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Optimal Stabilization of Periodic Orbits ⋆

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Abstract: In this contribution the optimal stabilization problem of periodic orbits is studied in a symplectic geometry setting. For this, the stable manifold theory for the point stabilization case is generalized to the case of periodic orbit stabilization. Sufficient conditions for the existence of a normally hyperbolic invariant manifold (NHIM) of the Hamiltonian system are derived. It is shown that the optimal control problem has a solution if the related periodic Riccati equation has a unique periodic solution. For the analysis of the stable and unstable manifold a coordinate transformation is used which is moving along the orbit. As an example, an optimal control problem is considered for a spring-mass oscillator system, which should be stabilized at a certain energy level.

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1. INTRODUCTION

Rendering a periodic orbit asymptotically stable employing the control input can be regarded as a very fundamental control problem. The problem statement is motivated by various examples in engineering such as, for example, the control of a satellite motion along a particular mission dependent orbit (Gurfil and Seidelmann (2016)) or in a hybrid setting to realize walking and running motions of bipedal robots (Westervelt et al. (2003)). Moreover, the control problem also arises in the context of Lotka-Volterra equations which describe chemical reactions, predator and pray populations in biology and models in economic theory.

Nevertheless, orbit stabilization can be still regarded as an open research topic of today especially when an optimal control is sought. The orbital stabilization problem was for example recently approached using IDA-PBC in Yi et al. (2020). Furthermore, the immersion and invariance framework was adapted to orbital stabilization (Ortega et al. (2020)). The optimal submanifold stabilization problem was studied by Montenbruck and Allgöwer (2017) which can also be applied to periodic orbits. The authors show that the adjoint equation of the Hamiltonian formalism associated with the submanifold stabilization problem exhibits structured solutions and results in structured state feedbacks.

The contributions of this paper are the following. Despite the fact that the Hamiltonian formalism was employed in Montenbruck and Allgöwer (2017), the approach was not using any tools from symplectic geometry. One of the main contributions of this paper is the formulation of the optimal orbit stabilization problem using symplectic geometric tools. The work is inspired and motivated by the stable manifold method proposed by Sakamoto and van der Schaft (2008), where it was shown that for the point stabilization the stable manifold and therefore the feedback can be approximated using an iterative algorithm. To show the existence of a stable manifold, the concept of normally hyperbolic invariant manifolds (NHIMs) is introduced, which generalizes the isolated hyperbolic equilibrium point for the regulation case. Another novelty of the present paper is to employ a moving coordinate system (see, e.g., Hale (1980)) along the periodic orbit, with which the stabilization of the periodic orbit can be handled more naturally than in the work of Habaguchi et al. (2015) on periodic orbit optimal stabilization where the closed loop trajectory tracks the periodic orbit with a specific initial time. Furthermore, the periodic Riccati equation studied by Kano and Nishimura (1979, 1985) plays a central role to find sufficient conditions for the existence of a solution of the periodic orbit stabilization problem. This coincides with the results established in periodic linear control. Here, the periodic Riccati equation arises in solving the periodic linear quadratic regulator problem with quadratic cost (see Gusev et al. (2010)). The proposed method can be thought of as a nonlinear extension of the orbit stabilization by a periodic linear quadratic regulator. It utilizes the periodic Riccati equation to determine the control law in the neighborhood of the periodic orbit and employs the Hamiltonian flow to extend the stabilizing control law to a larger region. Moreover, an example is presented to elaborate the theoretical approach and make it more accessible to control engineers, who are not familiar with symplectic geometry.

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The remainder of the paper is organized as follows. Section 2 introduces the geometric foundations of the orbit stabilization problem employing tools from symplectic geometry. Transversal coordinates are introduced in Section 3 in order to derive sufficient conditions for the existence of a NHIM. Thereupon, the presented theory is applied in Section 4 to a mass spring system which should be stabilized at the orbit determined by an energy level of one. The paper is summarized with concluding remarks in Section 5.

2. PROBLEM STATEMENT AND MOTIVATION

Let $\mathbb{R}^n$ be an n-dimensional euclidean state space with coordinates $z^1, \dots, z^n$ and consider a smooth dynamical system

$$\dot{z} = f(z). \tag{1}$$

Let $\mathcal{S} \subset \mathbb{R}^n$ be a closed curve in $\mathbb{R}^n$ representing the periodic orbit of (1). By normalizing time, the period is set to 1. In Section 3 a parametrization of $\mathcal{S}$ is used;

$$\mathcal{S} = \{z \in \mathbb{R}^n | z = \gamma(\theta),\ 0 \leqslant \theta \leqslant 1\}, \tag{2}$$

where γ is a periodic and continuous differentiable function, which is one-to-one on the interval $[0, 1)$. We consider the control problem for (1):

$$\dot{z} = f(z) + g(z)u, \tag{3}$$

where g is a $n \times m$ matrix consisting of m smooth vector fields as columns and $u \in \mathbb{R}^m$ represents the control inputs. More precisely, the control problem considered in the present paper is given as follows.

Problem 1. For system (3), design a feedback law that stabilizes $\mathcal{S}$ in an optimal manner such that the cost functional

$$J = \int_0^\infty q(z) + u^\top R u \, dt \tag{4}$$

is minimized.

Here, the function $q(z) \geq 0$ is used to ensure the convergence to the orbit and satisfies $q(z) = 0$ as well as $\frac{\partial q}{\partial z}(z) = 0$ for all $z \in \mathcal{S}$. Furthermore, it is assumed that q is positive semi-definite along $\mathcal{S}$ according to Definition 1 given in Section 3. Moreover, the input is weighted using the positive definite matrix R. By stabilizing it is meant to render $\mathcal{S}$ orbitally asymptotically stable.

One can immediately derive a Hamilton-Jacobi-Bellman equation (HJBE) for the optimal control problem (3)-(4), which is

$$p^\top f(z) - \frac{1}{4} p^\top g(z) R^{-1} g(z)^\top p + q(z) = 0. \tag{5}$$

Here, p denotes the momentum variable, i.e., the vector of (generalized) momenta. In what follows, we set up an appropriate geometry to analyze the HJBE (5) using symplectic geometry. The cotangent bundle $T^*\mathbb{R}^n$ of $\mathbb{R}^n$ equipped with the canonical symplectic form $\omega = \sum_{i=1}^n dz^i \wedge dp_i$ is a symplectic manifold, where (z, p) denote the coordinates of $T^*\mathbb{R}^n$. To embedd the cotangent bundle $T^*\mathcal{S}$ of $\mathcal{S}$ into $T^*\mathbb{R}^n$ the pullback by the differential of the inclusion map $di^* : T^*\mathcal{S} \to T^*\mathbb{R}^n$ is employed [1]. The set $di^*(T^*\mathcal{S})$ is a symplectic submanifold of $(T^*\mathbb{R}^n, \omega)$, which is contained in a single zero level set of a smooth real-valued function F defined on a subset of $T^*\mathbb{R}^n$. The function F, corresponding to the left-side of (5), can be employed to define a first order partial differential equation (PDE) of the form

$$F(z^1, \dots, z^n, p_1, \dots, p_n) = 0, \tag{6}$$

where $p_i = \frac{\partial V}{\partial z^i}$ for some smooth function $V \in \mathcal{C}^\infty(\mathbb{R}^n)$ with $i \in \{1, \dots, n\}$. This type of PDE that involves only the first derivatives of the unknown function V, but not the values of the function itself, is called Hamilton-Jacobi equation or eikonal equation, see for example Lee (2012). Solving (6) can be reduced to finding a closed 1-form α satisfying $F(z, \alpha(z)) = 0$. Having found such an α, the Poincaré lemma guarantees that locally $\alpha = dV$ for some smooth function V, which satisfies (6). The closedness property could alternatively be replaced by requiring that the image of α is a Lagrangian submanifold of $T^*\mathbb{R}^n$.

By the nondegeneracy of ω, the function F determines a unique vector field X_F defined by

$$dF = \iota_{X_F}\omega, \tag{7}$$

where $\iota_{X_F}\omega(\cdot) = \omega(X_F, \cdot)$. The 3-tuple $(T^*\mathbb{R}^n, \omega, F)$ defines a Hamiltonian system with Hamiltonian vector field X_F and flow $\Phi_F : \mathbb{R} \times T^*\mathbb{R}^n \to T^*\mathbb{R}^n$. The flow starting at (z, p) with $t = 0$ is denoted by $\Phi_F(t, (z, p))$. In coordinates the flow is given by

$$\dot{z} = f(z) - \frac{1}{2} g(z) R^{-1} g(z)^\top p, \tag{8}$$

$$\dot{p} = -\frac{\partial f(z)}{\partial z}^\top p + \frac{1}{4}\left(\frac{\partial p^\top g(z) R^{-1} g(z)^\top p}{\partial z}\right)^\top - \frac{\partial q(z)}{\partial z}^\top. \tag{9}$$

In the set point stabilization problem, which was for example considered in Sakamoto and van der Schaft (2008), a solution V of the HJBE is called a stabilizing solution if $p(0) = \partial V(0)/\partial z^\top = 0$ and 0 is an asymptotically stable equilibrium point of the vector field given by (8).

In this case, the origin given by $(z, p) = 0$ is a hyperbolic equilibrium point of the Hamiltonian flow. For the orbit stabilization case, there is not a single isolated equilibrium point but a set generated by the orbit $\mathcal{S}$ which is invariant under X_F. The following Lemma 2 puts this into concrete terms.

Lemma 2. Let M be the zero section of the subbundle $di^*(T^*\mathcal{S}) \subset T^*\mathbb{R}^n$, i.e., $M = \{(z, p) \in T^*\mathbb{R}^n | z \in \mathcal{S}, p = 0\}$. Then, M is invariant under the Hamiltonian flow of X_F.

Proof. One has to show that for all $(z, p) \in M$ the Hamiltonian flow $X_F(z, p) \in T_{(z,p)}M$. In M the momentum p is zero and it follows that $\dot{p} = -\frac{\partial q(z)}{\partial z}^\top = 0$. Furthermore, this implies $\dot{z} = f(z)$ in M. The fact that $\mathcal{S}$ is invariant under f completes the proof. □

The invariant set M of Lemma 2 can be regarded as a periodic orbit of the vector field X_F. Taking the point stabilization case as a pattern, one could require for a stabilizing solution of (5) that M is a hyperbolic periodic orbit. In the following, it is shown that this is not the case.

2.1 Stability of Periodic Orbits and First Integrals

The stability of periodic orbits is usually analyzed by the Poincaré map or Floquet theory. In the case of a hyperbolic

[1] The closed curve $\mathcal{S}$ is an embedded submanifold of $\mathbb{R}^n$, that is, an immersed submanifold for which the inclusion map $i : \mathcal{S} \to \mathbb{R}^n$ is a topological embedding.

periodic orbit the number one is a Floquet multiplier, i.e., an eigenvalue of the fundamental matrix solution, with algebraic multiplicity of one (Chicone (2006)). Therefore, the stability can be straightforwardly determined by analyzing whether the remaining Floquet multipliers are strictly less than one. The following theorem states that for the Hamiltonian system $(T^*\mathbb{R}^n, \omega, F)$ the periodic orbit is not hyperbolic. This is due to the fact, that the Hamiltonian system has a first integral given by the Hamiltonian itself. For more details the reader is referred to Chicone (2006) and Meyer and Offin (2017).

Theorem 3. The periodic orbit characterized by M defined in Lemma 2 with the Hamiltonian flow given by (8)-(9) has a Floquet multiplier of value one with algebraic multiplicity ≥ 2.

Proof. Let $\xi \in T^*\mathbb{R}^n$ and Σ be a section transverse to $X_F(\xi)$. Moreover, let $\Phi_F(t,\zeta)$ be the solution starting at ζ and $T(\zeta)$ is the time of first return for Σ. Furthermore, the variational equation is defined as

$$\dot{W} = DX_F(\Phi_F(t,\xi))W. \tag{10}$$

The principal fundamental matrix solution of the initial value problem for the variational equation with $W(0) = I$ is denoted by $\Psi(t,\xi)$ and given by $\Psi(t,\xi) = \partial\Phi_F(t,\xi)/\partial\zeta$. It is obvious, that $X_F(\Phi_F(t,\xi))$ is a solution of (10). As $\Psi(t,\xi)$ is the principal fundamental matrix solution it follows

$$\begin{aligned}\Psi(T(\xi),\xi)X_F(\xi) &= \Psi(T(\xi),\xi)X_F(\Phi_F(0,\xi))\\ &= X_F(\Phi_F(T(\xi),\xi))\\ &= X_F(\xi).\end{aligned}$$

Therefore, $X_F(\xi)$ is an eigenvector of $\Psi(T(\xi),\xi)$. Since F is a first integral, one has

$$F(P(\xi)) = F(\xi), \tag{11}$$

where $P(\zeta) = \Phi_F(T(\zeta),\zeta)$ is the Poincaré map and it follows that

$$\frac{\partial F}{\partial \zeta}\Psi(T(\xi),\xi) = \frac{\partial F}{\partial \zeta}. \tag{12}$$

Therefore, the $\partial F(\xi)/\partial\zeta$ is an eigenvector of the transposed fundamental matrix at $T(\xi)$ and ξ with eigenvalue one. The orthogonality of $X_F(\xi)$ and $\partial F(\xi)/\partial\zeta$ follows from the fact that the Hamiltonian is constant along $X_F(\xi)$. With this, algebraic multiplicity of at least two can be shown by Laplace expansion of $\det(\Psi(T(\xi),\xi)-\lambda I)$ in a coordinate system such that $X_F(\xi)$ and $\partial F(\xi)/\partial\zeta^\top$ correspond to the first and second unit vector, respectively. □

In the following, a change of coordinates is performed which splits the dynamics into four parts. This form enables new insights in the local dynamics near M. From this structure it becomes clear that a stabilizing solution requires that M is a NHIM. In particular, the orbit $\mathcal{S}$ is a hyperbolic and asymptotically stable periodic orbit of the vector field $f(z) - g(z)Rg(z)^\top \frac{\partial V}{\partial z}(z)^\top$.

3. EXISTENCE OF A STABLE MANIFOLD AND THE PERIODIC RICCATI EQUATION

3.1 Change to Transverse Coordinates

In the following, a local orthogonal coordinate system about the periodic orbit is used. Please see Chapter VI.1 of Hale (1980) for a detailed derivation of such a coordinate system. We use the parametrization $\gamma(t)$ of (2). As in Hale (1980), a moving orthonormal system about $\mathcal{S}$ is constructed using

$$e_1 = \frac{\frac{\partial\gamma(\theta)}{\partial\theta}}{\|\frac{\partial\gamma(\theta)}{\partial\theta}\|}$$

together with $n-1$ additional orthonormal vectors $e_2,\ldots,e_n$. With this, the new coordinates are defined by

$$z = \psi(x) = \gamma(x_1) + Z(x_1)x_2, \tag{13}$$

for $0 \leq x_1 \leq 1$ and $Z = [e_2,\ldots,e_n]$. The new coordinate vector x is decomposed of the two components $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}^{n-1}$. Applying the transformation (13) to (3) yields

$$\dot{x}_1 = 1 + f_1(x_1,x_2) + g_1(x_1,x_2)u, \tag{14a}$$
$$\dot{x}_2 = A(x_1)x_2 + f_2(x_1,x_2) + g_2(x_1,x_2)u, \tag{14b}$$

where $\|f_1(x_1,x_2)\| \in \mathcal{O}(\|x_2\|)$ as $\|x_2\| \to 0$ and $f_2(x_1,0) = 0$ as well as $\frac{\partial f_2}{\partial x_2}(x_1,0) = 0$ for all $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}^{n-1}$.

Fact 4. A, f_1, f_2, g_1, g_2 are all period-1 functions in x_1 for $x_2 = 0$.

Definition 1. The function q is called positive semi-definite along $\mathcal{S}$ if the Hessian with respect to x_2 is positive semi-definite and has the form $\frac{\partial^2 q\circ\psi}{\partial x_2\partial x_2}(x) = Q(x_1)$ for $x \in \mathcal{S}$.

The cost functional expressed in the new coordinates yields

$$J = \int_0^\infty q(\psi(x)) + u^\top Ru\,dt, \tag{15}$$

is considered to be minimized for input u and the corresponding control Hamiltonian is given by

$$\begin{aligned}H_d(x,p,u) = p_1(1 + f_1(x_1,x_2) + g_1(x_1,x_2)u)\\ +p_2^\top(A(x_1)x_2 + f_2(x_1,x_2) + g_2(x_1,x_2)u)\\ +q(\psi(x)) + u^\top Ru,\end{aligned}$$

where $p_1 \in \mathbb{R}$ and $p_2 \in \mathbb{R}^{n-1}$ are the momentum vectors corresponding to x_1, x_2, respectively. The optimality condition on u implies

$$\frac{\partial H_d}{\partial u} = p_1 g_1(x_1,x_2) + p_2^\top g_2(x_1,x_2) + 2u^\top R = 0.$$

It follows that a minimizing u is given by

$$u = -\frac{1}{2}R^{-1}(g_1(x_1,x_2)p_1 + g_2(x_1,x_2)^\top p_2). \tag{16}$$

The Hamilton-Jacobi equation in the new coordinates is

$$\begin{aligned}H(x,p) = p_1 + p_1 f_1(x_1,x_2) - \frac{1}{4}G(x,p)^\top R^{-1}G(x,p)\\ + p_2^\top(A(x_1)x_2 + f_2(x_1,x_2)) + q(\psi(x)) = 0,\end{aligned} \tag{17}$$

where $G(x,p) = g_1(x_1,x_2)p_1 + g_2(x_1,x_2)^\top p_2$. The corresponding Hamiltonian system is

$$\dot{x}_1 = 1 + f_1(x) - \frac{1}{2} g_1(x_1, x_2) R^{-1} G(x, p), \tag{18a}$$

$$\dot{x}_2 = A(x_1)x_2 + f_2(x) - \frac{1}{2} g_2(x_1, x_2) R^{-1} G(x, p), \tag{18b}$$

$$\dot{p}_1 = -p_1 \frac{\partial f_1}{\partial x_1}(x) + \frac{1}{4} \frac{\partial}{\partial x_1}(G(x,p)^\top R^{-1} G(x,p)) - p_2^\top A'(x_1)x_2 - p_2^\top \frac{\partial f_2}{\partial x_1}(x) - \frac{\partial q \circ \psi}{\partial x_1}(x), \tag{18c}$$

$$\dot{p}_2 = -p_1 \frac{\partial f_1}{\partial x_2}(x)^\top + \frac{1}{4}\left(\frac{\partial}{\partial x_2}(G(x,p)^\top R^{-1} G(x,p))\right)^\top - A(x_1)^\top p_2 - \frac{\partial f_2}{\partial x_2}(x)^\top p_2 - \frac{\partial q \circ \psi}{\partial x_2}(x)^\top. \tag{18d}$$

3.2 Existence of a Stable Manifold

Let $M = \{(x,p) \,|\, (x_1, x_2) = (x_1, 0) \in \mathcal{S}, p_1 = 0, p_2 = 0\}$ as previously defined in Lemma 2 and consider a period-1 differential Riccati equation

$$\dot{P}(t) + A(t)^\top P(t) + P(t)A(t) - P(t)\bar{R}(t)P(t) + Q(t) = 0, \tag{19}$$

where $\bar{R}(t) = \frac{1}{2} g_2(t,0) R^{-1} g_2(t,0)^\top$.

Proposition 5. Given the periodic Riccati equation (19) has a positive semi-definite period-1 solution $P(t)$ with $A(t) - \bar{R}(t)P(t)$ being asymptotically stable. Then, M is a normally hyperbolic invariant manifold (NHIM) of (18) according to Definition A.1.

Proof. The invariance of M is proved in Lemma 2. As M is the continuous image of $[0,1]$ under γ it is compact. As for Definition A.1, we only show that the flow of (18) satisfies (A.3). Showing that the second and third inequalities in (A.3) hold is equivalent to considering

$$\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} A(t) & -\bar{R}(t) \\ -Q(t) & -A(t)^\top \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}, \tag{20}$$

and identifying a stable fiber (corresponding to E_x^s in Definition A.1). We denote the $2(n-1) \times 2(n-1)$ period-1 matrix from above as $\mathrm{Ham}(t)$. Note that it satisfies

$$J\mathrm{Ham}(t) + \mathrm{Ham}(t)^\top J = 0, \tag{21}$$

where $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ and I is the identity matrix of $n-1$ dimension. Let $z_j(t)$, $j = 1, \ldots, n-1$, be independent solutions to

$$\dot{z} = (A(t) - \bar{R}(t)P(t))z.$$

It can be shown that the vectors $\left[\begin{smallmatrix} z_j(t) \\ P(t)z_j(t) \end{smallmatrix}\right]$, $j = 1, \ldots, n-1$ span E_x^s for $x \in M$ (note that from Fact 4 we identify x_1 with t). Indeed, for any solution $z(t)$ we have

$$\begin{aligned} &\frac{d}{dt}\begin{bmatrix} z(t) \\ P(t)z(t) \end{bmatrix} \\ &= \begin{bmatrix} (A(t) - \bar{R}(t)P(t))z(t) \\ \dot{P}z + P\dot{z} \end{bmatrix} \\ &= \begin{bmatrix} (A - \bar{R}P)z \\ -(PA + A^\top P - P\bar{R}P + Q)z + P(A - \bar{R}P)z \end{bmatrix} \\ &= \begin{bmatrix} A(t) - \bar{R}(t)P(t) \\ -Q(t) - A(t)^\top P(t) \end{bmatrix} z \\ &= \begin{bmatrix} A(t) & -\bar{R}(t) \\ -Q(t) & -A(t)^\top \end{bmatrix} \begin{bmatrix} z(t) \\ P(t)z(t) \end{bmatrix}, \end{aligned}$$

where we have used (19). Thus, we have $z_j(t) \to 0$ as $t \to \infty$ and since $P(t)$ is periodic, the $n-1$ vectors $\left[\begin{smallmatrix} z_j(t) \\ P(t)z_j(t) \end{smallmatrix}\right]$, $j = 1, \ldots, n-1$ span E_x^s for $x \in M$. Let $\Phi_H(t,s)$ be the fundamental matrix for the linear differential equation (20) and set $\zeta_j = \left[\begin{smallmatrix} z_j(0) \\ P(0)z_j(0) \end{smallmatrix}\right]$, $j = 1, \ldots, n-1$. Then we have

$$\begin{bmatrix} z_j(t) \\ P(t)z_j(t) \end{bmatrix} = \Phi_H(t,0)\zeta_j, \ j = 1, \ldots, n-1.$$

The fact we have just shown means that for $k \in \mathbb{N}$

$$\Phi_H(k,0)\zeta_j \to 0 \text{ as } k \to \infty, \ j = 1, \ldots, n-1.$$

From the periodicity, we have $\Phi_H(k,0) = \Phi_H(1,0)^k$. Now, we can show from (21) that if $\lambda \in \mathbb{C}$ is an eigenvalue of $\Phi_H(1,0)$, so is $1/\lambda$ (detail is omitted). Thus we conclude that $\zeta_1, \ldots, \zeta_{n-1}$ belong to the generalized eigenspace corresponding to the eigenvalues of $\Phi(1,0)$ with $|\lambda| < 1$. This enables one to show that $\dim E_x^s = n-1$ and there are constants $C > 0$ and $0 < \rho < 1$ such that for $k \in \mathbb{N}$

$$\left|\begin{bmatrix} z_j(k) \\ P(k)z_j(k) \end{bmatrix}\right| < C\rho^k, \ j = 1, \ldots, n-1,$$

from which the second estimate in (A.3) is derived. The derivation of the third estimate is omitted. □

Using Theorem A.1, we have the following result.

Theorem 6. Under the same assumption as in Proposition 5, $W^s(M)$, the stable manifold of M, is a Lagrangian submanifold of $T^*\mathbb{R}^n$. It also satisfies the following:

$$\pi|_{W^s(M)} \text{ is surjective locally around } M, \tag{22}$$

where $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$, $\pi(x,p) = x$ is the canonical projection.

Proof. The Lagrangian submanifold property can be proved in the same way as in van der Schaft (1991). To prove property (22), one notices that $z_j(t)$, $j = 1, \ldots, n-1$, in the proof of Proposition 5 are independent solutions and they sweep an n dimensional surface as t (or x_1) moves. □

Now we wish to express the solvability condition for Problem 1 in terms of system (3) and the cost function (4). To this end, we first note the following relationship between (3) and (14) in their linearizations.

Proposition 7. Let $A_0(t) = Df(\gamma(t))$, $B_0(t) = g(\gamma(t))$, $Q_0(t) = \frac{\partial^2 q}{\partial z^2}(\gamma(t))$ in (3). Then, $(A(t), g_2(t,0))$ is stabilizable if $(A_0(t), B_0(t))$ is stabilizable in the sense of Definition B.2. Also, $(Q(t), A(t))$ is detectable if $(Q_0(t), A_0(t))$ is detectable in the sense of Definition B.2.

Now, we are in the position to state our main result which is the sufficient condition for the solvability of Problem 1 in terms of the original system (3).

Theorem 8. If the pair $(A_0(t), B_0(t))$ is stabilizable and $(Q_0(t), A_0(t))$ is detectable, then HJBE (17) has a solution $V(x)$ defined in a neighborhood of $\mathcal{S}$, namely, Problem 1 has a local solution. More precisely, in the closed-loop system consisting of the dynamical system (3) with feedback given by

$$\begin{aligned} u^*(z) &= (u_{\mathrm{opt}} \circ \psi^{-1})(z); \\ u_{\mathrm{opt}}(x_1, x_2) &= -\frac{1}{2} R^{-1}(g_1(x_1,x_2)(\partial V/\partial x_1)(x_1,x_2) \\ &\quad + g_2(x_1,x_2)^\top (\partial V/\partial x_2)^\top (x_1,x_2)) \end{aligned}$$

(see (13) and (16)), the periodic orbit $\mathcal{S}$ is locally asymptotically orbitally stable and the stabilization is optimal in the sense that (4) is minimized.

Proof. From the conditions and Theorem B.1, (19) has a stabilizing solution and therefore M is an NHIM of the Hamiltonian system (18) by Proposition 5. Then, the Hamiltonian system has a Lagrangian submanifold Λ in a neighborhood of M which is surjectively projectable to the base space (x_1, x_2) (Theorem A.1). From this and the closedness of Λ, there is a function $V(x)$ defined in a neighborhood of $\mathcal{S}$ such that $p = dV$ on Λ. This $V(x)$ is a local solution of HJBE (17). The rest of the theorem has been shown from the construction of V.

4. ENERGY CONTROL EXAMPLE

In the following a mass-spring system is considered, which dynamics are given by

$$\dot{z}_1 = z_2, \tag{23}$$
$$\dot{z}_2 = -z_1 + u, \tag{24}$$

where u denotes the system input. The goal of the control is to stabilize the energy level corresponding to one, i.e., $z_1^2 + z_2^2 - 1 = 0$. The cost function to be minimized is defined as

$$J = \int_0^\infty (z_1^2 + z_2^2 - 1)^2 + u^2 dt. \tag{25}$$

In the following a transversal coordinate system along the orbit is used. For this, a point transformation $x = \phi(z)$ defined by

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\arctan(\frac{z_2}{z_1}) \\ \frac{1}{2}(z_1^2 + z_2^2) - 1 \end{bmatrix}, \tag{26}$$

is used. Here, x_1 and x_2 are the new coordinates along the orbit for the mass-spring system. In the new coordinates, the dynamics yields

$$\dot{x}_1 = 1 - \cos(x_1)u, \tag{27}$$
$$\dot{x}_2 = (2x_2 + 1)\sin(x_1)u. \tag{28}$$

The Hamilton-Jacob equation is straightforwardly derived as

$$H(x, p) = p_1 - \frac{1}{4}G(x, p)^2 + x_2^2, \tag{29}$$

where $G(x, p) = (-p_1 \cos(x_1) + p_2 (2x_2 + 1) \sin(x_1))$. Moreover, the Hamiltonian flow results in

$$\dot{x}_1 = 1 + \frac{1}{2}G(x, p)\cos(x_1), \tag{30}$$
$$\dot{x}_2 = -\frac{1}{2}(2x_2 + 1)G(x, p)\sin(x_1), \tag{31}$$
$$\dot{p}_1 = \frac{1}{4}(2p_1 \sin(x_1) + 2p_2(2x_2 + 1)\cos(x_1))G(x, p), \tag{32}$$
$$\dot{p}_2 = p_2 G(x, p)\sin(x_1) - 2x_2. \tag{33}$$

Correspondingly to (20), the linear transverse dynamics is

$$\begin{bmatrix} \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2}\sin^2(t) \\ -2 & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}. \tag{34}$$

Theorem 6 is employed to verify that there exits a stable manifold. For this, it is required that the pair of $A(t) = 0$ and $B(t) = \frac{\sin(t)}{\sqrt{2}}$ is stabilizable. Moreover, the pair of $C_0(t) = \sqrt{2}$ and $A(t) = 0$ has to be detectable. While the latter is obvious, the stabilizability is verified by showing that the controllability gramian $W_c(t_0, t_1)$ is invertible for some $t_1 > t_0$ in the following. The controllability gramian for $t_1 > t_0$ yields

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t)(t_1-\tau)} B(\tau)B(\tau)^\top e^{A(t)(t_1-\tau)^\top} d\tau,$$
$$= \frac{1}{2}\int_{t_0}^{t_1} \sin^2(\tau) d\tau > 0,$$

and it follows that $(A(t), B(t))$ is controllable (and therefore stabilizable). With Theorem B.1 in the Appendix one concludes that there is a unique periodic positive semi-definite solution of the periodic Riccati equation. Thus, $M = \{(x, p) \in T\mathbb{R}^2 | x_1^2 + x_2^2 = 1, p = 0\}$ is a NHIM. The optimal control problem was solved using numerical optimization and the resulting trajectories in $x_1 - x_2$ plane are depicted in Fig. 1. Each trajectory corresponds to the Hamiltonian flow along the stable manifold $W^s(M)$ projected onto $x_1 - x_2$ plane by the canonical projection π. The control input which is required for stabilization is plotted over time in Fig. 2 for selected initial conditions.

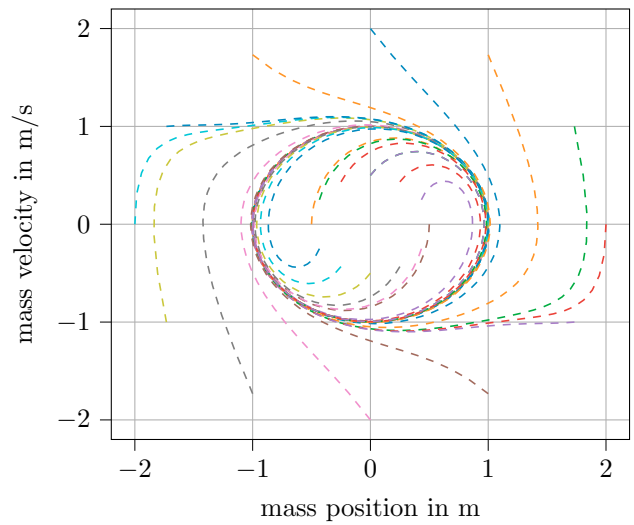

Fig. 1. Trajectories corresponding to the Hamilotian flow along the stable manifold projected onto x_1-x_2 plane.

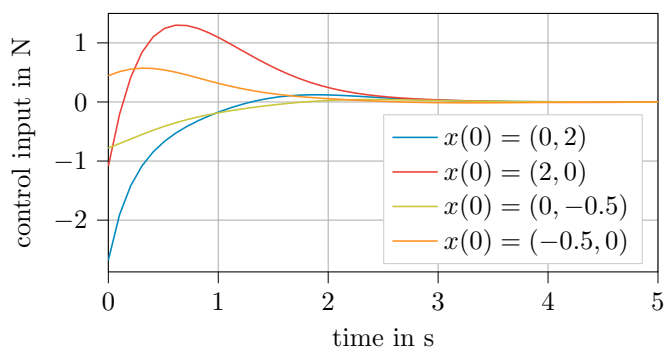

Fig. 2. Plot of the control input for selected initial conditions.

5. CONCLUSION AND OUTLOOK

We have proposed a framework for the optimal stabilization problem of periodic orbits that inherently exist in dynamical systems (control system with zero input).

It employs tools from symplectic geometry and dynamical system theory such as normally hyperbolic invariant manifolds. A sufficient condition for the solvability of the optimal control problem (Problem 1) is obtained in terms of stabilizability and detectability of the linearization of the periodic orbit with a given cost function, which is a natural extension of pioneering works by van der Schaft (1991, 1992) for optimal point stabilization.

The future work includes the application of the theory to practical (engineering) problems and numerical computations of optimal feedback laws, for which a generalization of the iterative computation in Sakamoto and van der Schaft (2008); Sakamoto (2013) is expected (see Horibe and Sakamoto (2017, 2019) for its applications with experiment).

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Appendix A. INVARIANT MANIFOLD THEORY

Let us consider a C^r dynamical system in $\mathbb{R}^N$

$$\dot{x} = F(x) \tag{A.1}$$

and let $\Phi(t,x)$ be its flow. Suppose that a manifold $M \subset \mathbb{R}^N$ is invariant under (A.1), namely, $\Phi(t,M) = M$ for $t \in \mathbb{R}$. Normal hyperbolicity of M with respect to (A.1) is defined as follows (see, e.g., Eldering (2013)).

Definition A.1. The invariant manifold M is called *normally hyperbolic invariant manifold* (NHIM) of (A.1) if Φ satifies the following properties: There exists a continuous Φ-invariant splitting

$$T\mathbb{R}^N\big|_M = TM \oplus E^s_M \oplus E^u_M \tag{A.2}$$

of the tangent bundle $T\mathbb{R}^N$ over M such that there exist positive constants α, C_M, C_s, C_u such that for $x \in M$,

$$\begin{aligned} |D\Phi(t,x)u| &\leqslant C_M e^{\alpha|t|}|u| \text{ for } u \in T_xM, t \in \mathbb{R} \\ |D\Phi(t,x)v| &\leqslant C_s e^{-\alpha t}|v| \text{ for } v \in E^s_x, t \geqslant 0 \\ |D\Phi(t,x)w| &\leqslant C_u e^{\alpha t}|w| \text{ for } w \in E^u_x, t \leqslant 0. \end{aligned} \tag{A.3}$$

Here, split (A.2) is called Φ-invariant if

$$\begin{aligned} D\Phi(t,x)(T_xM) = T_{\Phi(t,x)}M, \quad D\Phi(t,x)(E^s_x) = E^s_{\Phi(t,x)}, \\ D\Phi(t,x)(E^u_x) = E^u_{\Phi(t,x)} \end{aligned} \tag{A.4}$$

hold for $x \in M$ and $t \in \mathbb{R}$ and the continuity of the split means that as x varies in M one can find continuously varying bases in E^s_x and E^u_x.

The following theorem states that a compact NHIM M has a stable manifold, which consists of a collection of stable fibers (see e.g., (Guckenheimer and Holmes, 1985, p. 246)).

Theorem A.1. Let M be a compact NHIM for (A.1). Then, there is an $\varepsilon > 0$ and there is a collections of C^r manifold $W^s_\varepsilon(x)$, $x \in M$ with the following properties.

(i) $y \in W^s_\varepsilon(x) \iff |\Phi(t,x) - \Phi(t,y)| < \varepsilon$ for $t \geqslant 0$.
(ii) The tangent space of $W^s_\varepsilon(x)$ at x is E^s_x.
(iii) There are positive constants β, C such that if $y \in W^s_\varepsilon(x)$, then $|\Phi(t,x) - \Phi(t,y)| < Ce^{-\beta t}$ for $t \geqslant 0$.
(iv) $W^s_\varepsilon(x)$ is a k-dimensional disk, namely, homeomorphic to $\{(x_1, \ldots, x_k) \,|\, x_1^2 + \cdots + x_k^2 \leqslant 1\}$, where $k = \dim E^s_x$.

The stable manifold $W^s(M)$ of M can be represented using $W^s_\varepsilon(x)$ as

$$W^s(M) = \bigcup_{x \in M} \bigcup_{t \leqslant 0} \Phi(t, W^s_\varepsilon(x)).$$

Appendix B. THEORY OF LINEAR PERIODIC SYSTEMS

In this section, we review basic facts on linear periodic system theory such as periodic differential Riccati equations. For detail, see Kano and Nishimura (1979, 1985).

Let $A(t)$, $B(t)$, $C(t)$ be T-periodic real matrices of $n \times n$, $n \times m$ and $r \times n$ dimensions, respectively. Let $\Phi_A(t,s)$ be the fundamental matrix for the differential equation $\dot{x}(t) = A(t)x(t)$, namely,

$$\frac{\partial}{\partial t}\Phi_A(t,s) = A(t)\Phi_A(t,s), \quad \Phi_A(t,t) = I.$$

The Floquet theory (see, e.g., Hale (1980)) says that $\Phi_A(t,s)$ satisfies T-periodicity $\Phi_A(t+T, s+T) = \Phi_A(t,s)$ for $t, s \in \mathbb{R}$.

Definition B.1. A T-periodic square matrix $A(t)$ is said to be asymptotically stable if the corresponding linear differential equation $\dot{x} = A(t)x$ is asymptotically stable.

Now, let us consider a T-periodic linear control system

$$\dot{x} = A(t)x + B(t)u, \; y = C(t)x, \tag{B.1}$$

where $u(t) \in \mathbb{R}^m$ is the control input and $y(t) \in \mathbb{R}^r$ is the output. The stabilizability and detectability, which play an important role in linear time-invariant control systems, are defined for (B.1) as follows.

Definition B.2. The pair $(A(t), B(t))$ is said to be stabilizable if there exists a continuous T-periodic $m \times n$ matrix $K(t)$ such that $A(t) + B(t)K(t)$ is asymptotically stable, and the pair $(C(t), A(t))$ is said to be detectable if there exists a continuous T-periodic $n \times r$ matrix $G(t)$ such that $A(t) + G(t)C(t))$ is asymptotically stable.

The periodic Riccati equation, which plays the central role in optimal control for (B.1), takes the following form

$$-\dot{P}(t) = P(t)A(t) + A(t)^\top P(t) - P(t)R(t)P(t) - Q(t), \tag{B.2}$$

where $R(t)$, $Q(t)$ are T-periodic positive semi-definite matrices of $n \times n$ dimension.

Theorem B.1. The necessary and sufficient condition for (B.2) to have a unique T-periodic positive semi-definite solution with $A(t) - R(t)P(t)$ being asymptotically stable is that $(A(t), R(t))$ is stabilizable and $(Q(t), A(t))$ is detectable.