for almost all $T=\left(T_{1}, T_{2}, T_{3}\right) \in \underline{\mathcal{S}}^{\prime}$ we have $\nabla \cdot T=0$.

A proof of Theorem 7.1 will be given in Appendix B. Henceforth, we shall denote by $\nu$ the unique measure on $\underline{\mathcal{S}}^{\prime}$ satisfying (7.6).

Remark 7.2. We note that part (d) of Theorem 7.1 will not be needed. Nevertheless it is interesting in its own.

To formulate the next theorem we define

$$
\underline{\mathcal{S}}_{0}:=\{g \in \underline{\mathcal{S}}: \nabla \cdot g=0\} .
$$

By $\overline{(\cdot)}^{\mathrm{cl}}$ we shall denote the operator closure.
Theorem 7.3. There exists a unique unitary transformation $V_{\mathfrak{v}}: \mathcal{F}_{s}(\mathfrak{v}) \rightarrow L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$ with the following properties
(i) $V_{v} \Omega=1$,
(ii) $V_{\mathfrak{v}} \overline{\left(a^{*}\left(i_{\omega} f\right)+a\left(i_{\omega} f\right)\right)}{ }^{\mathrm{cl}} V_{\mathfrak{v}}^{-1}=\varphi(f)$, for all $f \in \underline{\mathcal{S}}_{0}$,
where $i_{\omega} f=\left(\omega^{-1 / 2} \hat{f}\right)^{\vee}$ and $\varphi(f)$ is understood as a multiplication operator. Moreover, we have $V_{\mathfrak{v}} \Gamma\left(\mathcal{K}_{\mathfrak{v}}\right)=J V_{\mathfrak{v}}$, where $J$ denotes complex conjugation in $L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$.

The proof of Theorem 7.3 will be given in Appendix B. Using Lemma 4.1 we obtain immediately the following corollary.

Corollary 7.4. Let the notation be as in in Theorem 7.3. There exists a unique unitary transformation $V_{\mathfrak{g}}: \mathcal{F}_{s}(\mathfrak{g}) \rightarrow L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$ with the following properties
(i) $V_{\mathfrak{g}} \Omega=1$,
(ii) $V_{\mathfrak{g}} \overline{\left(a^{*}\left(\tau_{\epsilon}^{-1} i_{\omega} f\right)+a\left(\tau_{\epsilon}^{-1} i_{\omega} f\right)\right)}{ }^{\mathrm{cl}} V_{\mathfrak{g}}^{-1}=\varphi(f)$, for all $f \in \underline{\mathcal{S}}_{0}$.

Moreover, we have $V_{\mathfrak{g}} \Gamma\left(\mathcal{K}_{\mathfrak{g}}\right)=J V_{\mathfrak{g}}$, where $J$ denotes complex conjugation in $L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$.

Next we define symmetries in Schrödinger representation. We will show in Theorem 7.6, below, that they agree by the unitary transformations of Theorem 7.3 and Corollary 7.4 with the definitions in Fock space representation. We define for $U \in S U(2)$ on $\underline{\mathcal{S}}$ the representation

$$
\left(\mathcal{U}_{\underline{\mathcal{S}}}(U) f\right)(x)=R f\left(R^{-1} x\right), \quad f \in \underline{\mathcal{S}}, x \in \mathbb{R}^{3}
$$

where $R=\pi(U)$. We define for $f \in \underline{\mathcal{S}}$

$$
\left(\mathcal{P}_{\underline{\mathcal{S}}} f\right)(x)=-f(-x) .
$$

As a consequence of the definition $\mathcal{P}_{\underline{\mathcal{s}}}^{-1}=\mathcal{P}_{\underline{\mathcal{S}}}$. Then this defines by duality a transformation on $\underline{\mathcal{S}}^{\prime}$ by

$$
\left(\mathcal{U}_{\underline{\mathcal{S}}^{\prime}}(U) T\right)(f)=T\left(\mathcal{U}_{\underline{\mathcal{S}}}(U)^{-1} f\right)
$$

and

$$
\left(\mathcal{P}_{\underline{\mathcal{S}}^{\prime}} T\right)(f)=T\left(\mathcal{P}_{\underline{\mathcal{s}}}^{-1} f\right)
$$

for all $T \in \underline{\mathcal{S}}^{\prime}$ and $f \in \underline{\mathcal{S}}$. On $L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$ we define for any $F \in L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$

$$
\begin{aligned}
\left(\mathcal{U}_{\mathrm{Sch}}(U) F\right)(T) & =F\left(\mathcal{U}_{{\underline{\mathcal{S}^{\prime}}}^{\prime}}(U)^{-1} T\right), \quad U \in S U(2), \\
\left(\mathcal{P}_{\mathrm{Sch}} F\right)(T) & =F\left(\mathcal{P}_{\left.{\underline{\mathcal{S}^{\prime}}}^{-1} T\right),}\right. \\
\left(\mathcal{K}_{\mathrm{Sch}} F\right)(T) & =\overline{F(T)} \\
\left(\Theta_{\mathrm{Sch}} F\right)(T) & =F(-T)
\end{aligned}
$$

for all $T \in \underline{\mathcal{S}}^{\prime}$.
Lemma 7.5. Let $U \in S U(2)$. The measure $\nu$ is invariant with respect to $\mathcal{U}_{\underline{s}^{\prime}}(U)$ and $\mathcal{P}_{\underline{\mathcal{S}}^{\prime}}$. The transformations $\mathcal{U}_{\text {Sch }}(U), \mathcal{P}_{\text {Sch }}$ are unitary transformations on $L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$. The transformation $\mathcal{K}_{\text {Sch }}$ is an anti-unitary transformation on $L^{2}\left(\underline{\mathcal{S}}^{\prime}\right.$,d $)$, which squares to one. The measure $\nu$ is invariant with respect to $-1_{\underline{S}^{\prime}}$, and $\Theta_{\text {Sch }}$ is a unitary transformation on $L^{2}\left(\mathcal{S}^{\prime}, d \nu\right)$, which squares to one.

Proof. Let $G$ stand for $\mathcal{U}_{\underline{\mathcal{S}}^{\prime}}(U)$ and $\mathcal{P}_{\underline{\mathcal{S}}^{\prime}}$ and $g$ for $\mathcal{U}_{\underline{\underline{S}}}(U)$ and $\mathcal{P}_{\underline{\mathcal{S}}}$, respectively. Then $G$ leaves the set of cylinder sets invariant, and hence the $\sigma$-algebra generated by the cylinder sets. Since the form $B$ is invariant with respect to $G$, so is the measure $\nu$. To see this define $\nu_{G}(A)=\nu(G(A))$ for any measurable set $A$. Then for any $f \in \underline{\mathcal{S}}$ we find from the definition of the integral

$$
\begin{aligned}
\exp \left(-\frac{1}{4} B(f, f)\right) & =\exp \left(-\frac{1}{4} B(g f, g f)\right)=\int \exp (i \varphi(g f)) d \nu=\int \exp \left(i\left(G^{-1} T\right)(f)\right) d \nu(T) \\
& =\int \exp (i T(f)) d \nu_{G}(T)=\int \exp (i \varphi(f)) d \nu_{G}
\end{aligned}
$$

Thus it follows $\nu=\nu_{G}$ from the uniqueness property in Theorem 7.1. Thus the unitarity properties of $\mathcal{U}_{\text {Sch }}(U)$ and $\mathcal{P}_{\text {Sch }}$ on $L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$ now follow by the definition of the integral as a limit of simple functions. The anti-unitarity of $\mathcal{K}_{\text {Sch }}$ is obvious. The last statement about $\Theta_{\text {Sch }}$ follows analogously as above with $G=-1_{\underline{\mathcal{S}^{\prime}}}$ and $g=-1_{\underline{\mathcal{S}}}$.

The following theorem relates the symmetries in the Fock representation to the symmetries in the Schrödinger representation.

Theorem 7.6. Let $V_{\mathfrak{v}}$ and $V_{\mathfrak{g}}$ be the unique unitary transformations satisfying (i) and (ii) of Theorem 7.3 and Corollary 7.4, respectively. Then the following identities hold.
(a) $V_{\mathfrak{g}} \mathcal{U}_{\mathfrak{f}}(U) V_{\mathfrak{g}}^{-1}=\mathcal{U}_{\text {Sch }}(U)$ and $V_{\mathfrak{v}} \Gamma\left(\mathcal{U}_{\mathfrak{v}}(\pi(U)) V_{\mathfrak{v}}^{-1}=\mathcal{U}_{\text {Sch }}(U)\right.$, for $U \in S U(2)$,
(b) $V_{\mathfrak{g}} \mathcal{P}_{\mathrm{f}} V_{\mathfrak{g}}^{-1}=\mathcal{P}_{\text {Sch }}$ and $V_{\mathfrak{v}} \Gamma\left(\mathcal{P}_{\mathfrak{v}}\right) V_{\mathfrak{v}}^{-1}=\mathcal{P}_{\text {Sch }}$,
(c) $V_{\mathfrak{g}} \Gamma\left(\mathcal{K}_{\mathfrak{g}}\right) V_{\mathfrak{g}}^{-1}=\mathcal{K}_{\text {Sch }}$ and $V_{\mathfrak{v}} \Gamma\left(\mathcal{K}_{\mathfrak{v}}\right) V_{\mathfrak{v}}^{-1}=\mathcal{K}_{\text {Sch }}$.
(d) $V_{\mathfrak{g}} \Gamma\left(-1_{\mathfrak{g}}\right) V_{\mathfrak{g}}^{-1}=\Theta_{\text {Sch }}$ and $V_{\mathfrak{v}} \Gamma\left(-1_{\mathfrak{v}}\right) V_{\mathfrak{v}}^{-1}=\Theta_{\text {Sch }}$.
(e) $V_{\mathfrak{g}} \mathcal{T}_{\mathrm{f}} V_{\mathfrak{g}}^{-1}=\Theta_{\text {Sch }} \mathcal{K}_{\text {Sch }}$.

Proof. We only discuss the case for $\mathfrak{v}$, the case for $\mathfrak{g}$ then follows using Lemma 4.1.
(a) Let $W=\mathcal{U}_{\text {sch }}(U) V_{\mathfrak{v}} \Gamma\left(\mathcal{U}_{\mathfrak{v}}(\pi(U))^{-1}\right)$. Then it follows from the definitions that $W \Omega=1$. Furthermore, it follows for all $f \in \underline{\mathcal{S}}_{0}$ using (3.4), the invariance of $\omega$ and Theorem 7.3 (ii)

$$
\begin{align*}
W\left(\overline{a^{*}\left(i_{\omega} f\right)+a\left(i_{\omega} f\right)}\right)^{\mathrm{cl}} W^{-1} & =\mathcal{U}_{\text {Sch }}(U) V_{\mathfrak{v}}\left(\overline{a^{*}\left(i_{\omega} \mathcal{U}_{\mathfrak{v}}(U)^{-1} f\right)+a\left(i_{\omega} \mathcal{U}_{\mathfrak{v}}(U)^{-1} f\right)}\right)^{\mathrm{cl}} V_{\mathfrak{v}}^{-1} \mathcal{U}_{\text {sch }}(U)^{-1} \\
& =\mathcal{U}_{\text {Sch }}(U) \varphi\left(\mathcal{U}_{\mathfrak{v}}(U)^{-1} f\right) \mathcal{U}_{\text {Sch }}(U)^{-1}=\varphi(f), \tag{7.7}
\end{align*}
$$

where the last equality can be seen as follows. For any $F \in L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$ we find with $F^{\prime}:=\mathcal{U}_{\text {Sch }}(U)^{-1} F$ using $\mathcal{U}_{\mathfrak{v}}(U) f=\mathcal{U}_{\underline{\mathcal{S}}}(U) f$ and inserting into the definitions, e.g. (7.4), that

$$
\begin{aligned}
& \left(\mathcal{U}_{\mathrm{Sch}}(U)\left(\varphi\left(\mathcal{U}_{\mathfrak{v}}(U)^{-1} f\right) F^{\prime}\right)\right)(T)=\left(\varphi\left(\mathcal{U}_{\underline{\mathcal{S}}}(U)^{-1} f\right) F^{\prime}\right)\left(\mathcal{U}_{\underline{\mathcal{S}}^{\prime}}(U)^{-1} T\right) \\
& =\left(\mathcal{U}_{\underline{\mathcal{S}}^{\prime}}(U)^{-1} T\right)\left(\mathcal{U}_{\underline{S}}(U)^{-1} f\right) F^{\prime}\left(\mathcal{U}_{\underline{S}^{\prime}}(U)^{-1} T\right) \\
& =T(f) F^{\prime}\left(\mathcal{U}_{\underline{\mathcal{S}}^{\prime}}(U)^{-1} T\right)=\varphi(f) F(T)
\end{aligned}
$$

This show the last equality in (7.7). It now follows from (7.7) that $W=V_{\mathfrak{v}}$ by the uniqueness statement of Theorem 7.3. This shows (a). Now (b) is shown similarly as (a). (c) Let $W=\mathcal{K}_{\mathrm{Sch}} V_{\mathfrak{v}} \Gamma\left(\mathcal{K}_{\mathfrak{v}}\right)$. Then it follows from the definitions that $W \Omega=1$. Furthermore, it follows for all $f \in \underline{\mathcal{S}}_{0}$ using (3.4), the reality and symmetry assumptions of $\omega$, and Theorem 7.3 (ii) that

$$
\begin{aligned}
W\left(\overline{a^{*}\left(i_{\omega} f\right)+a\left(i_{\omega} f\right)}\right)^{\mathrm{cl}} W^{-1} & =\mathcal{K}_{\text {Sch }} V_{\mathfrak{v}}\left(\overline{a^{*}\left(\mathcal{K}_{\mathfrak{v}} i_{\omega} f\right)+a\left(\mathcal{K}_{\mathfrak{v}} i_{\omega} f\right)}\right)^{\mathrm{cl}} V_{\mathfrak{v}}^{-1} \mathcal{K}_{\text {Sch }}^{-1} \\
& =\mathcal{K}_{\text {Sch }} V_{\mathfrak{v}}\left(\bar{a}^{*}\left(i_{\omega} \mathcal{K}_{\mathfrak{v}} f\right)+a\left(i_{\omega} \mathcal{K}_{\mathfrak{v}} f\right)\right)^{\mathrm{cl}} V_{\mathfrak{v}}^{-1} \mathcal{K}_{\text {Sch }}^{-1} \\
& =\mathcal{K}_{\text {Sch }} \varphi\left(\mathcal{K}_{\mathfrak{v}} f\right) \mathcal{K}_{\text {Sch }}^{-1}=\varphi(f) .
\end{aligned}
$$

As in (a) it now follows that $W=V_{\mathfrak{v}}$ by the uniqueness statement of Theorem 7.3. This shows (c), since $\mathcal{K}_{\mathfrak{v}}^{-1}=\mathcal{K}_{\mathfrak{v}}$. Now (d) follows analogously to (c) by considering $W=$
$\Theta_{\text {Sch }} V_{\mathfrak{v}} \Gamma\left(-1_{\mathfrak{v}}\right)$ and observing that

$$
\begin{aligned}
W\left(\overline{a^{*}\left(i_{\omega} f\right)+a\left(i_{\omega} f\right)}\right)^{\mathrm{cl}} W^{-1} & =\Theta_{\mathrm{Sch}} V_{\mathfrak{v}}\left(\overline{a^{*}\left(-i_{\omega} f\right)+a\left(-i_{\omega} f\right)}\right)^{\mathrm{cl}} V_{\mathfrak{v}}^{-1} \Theta_{\mathrm{Sch}}^{-1} \\
& =\Theta_{\mathrm{Sch}} V_{\mathfrak{v}}\left(\overline{a^{*}\left(i_{\omega}(-f)\right)+a\left(i_{\omega}(-f)\right.}\right)^{\mathrm{cl}} V_{\mathfrak{v}}^{-1} \Theta_{\mathrm{Sch}}^{-1} \\
& =\Theta_{\mathrm{Sch}} \varphi(-f) \Theta_{\mathrm{Sch}}^{-1}=\varphi(f)
\end{aligned}
$$

Again by uniqueness $W=V_{\mathfrak{v}}$. This shows (d). Finally, (e) follows from (c) and (d).
Remark 7.7. We see from Subsection 4.4 and Theorem 7.6 that $\mathcal{U}_{\text {Sch }}, \mathcal{P}_{\text {Sch }}$ and $\mathcal{T}_{\text {Sch }}:=$ $\Theta_{\text {Sch }} \mathcal{K}_{\text {Sch }}$ correspond to the rotation, parity and time reversal symmetries in the Schrödinger representation. Alternatively, one could redefine the field operators in the Hamiltonian so that $\mathcal{K}_{\text {Sch }}$ has the property of a time reversal symmetry, cf. [21].

## Acknowledgements

Both authors acknowledge financial support by the Research Training Group (1523/2) "Quantum and Gravitational Fields" when this project was initiated. D. Hasler wants to thank I. Herbst for valuable discussions on the subject. M. Lange also acknowledges financial support from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC StG MaMBoQ, grant agreement n.802901).

## A Gaussian Random Processes

In this appendix we review notations and results about so called Gaussian random processes. We follow [26]. The main result is Theorem A.6, which will be used in the proof of Theorem 7.3 in Appendix B. First we introduce the following definitions.

Definition A.1. Let $(M, \mu)$ be a probability measure space. Let $V$ be a real vector space. $A$ random process indexed by $V$ is a map $\phi$ from $V$ to the random variables on $M$, so that almost everywhere

$$
\begin{aligned}
\phi(v+w) & =\phi(v)+\phi(w) \quad \forall v, w \in V \\
\phi(\alpha v) & =\alpha \phi(v) \quad \forall \alpha \in \mathbb{R}, \forall v \in V
\end{aligned}
$$

For a random variable $Y$ on probability measure space $(M, \mu)$ we will use the notation

$$
\langle Y\rangle:=\int Y d \mu
$$

Definition A.2. Let $\mathfrak{r}$ be a real Hilbert space with inner product $\langle\cdot, \cdot\rangle_{\mathfrak{r}}$. A Gaussian random process indexed by $\mathfrak{r}$ is a random process $\phi$ indexed by $\mathfrak{r}$ so that the following holds.
(a) The set $\left\{F\left(\phi\left(v_{1}\right), \ldots, \phi\left(v_{n}\right)\right): v_{1}, \ldots, v_{n} \in \mathfrak{r}, F \in \mathcal{S}\left(\mathbb{R}^{n}\right)\right\}$ is dense in $L^{2}(M, d \mu)$, where $(M, \mu)$ is the probability measure space of the random process $\phi$.
(b) Each $\phi(v)$ is a Gaussian random variable.
(c) $\langle\phi(v) \phi(w)\rangle=\frac{1}{2}\langle v, w\rangle_{\mathrm{r}}$.

## Remark A.3.

(a) We note that in (a) of Definition A.2, we use a different assumption than in the definition of a Gaussian random process indexed by a Hilbert space in [26]. However, in view of [26, Lemma I.5] this is equivalent.
(b) One can show that two Gaussian random processes indexed by the same real Hilbert space are unique up to isomorphisms of probability measure spaces, see for example [26, Theorem I.6].
(c) For any real Hilbert space $\mathfrak{r}$, a Gaussian process indexed by $\mathfrak{r}$ exists. For a proof see Theorem I. 9 in [26].

Let $\mathfrak{r}$ be the complexification of $\mathfrak{r}$, i.e., $\mathfrak{r}_{\mathbb{C}}=\mathfrak{r} \oplus \mathfrak{r}$ as a real Hilbert space with a complex structure given by $i(u, v)=(-v, u)$. We define

$$
\begin{equation*}
J: \mathfrak{r}_{\mathbb{C}} \rightarrow \mathfrak{r}_{\mathbb{C}}, \quad J(u, v)=(u,-v) \tag{A.1}
\end{equation*}
$$

Then $J$ is anti-linear and satisfies $J^{2}=1$. Without mention we shall imbed $\mathfrak{r}$ in $\mathfrak{r}_{\mathbb{C}}$ by the map $\iota: u \mapsto(u, 0)$. For the operator introduced in (3.3) we shall write for notational convenience $a^{\#}(f)=a^{\#}(\iota f)$ for $f \in \mathfrak{r}$.

Next we introduce the notion of Wick powers and Wick product of random variables. To this end we introduce the following multi-index notation. For $k \in \mathbb{N}, \underline{n} \in \mathbb{N}_{0}^{k}$ and $\alpha, \beta \in \mathbb{C}^{k}$ we define

$$
\alpha^{\underline{n}}=\prod_{j=1}^{k} \alpha^{n_{j}}, \quad \alpha \beta=\sum_{j=1}^{k} \alpha_{j} \beta_{j}, \quad|\underline{n}|=\sum_{j=1}^{k} n_{j}, \quad \underline{n}!=\prod_{j=1}^{k} n_{j}!.
$$

Given a formal power series in random variables $f_{1}, \ldots, f_{k}$ with finite moments on a measure space $(M, \mu)$, which we denote by $\sum_{\underline{n} \in \mathbb{N}_{0}^{k}} a_{\underline{n}} f \underline{n}$, where $a_{\underline{n}} \in \mathbb{C}$ and

$$
f^{\underline{n}}:=\prod_{j=1}^{k} f_{j}^{n_{j}}
$$

we define the formal derivative

$$
\frac{\partial}{\partial f_{i}} \sum_{\underline{n} \in \mathbb{N}_{0}^{k}} a_{\underline{n}} f^{\underline{n}}=\sum_{\underline{n} \in \mathbb{N}_{0}^{k}} a_{\underline{n}} n_{i} f^{\underline{n}-\underline{e}_{i}} .
$$

where $\underline{e}_{i} \in \mathbb{N}^{k}$ is defined such that all components vanish except the $i$-th, which equals 1 .

Remark A.4. As in [26] we don't identify two series which are identical by virtue of substituting in specific arguments (e.g. $f$ and $f^{2}$ are distinct as formal power series even if $f=1$ ).

Definition A.5. Let $f_{1}, \ldots, f_{k}$ be random variables with finite moments on a measure space $(M, \mu)$. The Wick product : $f^{\underline{n}}$ : is defined inductively in $n=|\underline{n}|$ by
(i) : $f \underline{0}:=1$, where $\underline{0}=(0, \ldots, 0)$,
(ii) $\left\langle: f^{\underline{n}}:\right\rangle=0$ if $n \neq 0$,
(iii) $\frac{\partial}{\partial f_{i}}: f^{\underline{n}}:=n_{i}: f^{\underline{n}-\underline{e}_{i}}:$.

The following theorem is the main theorem of this section.
Theorem A.6. Let $\phi$ be a Gaussian random process indexed by a separable real Hilbert space $\mathfrak{r}$ on the probability measure space $(M, \mu)$, and let $D$ be a dense subset of $\mathfrak{r}$. Then there exists a unique unitary transformation $V: \mathcal{F}_{s}\left(\mathfrak{r}_{\mathbb{C}}\right) \rightarrow L^{2}(M, d \mu)$ satisfying
(i) $V \Omega=1$
(ii) $V\left(\overline{a^{*}(f)+a(f)}\right)^{\mathrm{cl}} V^{-1}=\phi(f)$ for all $f \in D$.

Moreover, the following holds. We have
(a) $V\left(\overline{a^{*}(f)+a(f)}\right)^{\mathrm{cl}} V^{-1}=\phi(f)$ holds for all $f \in \mathfrak{r}$.
(b) $\mathcal{J} V=V \Gamma(J)$, where $J$ is defined in (A.1) and $\mathcal{J}$ denotes ordinary complex conjugation in $L^{2}(M, d \mu)$.
(c) For all $f_{j} \in \mathfrak{r}$ we have

$$
\begin{equation*}
V a^{*}\left(f_{1}\right) \cdots a^{*}\left(f_{n}\right) \Omega=: \phi\left(f_{1}\right) \cdots \phi\left(f_{n}\right): . \tag{A.2}
\end{equation*}
$$

A proof of Theorem A. 6 can be found in Theorems I. 6 and I. 11 in [26]. For the convenience of the reader, we shall outline a proof below. First, we need a few lemmas. For random variables $f_{1}, \cdots, f_{k}$ with finite moments we define the formal power series for $\alpha \in \mathbb{C}^{k}$ by

$$
\begin{equation*}
: \exp (\alpha f):=\sum_{\underline{n} \in \mathbb{N}_{0}^{k}}^{\infty} \frac{\alpha^{\underline{n}}: f^{\underline{n}}:}{\underline{n}!} \tag{A.3}
\end{equation*}
$$

Lemma A.7. Let $f_{1}, \ldots, f_{k}$ be random variables with finite moments on a probability measure space $(M, \mu)$. Then for all $\alpha \in \mathbb{C}^{k}$ the following holds
(a) $\langle: \exp (\alpha f):\rangle=1$
(b) $: \exp (\alpha f):=\exp (\alpha f)\langle\exp (\alpha f)\rangle^{-1}$
(c) If $f$ is a Gaussian random variable, then (A.3) converges in $L^{1}(M, d \mu)$ and

$$
: \exp (\alpha f):=\exp (\alpha f) \exp \left(-\frac{1}{2} \sum_{i, j} \alpha_{i} \alpha_{j}\left\langle f_{i} f_{j}\right\rangle\right)
$$

Proof. (a) This follows from (i) and (ii) of Definition A.5. (b) By (iii) of Definition A.5, we find $\frac{\partial}{\partial f_{j}}: \exp (\alpha f):=\alpha_{j}: \exp (\alpha f)$ :. Thus $\frac{\partial}{\partial f_{j}}: \exp (\alpha f): \exp (-\alpha f)=0$ and so $: \exp (\alpha f):$ $\exp (-\alpha f)=C$ for some constant $C$. Thus from (a) it follows that $C=\langle\exp (\alpha f)\rangle^{-1}$. (c) The $L^{1}$ convergence follows from dominated convergence. Using that $f$ is Gaussian one finds $\langle\exp (\alpha f)\rangle=\exp \left(\frac{1}{2} \sum_{i, j} \alpha_{i} \alpha_{j}\left\langle f_{i} f_{j}\right\rangle\right)$ (e.g. by calculating the Fourier transform for $\alpha=i t$, with $t \in \mathbb{R}^{k}$, and then using analytic continuation). Thus (c) follows from (b).

The following Lemma is from [26, Theorem I.3, Corollary I.4].
Lemma A.8. The following holds.
(a) If $f$ and $g$ are Gaussian random variables, then for $m, n \in \mathbb{N}_{0}$

$$
\left\langle: f^{n}:: g^{m}:\right\rangle=\delta_{n, m} n!\langle f g\rangle^{n} .
$$

(b) If $f_{1}, \ldots, f_{n}$ and $g_{1}, \ldots, g_{m}$ are Gaussian random variables and $n \neq m$, then

$$
\left\langle: f_{1} \cdots f_{n}:: g_{1} \cdots g_{m}:\right\rangle=0
$$

(c) If $f_{1}, \ldots, f_{k}$ are Gaussian random variables with $\left\langle f_{i} f_{j}\right\rangle=\delta_{i, j}$, then for $\underline{n}, \underline{m} \in \mathbb{N}_{0}^{k}$

$$
\left\langle: f^{\underline{n}}:: f^{\underline{m}}:\right\rangle=\delta_{\underline{n}, \underline{m}} \underline{n}!.
$$

Proof. (a). By (c) of Lemma A. 7 we find

$$
\begin{aligned}
: \exp (\alpha f):: \exp (\beta g): & =\exp (\alpha f+\beta g) \exp \left(-\frac{1}{2}\left[\alpha^{2}\left\langle f^{2}\right\rangle+\beta^{2}\left\langle g^{2}\right\rangle\right]\right) \\
& =: \exp (\alpha f+\beta g): \exp (\alpha \beta\langle f g\rangle) .
\end{aligned}
$$

Thus by (a) of Lemma A. 7

$$
\langle: \exp (\alpha f):: \exp (\beta g):\rangle=\exp (\alpha \beta\langle f g\rangle)
$$

Thus (a) now follows by expanding exponentials and equating coefficients. (b,c) follow from the multinomial theorem and (a).

Lemma A.9. Let $\phi$ be a Gaussian random process indexed by the real Hilbert space $\mathfrak{r}$. Let

$$
\Gamma_{n}(\mathfrak{r})=\overline{\operatorname{lin}_{\mathbb{C}}\left\{: \phi\left(f_{1}\right) \cdots \phi\left(f_{n}\right): \quad \mid \quad f_{1}, \ldots, f_{n} \in \mathfrak{r}\right\}}, \quad n \in \mathbb{N}
$$

and $\Gamma_{0}(\mathfrak{r})=\mathbb{C}$. Then the following holds.
(a) $\Gamma_{n}(\mathfrak{r}) \perp \Gamma_{m}(\mathfrak{r})$ for $n \neq m$.
(b) $L^{2}(M, d \mu)=\bigoplus_{n=0}^{\infty} \Gamma_{n}(\mathfrak{r})$.

Proof. (a) This follows from (b) of Lemma A.8. (b) For any $f \in \mathfrak{r}$, a direct computation shows that the formal power series : $e^{i \phi(f)}$ : converges in $L^{2}(M, d \mu)$. We shall denote the limit by the same symbol. Thus by definition $\bigoplus_{n=0}^{\infty} \Gamma_{n}(\mathfrak{r})$ contains : $e^{i \phi(f)}$ : and so $e^{i \phi(f)}$ in view of (c) of Lemma A.7. In particular, for any $F \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ and $f_{1}, \ldots, f_{n} \in \mathfrak{r}$ we find that

$$
\begin{equation*}
F\left(\phi\left(f_{1}\right), \cdots, \phi\left(f_{n}\right)\right)=(2 \pi)^{-n / 2} \int \widehat{F}(t) \exp \left(\sum_{j=1}^{n} t_{j} \phi\left(f_{j}\right)\right) d^{n} t \tag{A.4}
\end{equation*}
$$

is in $\bigoplus_{n=0}^{\infty} \Gamma_{n}(\mathfrak{r})$. But the set of random variables of the form as on the left hand side of (A.4) are dense in $L^{2}(M, d \mu)$ by the assumptions of an indexed Gaussian random process. Thus (b) follows.

Proof of Theorem A.6. First we show uniqueness. To this end we define for $f \in \mathfrak{r}_{\mathbb{C}}$ the operator $\phi_{\mathcal{F}}(f)$ in $\mathcal{F}_{s}\left(\mathfrak{r}_{\mathbb{C}}\right)$ by

$$
\begin{equation*}
\phi_{\mathcal{F}}(f)=\overline{a^{*}(f)+a(f)}{ }^{\mathrm{cl}} . \tag{A.5}
\end{equation*}
$$

We claim that for any $m \in \mathbb{N}_{0}$ the set

$$
\left\{\phi_{\mathcal{F}}\left(f_{1}\right) \cdots \phi_{\mathcal{F}}\left(f_{n}\right) \Omega: f_{i} \in D, n=0,1, \ldots, m\right\}
$$

is dense in $\bigoplus_{n=0}^{m} S_{n}\left(\mathfrak{r}_{\mathbb{C}}^{\otimes n}\right)$. To show this, we use induction in $m$. The claim clearly holds for $m=0$. Suppose it holds for $m$. Then multiplying out, we find

$$
\phi_{\mathcal{F}}\left(f_{1}\right) \cdots \phi_{\mathcal{F}}\left(f_{m+1}\right) \Omega=a^{*}\left(f_{1}\right) \cdots a^{*}\left(f_{m+1}\right) \Omega+h
$$

where $h \in \bigoplus_{n=0}^{m} S_{n}\left(\mathfrak{r}_{\mathbb{C}}^{\otimes n}\right)$. Since the linear span of $a^{*}\left(f_{1}\right) \cdots a^{*}\left(f_{m+1}\right) \Omega$ is dense in $S_{m+1}\left(\mathfrak{r}_{\mathbb{C}}^{\otimes n}\right)$ the claim follows for $m+1$. Since

$$
V \phi_{\mathcal{F}}\left(f_{1}\right) \cdots \phi_{\mathcal{F}}\left(f_{n}\right) \Omega=\left(V \phi_{\mathcal{F}}\left(f_{1}\right) V^{-1}\right) \cdots\left(V \phi_{\mathcal{F}}\left(f_{n}\right) V^{-1}\right) V \Omega
$$

properties (i) and (ii) determine the action of $V$ uniquely on a dense set.
Let us now show existence. First choose an o.n.b. $\mathcal{B}$ of $\mathfrak{r}$. Define $V$ by $V \Omega=1$ and

$$
V a^{*}\left(f_{1}\right) \cdots a^{*}\left(f_{n}\right) \Omega=: \phi\left(f_{1}\right) \cdots \phi\left(f_{n}\right):
$$

where $f_{j} \in \mathcal{B}$ (this is well defined by the symmetry property of the Wick product) and extend it by linearity. It is straight forward to see that the map $V$ is an isometry using on the one hand side the canonical commutation relations for creation and annihilation operators in Fock space and on the other hand Lemma A.8. Surjectivity, and hence unitarity, follows from Lemma A.9. Obviously, $V$ satisfies (i) by construction. Let us now show, that it
satisfies (a) and hence (ii). Using the definition, (A.5), and the canonical commutation relations we find for $f_{j} \in \mathcal{B}$

$$
\begin{align*}
& \phi_{\mathcal{F}}\left(f_{1}\right) a^{*}\left(f_{1}\right)^{n_{1}} \cdots a^{*}\left(f_{k}\right)^{n_{k}} \Omega  \tag{A.6}\\
& \quad=a^{*}\left(f_{1}\right)^{n_{1}+1} \cdots a^{*}\left(f_{k}\right)^{n_{k}} \Omega+n_{1} a^{*}\left(f_{1}\right)^{n_{1}-1} \cdots a^{*}\left(f_{k}\right)^{n_{k}} \Omega
\end{align*}
$$

On the other hand we will show that

$$
\begin{align*}
& \phi\left(f_{1}\right): \phi\left(f_{1}\right)^{n_{1}} \cdots \phi\left(f_{k}\right)^{n_{k}}:  \tag{A.7}\\
& \quad=: \phi\left(f_{1}\right)^{n_{1}+1} \cdots \phi\left(f_{k}\right)^{n_{k}}:+n_{1}: \phi\left(f_{1}\right)^{n_{1}-1} \cdots \phi\left(f_{k}\right)^{n_{k}}: .
\end{align*}
$$

To see (A.7), we first note that using (c) of Lemma A. 7 we obtain

$$
\begin{equation*}
\phi\left(f_{1}\right): \exp \left(\sum_{j=1}^{n} \alpha_{j} \phi\left(f_{j}\right)\right):=\left(\frac{\partial}{\partial \alpha_{1}}+\alpha_{1}\right): \exp \left(\sum_{j=1}^{n} \alpha_{j} \phi\left(f_{j}\right)\right): . \tag{A.8}
\end{equation*}
$$

Now expanding (A.8) in a power series, calculating the derivative, and equating coefficients, we obtain (A.7). Thus it follows in view of (A.6), (A.7), and from the definition of $V$ that for all $f \in \mathcal{B}$

$$
\begin{equation*}
\left.\overline{V\left(a^{*}(f)+a^{*}(f)\right)}\right)^{\mathrm{cl}} V^{-1}=\phi(f) . \tag{A.9}
\end{equation*}
$$

This implies (a) (and hence (ii)) by linearity and continuity. Clearly, (c) follows from uniqueness of the above construction and multi-linearity. To show (b) observe that from (A.2) we find for any $f_{j} \in \mathfrak{r}$ that

$$
\begin{aligned}
& \left.\mathcal{J} V \Gamma(J) a^{*}\left(f_{1}\right) \cdots a^{*}\left(f_{n}\right) \Omega=\mathcal{J} V a^{*}\left(J f_{1}\right) \cdots a^{*}\left(J f_{n}\right)\right) \Omega \\
& \left.=\mathcal{J} V a^{*}\left(f_{1}\right) \cdots a^{*}\left(f_{n}\right)\right) \Omega=\mathcal{J}: \phi\left(f_{1}\right) \cdots \phi\left(f_{n}\right): \\
& =: \phi\left(f_{1}\right) \cdots \phi\left(f_{n}\right):=V a^{*}\left(f_{1}\right) \cdots a^{*}\left(f_{n}\right) \Omega .
\end{aligned}
$$

Thus by density and $\mathbb{C}$-linearity it follows that $\mathcal{J} V \Gamma(J)=V$. Thus (b) follows, since $J^{-1}=J$.

## B An Application of Minlos' theorem

In this appendix we will prove Theorems 7.1 and 7.3. For this we shall introduce the following definitions from [26]. Let us first recall the definition

$$
c(f)=e^{-\frac{1}{4} B(f, f)}
$$

for $f \in \underline{\mathcal{S}}$ with $B$ defined in (7.2).
Lemma B.1. The following holds.
(i) $c(0)=1$.
(ii) $f \mapsto c(f)$ is continuous.
(iii) For any $f_{1}, \ldots, f_{n} \in \underline{\mathcal{S}}$ and $z_{1}, \ldots, z_{n} \in \mathbb{C}$ we have

$$
\sum_{i, j=1}^{n} z_{i} \bar{z}_{j} c\left(f_{i}-f_{j}\right) \geq 0
$$

Proof. (i) This follows from $B(0,0)=0$. (ii) It is straight forward to see that $f \mapsto B(f, f)$ is continuous on $\underline{\mathcal{S}}$, and hence also the function $c: f \mapsto \exp \left(-\frac{1}{4} B(f, f)\right)$. (iii) Let $V=$ $\operatorname{lin}_{\mathbb{R}}\left\{f_{1}, \ldots, f_{n}\right\}$. Then there exists a basis $\left(e_{j}\right)_{j=1, \ldots, m}$ of $V$, with dual basis $\left(b_{j}\right)_{j=1}^{m}$, such that $B\left(e_{i}, e_{j}\right)=\lambda_{i} \delta_{i, j}$ with $\lambda_{1}=\cdots=\lambda_{p}=1$ and $\lambda_{p+1}=\cdots \lambda_{m}=0$ for some $1 \leq p \leq m$. Using that the Fourier transform of a Gaussian is a Gaussian we find for any $f \in V$ with $f_{j}=b_{j}(f)$

$$
c(f)=e^{-\frac{1}{4} B(f, f)}=e^{-\frac{1}{4} \sum_{j=1}^{p} f_{j}^{2}}=(\pi)^{-p / 2} \int e^{-i \sum_{j=}^{p} y_{j} b_{j}(f)} e^{-\sum_{j=1}^{p} y_{j}^{2}} d^{p} y
$$

So positivity of $c(f)$ now follows from Bochner's theorem [23, Theorem IX.9].
Proof of Theorem 7.1. The existence and uniqueness of the measure $\nu$ follows in view of Lemma B. 1 from Minlos theorem [9, Theorem 3.4.2] see also [1,2,3,22, 28]. To this end, we extend the seminorms (7.1) to $\underline{\mathcal{S}}$ as follows. For $f=\left(f_{1}, f_{2}, f_{3}\right) \in \underline{\mathcal{S}}$ we define $\|f\|_{\alpha, \beta}:=$ $\left\|f_{1}\right\|_{\alpha, \beta}+\left\|f_{2}\right\|_{\alpha, \beta}+\left\|f_{3}\right\|_{\alpha, \beta}$. Then it is straight forward to see that $\underline{\mathcal{S}}$ with these seminorms is a nuclear space.
(a) This follows since for $f \in \underline{\mathcal{S}}$ and each $t \in \mathbb{R}$ we have by (7.6)

$$
\int \exp (i t \varphi(f)) d \nu=\exp \left(-\frac{1}{4} t^{2} B(f, f)\right)
$$

and so $\varphi(f)$ is a Gaussian random variable with mean zero, see [26]. (b) This follows from (a) and linearity (7.5), see [26]. (c) We argue similarly as in [28]. First observe that for all measurable sets $E$ we have

$$
\begin{equation*}
\forall \epsilon>0, \exists C \text { a cylinder set, } \quad \nu(C \triangle E)<\epsilon . \tag{B.1}
\end{equation*}
$$

Here, $\triangle$ stands for the symmetric difference. To this end, let $\mathcal{E}$ be the set of all measurable $E$ which satisfy (B.1). It is straight forward to verify that $\mathcal{E}$ is a $\sigma$-algebra containing all cylinder sets. Hence $\mathcal{E}$ equals the set of all measurable sets. It follows by definition of the integral that $\left\{1_{\Omega}\left(\varphi\left(f_{1}\right), \ldots, \varphi\left(f_{n}\right)\right): \Omega \subset \mathbb{R}^{n}\right.$ Borel measurable, $\left.f_{1}, \ldots, f_{n} \in \underline{\mathcal{S}}\right\}$ is dense in $L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$. Now it is well known that $\mathcal{S}\left(\mathbb{R}^{n} ; \mathbb{C}\right)$ is dense in $L^{1}\left(\mathbb{R}^{n}, d \mu_{C}\right)$, where $\mu_{C}$ denotes Gaussian measure with covariance $C$ (with possibly matrix elements which are infinite). This shows the density. (d) If $f \in \underline{\mathcal{S}}$ with $P \widehat{f}=0$, then $B(f, f)=0$, so $\varphi(f)$ is by (a) a Gaussian random variable with variance zero. Thus for all $f \in \underline{\mathcal{S}}$ with $P \widehat{f}=0$ it follows
that $\varphi(f)=0$ almost everywhere. Now let $h \in \mathcal{S}\left(\mathbb{R}^{3} ; \mathbb{R}\right)$. Then $\nabla h \in \underline{\mathcal{S}}$ and for all $T \in \underline{\mathcal{S}}^{\prime}$ we have

$$
\varphi(\nabla h)(T)=0 \Leftrightarrow T(\nabla h)=0 \Leftrightarrow(\nabla \cdot T)(h)=0
$$

Since $P \widehat{\nabla h}=0$, we find $(\nabla \cdot T)(h)=0$ for almost all $T \in \underline{\mathcal{S}}^{\prime}$. Since $\mathcal{S}\left(\mathbb{R}^{3} ; \mathbb{R}\right)$ is separable, there exists a countable dense subset $\mathcal{Q}$. It follows that for almost all $T \in \underline{\mathcal{S}}^{\prime}$ we have $(\nabla \cdot T)(h)=0$ for all $h \in \mathcal{Q}$. Since $\nabla \cdot T$ is continuous it follows that for almost all $T \in \underline{\mathcal{S}}^{\prime}$ we have $(\nabla \cdot T)(h)=0$ for all $h \in \mathcal{S}\left(\mathbb{R}^{3} ; \mathbb{R}\right)$. This shows the claim.

As an immediate consequence of Theorem 7.1 we obtain the following lemma, which we shall use for the proof of Theorem 7.3.

Lemma B.2. Let $\mathfrak{h}_{B}$ denote the real Hilbert space obtained by the completion of the inner product space $\left(\underline{\mathcal{S}}_{0}, B(\cdot, \cdot)\right)$ with the imbedding $\iota: \underline{\mathcal{S}}_{0} \rightarrow \mathfrak{h}_{B}$ having dense range. Let $v \in \mathfrak{h}_{B}$, and let $\left(v_{n}\right)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\underline{\mathcal{S}}_{0}$ such that $\iota\left(v_{n}\right) \rightarrow v$. Then the following limit exists in $L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$

$$
\varphi(v):=\lim _{n \rightarrow \infty} \varphi\left(v_{n}\right),
$$

is independent of the Cauchy sequence. Furthermore, $\varphi(v)$ is a Gaussian random process indexed by $\mathfrak{h}_{B}$ with $\left(\underline{\mathcal{S}}^{\prime}, \nu\right)$ the probability measure space of the random process.

Proof. First observe that $\left(\underline{\mathcal{S}}_{0}, B(\cdot, \cdot)\right)$ is indeed an inner product space, since $\nabla \cdot f=0$ implies $P \widehat{f}=\widehat{f}$. Clearly, $\varphi\left(v_{n}\right)$ is a Cauchy sequence in $L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$, since $\int \mid \varphi\left(v_{n}\right)-$ $\left.\varphi\left(v_{m}\right)\right|^{2} d \nu=\frac{1}{2} B\left(v_{n}-v_{m}, v_{n}-v_{m}\right)$ by Theorem 7.1 (a), and hence converges to a unique limit. With regard to Definition A. 2 the statement of the last sentence is straight forward to show using Theorem 7.1 and the fact that limits of Gaussians are Gaussian.

Proof of Theorem 7.3. The map $i_{\omega}:\left(\underline{\mathcal{S}}_{0}, B(\cdot, \cdot)\right) \rightarrow\{v \in \mathfrak{v}: \operatorname{Im} v=0\}$ is an isometry of real inner product spaces, which follows directly from the definitions. Furthermore, $i_{\omega}$ has dense range. To see this, observe that for any real $v \in \mathfrak{v}$ there exists by well known construction a real $v_{n} \in \underline{\mathcal{S}}$ such that $v_{n} \rightarrow v$ in the $L^{2}\left(\mathbb{R}^{3} ; \mathbb{C}^{3}\right)$ norm. Now define $w_{n}=\left(\omega^{1 / 2}\left(1-\chi_{n}\right) P \hat{v}_{n}\right)^{\vee}$ for $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{3} ;[0,1]\right)$ with $\chi=1$ on $B_{1 / 2}(0)$ and $\chi=0$ outside of $B_{1}(0)$, and $\chi_{n}(x)=\chi(n x)$. Then it is straight forward to see that $w_{n} \in \underline{\mathcal{S}}_{0}$ and (by unitarity of the Fourier transform and dominated convergence)

$$
\begin{aligned}
& \left\|i_{\omega} w_{n}-v\right\|=\left\|\left(\left(1-\chi_{n}\right) P \hat{v}_{n}\right)^{v}-v\right\|=\|\left(\left(1-\chi_{n}\right) P \hat{v}_{n}-\hat{v}\|=\|\left(\left(1-\chi_{n}\right) P \hat{v}_{n}-P \hat{v} \|\right.\right. \\
& \leq\left\|\chi_{n} P \hat{v}\right\|+\left\|\left(1-\chi_{n}\right) P\left(\hat{v}_{n}-\hat{v}\right)\right\| \leq\left\|\chi_{n} P \hat{v}\right\|+\left\|\hat{v}_{n}-\hat{v}\right\| \rightarrow 0,
\end{aligned}
$$

as $n \rightarrow \infty$, by construction. This shows that $i_{\omega}$ has dense range. So the map $i_{\omega}$ extends to $\mathfrak{h}_{B}$ the closure of $\left(\underline{\mathcal{S}}_{0}, B(\cdot, \cdot)\right)$ and yields a bijective isometry $\mathfrak{h}_{B} \rightarrow\{v \in \mathfrak{v}: \operatorname{Im} v=0\}$. It follows using Lemma B. 2 that $\varphi \circ i_{\omega}^{-1}$ is a Gaussian random process indexed by $\{v \in \mathfrak{v}$ : $\operatorname{Im} v=0\}$ with probability measure space $\left(\underline{\mathcal{S}}^{\prime}, \nu\right)$. Thus it follows from Theorem A. 6 that there exists a unique unitary transformation $V_{\mathfrak{v}}: \mathcal{F}_{s}(\mathfrak{v}) \rightarrow L^{2}\left(\underline{\mathcal{S}}^{\prime}, d \nu\right)$ with $V_{\mathfrak{v}} \Omega=1$ and
$V_{\mathfrak{v}}\left(\overline{a^{*}(h)+a(h)}\right) V_{\mathfrak{v}}^{-1}=\varphi(h)$ for all $h \in i_{\omega} \underline{\mathcal{S}}_{0}\left(\right.$ since $\{v \in \mathfrak{v}: \operatorname{Im} v=0\}_{\mathbb{C}}=\mathfrak{v}$ and $i_{\omega} \underline{\mathcal{S}}_{0}$ is dense in $\{v \in \mathfrak{v}: \operatorname{Im} v=0\}$ ). This shows the first part of the theorem. The last statement of the theorem now follows form part (b) of Theorem A.6.

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