Error Detection Strategies for CRC-Concatenated Polar Codes under Successive Cancellation List Decoding

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***Abstract*—In this work we introduce a framework to study the trade-off between the undetected error rate (UER) and overall frame error rate (FER) of CRC-concatenated polar codes in the short blocklength regime. Three approaches to improve the tradeoff under successive cancellation list (SCL) decoding are outlined. Two techniques are based on the optimum threshold test introduced by Forney in 1968, whereas a third technique partitions the CRC code parity bits in two sets, where one set is used to prune the SCL decoder list, and the other set is used for error detection. The performance of the three schemes is analyzed via Monte Carlo simulations, and compared with a finite-length achievability bound based on Forney's random coding bound.**

I. Introduction

In the past years, interest in short codes has been renewed due to a set of emerging applications which require the use of relatively small data units [1], [2]. A prominent example is the 3GPP 5G cellular standard. Among its application scenarios, the 5G standard targets enhanced mobile broadband (eMBB) communications, massive machine-type communication (mMTC) systems, and ultra-reliable low-latency communications (URLLC). Short packet transmission is relevant for all three use classes. While for eMBB small data units are mostly relevant for the control channel, mMTC envisions a large number of devices which regularly transmit small amount of data. For URLLC, delay constraints require the use of short, well-protected data packets. URLLC covers mission-critical applications, e.g., in the context of intelligent mobility, industry automation, and wireless telecommand systems (see e.g. [3] [4, Ch. 4]). On the physical layer of communication systems, reliability is measured by the frequency of decoding errors. In some cases, decoding errors can cause undesired system behaviors [5]. This can happen, for instance, when the decoder outputs a codeword different from the transmitted one and the system is not aware of this incorrect decision. Such undetected errors can be harmful, in particular for mission-critical applications. Hence, the code design may not only aim at controlling the overall frame error rate (FER), but also at keeping the undetected error rate (UER) low.

When long packets are used, an error detection capability is typically ensured by protecting a packet with a cyclic redundancy check (CRC) code, prior to encoding with the error correction code. The CRC code is here used to provide an error detection capability after channel decoding. The addition of the CRC code parity bits causes, for long packets, a negligible rate loss. On the contrary, the use of a CRC code as an error detection code is unappealing for short packets, since acceptable error detection capabilities come at the expense of non-negligible rate losses. An alternative approach to the use of CRC codes can be obtained by embedding an error detection mechanism in the decoding algorithm of the error correcting code. In fact, all incomplete decoders [6, Ch. 1] provide naturally an error detection capability. An optimum incomplete decoding algorithm was introduced and analyzed in [7]. The approach of [7] can be seen as the application of (complete) maximum likelihood (ML) decoding, followed by a post-decoding threshold test. The test is used to either accept or discard the ML decoder decision. A criterion to discard the decision, based on the Neyman-Pearson theorem, was derived and analyzed in terms of error exponents [7], showing its optimality in the sense of minimizing the UER for a given FER (and viceversa). The evaluation of the metric used for the test can be done efficiently for certain classes of linear block codes (e.g., for terminated convolutional codes [8], [9]) or it can be well approximated with limited complexity for other code classes (e.g., for codes based on compact tail-biting trellises [10], [11]). A performance analysis of code ensembles under the generalized decoding rule of [7] was presented in [12]. Sub-optimum post-decoding tests were proposed and analyzed in [13], [14], where they were compared with the performance of the optimum criterion. Heuristic threshold tests were introduced in [15] for the special case of CRC-concatenated polar codes [16], [17] under successive cancellation list (SCL) decoding [18].

In this work, we consider approaches to improve the error detection capability of short CRC-concatenated polar codes. The focus on this code class stems from its excellent performance in the short block length regime with low-complexity SCL decoding [19]. Two error detection methods are based on the threshold test of [7], adapted to SCL decoding. A third approach relies on "splitting" the parity bits of the CRC code: A portion of the bits is used to prune the SCL decoder list, whereas the remaining parity bits are used for error detection. The FER and UER performance of short CRC-concatenated polar codes under the three approaches is analyzed, and

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compared on a finite-length achievability bound based on the Gallager-type random coding bound (RCB) derived in [7].

The rest of paper is organized as follows. Section II provides preliminary definitions. Forney's generalized decoding rule is discussed in Section III, where an achievable finite-length signal-to-noise ratio (SNR) threshold for a given target UER/FER pair is presented. Error detection strategies for CRC-concatenated polar codes under SCL decoding are introduced in Section IV. Numerical results comparing the performance achievable by short concatenated polar codes with the SNR threshold of Section III are illustrated in Section V. A discussion on the role of code design is also outlined. Concluding remarks follow in Section VI.

II. Preliminaries

We denote vectors by small bold letters, e.g., $\boldsymbol{x} = [x_1, ..., x_N]$. Matrices are denoted by capital bold letters, e.g., $\boldsymbol{G}$, sets by capital calligraphic letters, e.g., $\mathcal{L}$. We use $\mathbb{F}_2$ to denote the binary finite field with elements $\{0, 1\}$. Consider transmission over a biAWGN channel with an (N, K) binary linear block code $\mathcal{C}$ with rate $R = K/N$. The channel input alphabet is $\mathcal{X} = \{-1, +1\}$, the noise variance is σ^2, and the channel SNR is given by $E_b/N_0 = 1/(2R\sigma^2)$ where E_b is the energy per information bit and N_0 is the single-sided noise power spectral density. Denote by $\boldsymbol{c} \in \mathbb{F}_2^N$ a codeword of $\mathcal{C}$. The corresponding "modulated codeword" that is input to the biAWGN channel is $\boldsymbol{x}$ with $x_i = \mathsf{B}(c_i)$ for $i = 1, \ldots, N$ where $\mathsf{B}(c_i) := 1 - 2c_i$ denotes the binary antipodal mapping. With a slight abuse of notation, we will write $\boldsymbol{x} \in \mathcal{C}$ when $\boldsymbol{x}$ is a modulated codeword.

A. Polar Codes

Let the binary polar transform be defined by the $N \times N$ matrix

$$\boldsymbol{G}_N = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\otimes n} \tag{1}$$

where the superscript $^{\otimes n}$ denotes the n-fold Kronecker product and $n = \log_2 N$. For an (N, K_I) polar code, the K_I information bits are copied to K_I positions of a length-N input vector $\boldsymbol{u}$. Let the indices of the K_I information bits in $\boldsymbol{u}$ be $\mathcal{A}$ and the complementary set of frozen bit indices be $\mathcal{A}_\mathsf{c} = \{1, ..., N\} \setminus \mathcal{A}$. The input vector $\boldsymbol{u}$ can be split into a vector $u_\mathcal{A}$ of information bits and $u_{\mathcal{A}_\mathsf{c}}$ of frozen bits. For a given channel, the set $\mathcal{A}$ is the set of bit coordinates in $\boldsymbol{u}$ with highest reliability under genie-aided successive cancellation (SC) decoding [17], and it can be determined via density evolution (DE) analysis [20]–[22]. A polar code codeword is obtained as $\boldsymbol{x} = \mathsf{B}(\boldsymbol{u}\boldsymbol{G}_N)$.

B. CRC-Concatenated Polar Codes

In CRC-concatenated polar codes [18] an outer (shortened) CRC code $\mathcal{C}_\mathsf{O}(N_\mathsf{O}, K)$ is serially concatenated with an inner polar code $\mathcal{C}_\mathsf{I}(N, K_\mathsf{I})$ with $K_\mathsf{I} = N_\mathsf{O}$. Let the length-$K$ information word at the input of the CRC encoder be $\boldsymbol{w}$ and the length-N_O CRC code codeword be $\boldsymbol{v} = \boldsymbol{w}\boldsymbol{G}_\mathsf{O}$, where $\boldsymbol{G}_\mathsf{O}$ is the outer CRC code generator matrix. We assume systematic encoding, i.e., $\boldsymbol{v} = [\boldsymbol{w}|\boldsymbol{d}]$ where $\boldsymbol{d}$ is the length-$(N_\mathsf{O} - K)$ vector of parity bits introduced by the CRC encoder. We denote by $m = N_\mathsf{O} - K$ the number of parity bits of the CRC code. The rate of the outer code is $R_\mathsf{O} = K/N_\mathsf{O}$ and the code rate of the concatenated scheme is $R = R_\mathsf{O}R_\mathsf{I} = K/N$. CRC generator polynomials can be described in hexadecimal notation. For instance, $\mathrm{x}^7 + \mathrm{x}^3 + 1$ is denoted by 0x9. We denote the operator that maps the input of the CRC encoder $\boldsymbol{w} \in \mathbb{F}_2^K$ onto the input of the polar encoder $\boldsymbol{u} \in \mathbb{F}_2^N$ as $\phi(\cdot)$, i.e., $\boldsymbol{u} = \phi(\boldsymbol{w})$. The ϕ-operator is the composition of the linear CRC encoding step $\boldsymbol{v} = \boldsymbol{w}\boldsymbol{G}_\mathsf{O}$ and of the mapping $\boldsymbol{v} \mapsto \boldsymbol{u}$ through frozen bits insertion. SCL is typically used to decode CRC-concatenated polar codes [18]. In particular, an SCL decoder with list size L is used to decode the inner polar code. The output list $\mathcal{L}$ is pruned by imposing the outer CRC code constraints, resulting in the list $\mathcal{L}_\mathsf{P}$. If $\mathcal{L}_\mathsf{P}$ turns to be empty, a decoding failure is declared. Otherwise, the final decision is taken by means of a ML search within $\mathcal{L}_\mathsf{P}$. The concatenated scheme is illustrated in Figure 1. In the figure, the block 'Insertion' refers to the generation of the length-N input sequence $\boldsymbol{u}$ out of the length-K_I vector $\boldsymbol{v}$ by inserting $N - K_\mathsf{I}$ frozen bits. The inverse operation on the decoder side is called 'Frozen Bit Removal'. The block 'Polar Mapping' performs both polar transform in (1) and the mapping onto the channel the channel input alphabet via the function B.

III. Forney's Generalized Decoding Rule

ML decoding of an (N, K) binary linear block code $\mathcal{C}$ reduces to

$$\hat{\boldsymbol{x}}_\mathsf{ML} = \underset{\boldsymbol{x} \in \mathcal{C}}{\operatorname{argmax}}\, p(\boldsymbol{y}|\boldsymbol{x}). \tag{2}$$

In (2), $p(\boldsymbol{y}|\boldsymbol{x})$ is the channel output conditional probability (density) given the channel input $\boldsymbol{x}$. The rule (2) defines a complete decoder. In [7], Forney introduced a generalized decoding rule, which relies on a threshold test that – depending on the choice of the threshold – modifies (2) resulting either in a list decoder (a decoder that outputs a set of decisions) or in an erasure decoder (an incomplete decoder that outputs a single decision or an error flag). Restricting our attention to the latter, the decoding rule of [7] can be cast as follows. Initially, ML decoding is performed according to (2). Let

$$\Lambda(\boldsymbol{y}, \boldsymbol{x}) := \frac{p(\boldsymbol{y}|\boldsymbol{x})}{\sum_{\boldsymbol{x}' \in \mathcal{C} \setminus \boldsymbol{x}} p(\boldsymbol{y}|\boldsymbol{x}')}.$$

The ML decoder output is tested by comparing $\Lambda(\boldsymbol{y}, \hat{\boldsymbol{x}}_\mathsf{ML})$ with a threshold. In particular, the ML decoder output $\hat{\boldsymbol{x}}_\mathsf{ML}$ is *accepted* if

$$\Lambda(\boldsymbol{y}, \hat{\boldsymbol{x}}_\mathsf{ML}) \geq 2^{NT} \tag{3}$$

whereas the decision is *rejected* (returning an error flag) if (3) is not satisfied. Note that in (3) the threshold parameter T is positive. The threshold T regulates the tradeoff between FER and UER. In fact, if T is set to a large value, the test in (3) will reject a decision with high probability, resulting in a large FER and a low UER. On the contrary, if T is set close

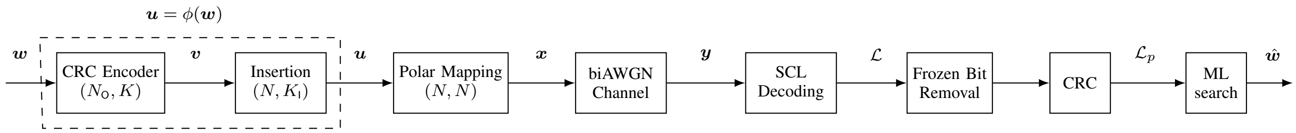

Fig. 1. Reference model describing the encoding with a CRC-concatenated polar code, transmission over the biAWGN channel, and SCL decoding.

to zero, most of the ML decisions will be accepted, resulting in a reduced gap between FER and UER.

A. Bounds on the Error Probabilities

Denote by ϵ_{U} the undetected error probability, by ϵ_{D} the probability of detected errors, and by $\epsilon_{\mathsf{E}} = \epsilon_{\mathsf{U}} + \epsilon_{\mathsf{D}}$ the overall error probability. Remarkably, the test (3) is optimal in the sense that, for a given ϵ_{E}, it minimizes ϵ_{U} (and viceversa) [7]. A characterization of the achievable $(\epsilon_{\mathsf{E}}, \epsilon_{\mathsf{U}})$ pairs under (2)-(3) over a discrete memoryless channel (DMC) was provided in [7] through a random coding error exponent analysis. In particular, it was shown that there exists a block code of length N and rate R that under (2)-(3) simultaneously satisfies

$$\epsilon_{\mathsf{E}} < 2^{-NE_1(R,T)} \tag{4}$$

$$\epsilon_{\mathsf{U}} < 2^{-NE_2(R,T)} \tag{5}$$

where the error exponents $E_1(R,T)$ and $E_2(R,T)$ are

$$E_1(R,T) := \max_{0\leq s\leq\rho\leq 1,\boldsymbol{q}} \left[E_0(s,\rho,\boldsymbol{q}) - \rho R - sT\right]$$

$$E_2(R,T) := E_1(R,T) + T$$

with

$$E_0(s,\rho,\boldsymbol{q}) := -\log_2 \sum_j \left(\sum_k q_k q_{jk}^{1-s}\right)\left(\sum_{k'} q_{k'} q_{jk'}^{s/\rho}\right)^{\rho}$$

where $\boldsymbol{q}$ is the vector of input symbol probabilities. Moreover, q_{jk} is the DMC transition probability. Note that the considered biAWGN channel has a continuous output alphabet. For the evaluation of the upper bounds (4) - (5) we uniformly quantize the channel output with 1024 quantization levels, yielding a binary-input output-symmetric DMC with channel transition probabilities q_{jk}.

With reference to the biAWGN channel, by leveraging on (4) - (5) we can derive finite-length limits on the (undetected) error probability achievable by a code. In particular, consider the case where a system requires the overall error probability to be at most $\epsilon_{\mathsf{E}}^\star$, and the undetected error probability to be not larger than $\epsilon_{\mathsf{U}}^\star$. For a given block length N and rate R, we introduce the notion of *SNR threshold*.

Definition 1 (SNR threshold)**.** *Given a block length N, a rate R, and a channel SNR E_b/N_0, denote by $\mathcal{P}_{N,R}(E_b/N_0)$ the set of achievable pairs $(\epsilon_{\mathsf{E}}, \epsilon_{\mathsf{U}})$ according to* (4) - (5)*. For the target error probabilities $\epsilon_{\mathsf{E}}^\star$ and $\epsilon_{\mathsf{U}}^\star$, the SNR threshold is defined as*

$$\gamma(\epsilon_{\mathsf{E}}^\star, \epsilon_{\mathsf{U}}^\star) := \min\left\{\frac{E_b}{N_0}\middle| (\epsilon_{\mathsf{E}}^\star, \epsilon_{\mathsf{U}}^\star) \in \mathcal{P}_{N,R}(E_b/N_0)\right\}. \tag{6}$$

The SNR threshold yields an upper bound on the minimum SNR for which a length-N, rate-R code can attain $\epsilon_{\mathsf{E}} \leq \epsilon_{\mathsf{E}}^\star$ and $\epsilon_{\mathsf{U}} \leq \epsilon_{\mathsf{U}}^\star$. Figure 2 depicts the SNR threshold $\gamma(\epsilon_{\mathsf{E}}^\star, \epsilon_{\mathsf{U}}^\star)$ for $\epsilon_{\mathsf{E}}^\star = 10^{-3}$ and $\epsilon_{\mathsf{U}}^\star = 10^{-5}$ as a function of $K = RN$ for rate-$1/2$ codes.

IV. Error Detection Strategies for CRC-Concatenated Polar Codes

In this section, we describe four strategies to embed an error detection capability for CRC-concatenated polar codes under (CRC-aided) SCL decoding. The first strategy consists of the plain application of SCL decoding, which yields a detected error when the pruned list is empty. For a given CRC-concatenated polar code, this approach does not allow to vary the trade-off between the FER and the UER. We will adopt this approach as a reference for three more sophisticated strategies that will be outlined next.

A. SCL Decoding with Threshold Test

The first approach is inspired by the threshold test (3) to improve upon the error detection capability of the SCL decoder. Consider SCL decoding followed by the CRC check. We must distinguish three cases:

i. If the pruned list is empty, the decoder declares a decoding failure.
ii. If the pruned list contains only one element, the decoder outputs the only element of $\mathcal{L}_{\mathsf{P}}$ as final decision.
iii. If the pruned list cardinality is larger than 1 ($|\mathcal{L}_{\mathsf{P}}| > 1$), the decoder selects the information vector $\boldsymbol{w}$ that maximizes the likelihood

$$\hat{\boldsymbol{w}} = \operatorname*{argmax}_{\boldsymbol{w}\in\mathcal{L}_{\mathsf{P}}} p(\boldsymbol{y}|\boldsymbol{x}(\boldsymbol{w})) \tag{7}$$

where we use the shorthand $\boldsymbol{x}(\boldsymbol{w})$ for $\mathsf{B}\left(\phi(\boldsymbol{w})\boldsymbol{G}_N\right)$. Let $\hat{\boldsymbol{x}} = \mathsf{B}\left(\phi(\hat{\boldsymbol{w}})\boldsymbol{G}_N\right)$ (i.e., the codeword corresponding to the decision) and

$$\Lambda^{\mathsf{SCL}}(\boldsymbol{y}, \boldsymbol{x}(\boldsymbol{w})) := \frac{p\left(\boldsymbol{y}|\boldsymbol{x}(\boldsymbol{w})\right)}{\displaystyle\sum_{\substack{\boldsymbol{x}'=\mathsf{B}\left(\phi(\boldsymbol{w}')\boldsymbol{G}_N\right)\\ \boldsymbol{w}'\in\mathcal{L}_{\mathsf{P}}\setminus\boldsymbol{w}}} p\left(\boldsymbol{y}|\boldsymbol{x}'\right)}.$$

The decision (7) is accepted if

$$\Lambda^{\mathsf{SCL}}(\boldsymbol{y}, \hat{\boldsymbol{x}}) \geq 2^{NT} \tag{8}$$

and it is discarded otherwise, declaring a decoding failure.

Note that this strategy is an approximation of the rule (3), which becomes increasingly tight as the list size grows. In the limiting (and impractical) case where $|\mathcal{L}_{\mathsf{P}}| = 2^K$, i.e., when the SCL decoder outputs the entire code book, (8) coincides with (3).

B. Augmented SCL Decoding with Threshold Test

Intuitively, the approach of Section IV-A may provide limited gains on the reference approach (plain SCL decoding, followed by list pruning) when the list size is small. In fact, in this case we expect the pruned list to contain with high probability at most one element, reducing the number of occurrences of case (iii) above. The second approach tries to circumvent the problem by enforcing a larger list after applying the CRC constraints at the decoder side. We refer to this approach as augmented successive cancellation list (ASCL) decoding. Recall that for the outer code we adopt systematic encoding, i.e., with reference to Figure 1 (see also Section II-B) we have $\boldsymbol{v} = [\boldsymbol{w}|\boldsymbol{d}]$. Moreover, assume for simplicity that the bits composing $\boldsymbol{d}$, that is, the last m bits in $\boldsymbol{v}$ are mapped onto the last m coordinates of $\boldsymbol{u}$.[1] ASCL decoding works as follows. SCL decoding is performed for the inner polar code, up to the bit index $\ell := N-m$. With a slight abuse of notation, we denote by $\mathcal{L}$ the set of K-bits vectors produced by the truncated SCL decoder, after removing the frozen bits. We construct an augmented list

$$\mathcal{L}_{\mathsf{A}} = \left\{\boldsymbol{x} = \mathsf{B}\left(\phi(\boldsymbol{w})\boldsymbol{G}_N\right)\middle|\boldsymbol{w} \in \mathcal{L}\right\}.$$

that contains all the modulated codewords associated with the K-bit information vectors in $\mathcal{L}$. It is easy to verify that this list contains all the codewords associated with the information vectors in the pruned list of the standard SCL decoder. The cardinality of $\mathcal{L}_{\mathsf{A}}$ is equal to L (i.e., the cardinality of $\mathcal{L}$). The decoder selects first the ML codeword in $\mathcal{L}_{\mathsf{A}}$ yielding

$$\hat{\boldsymbol{x}} = \operatorname*{argmax}_{\boldsymbol{x}\in\mathcal{L}_{\mathsf{A}}} p(\boldsymbol{y}|\boldsymbol{x}). \tag{9}$$

Denote by

$$\Lambda^{\mathsf{ASCL}}(\boldsymbol{y},\boldsymbol{x}) := \frac{p\left(\boldsymbol{y}|\boldsymbol{x}\right)}{\sum\limits_{\boldsymbol{x}'\in\mathcal{L}_{\mathsf{A}}\setminus\boldsymbol{x}} p\left(\boldsymbol{y}|\boldsymbol{x}'\right)}.$$

The decision (9) is accepted if

$$\Lambda^{\mathsf{ASCL}}(\boldsymbol{y},\hat{\boldsymbol{x}}) \geq 2^{NT}$$

and it is discarded otherwise, declaring a decoding failure. Observe that the bit values of $\boldsymbol{d}$ depend on the information bits corresponding to $\boldsymbol{u}$. Therefore, ASCL decoding is equivalent to SCL decoding of a polar code with dynamic frozen bits [23] defined by the CRC code constraints.

C. SCL Decoding with Split CRC

The third approach departs from the idea of introducing a threshold test to accept/reject the decision of the SCL decoder, and it relies solely on the error detection capability inherent to the outer CRC code. In particular, we partition the parity vector $\boldsymbol{d}$ introduced by the CRC code into two subvectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ where the length of $\boldsymbol{d}_1$ is m_1 and the length of $\boldsymbol{d}_2$ is m_2 (with $m_1+m_2=m$). The parity bits in $\boldsymbol{d}_1$ are then used to prune the list produced by the SCL decoder. The resulting pruned list is either empty, or it contains one or more vectors with length $K+m_2$. We distinguish the two cases:

i. The resulting pruned list $\mathcal{L}_{\mathsf{P}}$ is empty. In this case, the decoder declares a decoding failure.
ii. The pruned list is nonempty. The decoder computes

$$\hat{\boldsymbol{z}} = \operatorname*{argmax}_{\boldsymbol{z}\in\mathcal{L}_{\mathsf{P}}} p(\boldsymbol{y}|\boldsymbol{x}(\boldsymbol{z})) \tag{10}$$

where we make use again of the shorthand $\boldsymbol{x}(\boldsymbol{z})$ to denote the modulated codeword associated with $\boldsymbol{z}$. The decision $\hat{\boldsymbol{z}}$ is finally checked through the remaining m_2 CRC code constraints (i.e., the ones associated with the bits in $\boldsymbol{d}_2$). If the constraints are satisfied, then the decision (10) is accepted and the decoder outputs $\hat{\boldsymbol{w}} = [\hat{z}_1,\ldots,\hat{z}_K]$. Otherwise the decision is rejected, declaring a decoding failure.

Observe that for $m_1 = m$ the approach reduces to the reference scheme. For $m_1 = 0$, $|\mathcal{L}_{\mathsf{P}}|$ is equal to $|\mathcal{L}|$ and all CRC code constraints are dedicated to error detection. Intermediate values of m_1 can be used to achieve a different tradeoff between the FER and the UER.

V. Numerical Results

In this section, we compare the performance of the decoding strategies outlined in Section IV with the finite-length benchmark provided by the SNR threshold of Definition 1. We consider the case where $\epsilon_{\mathsf{E}}^\star = 10^{-3}$ and $\epsilon_{\mathsf{U}}^\star = 10^{-5}$, and we focus on rate-$1/2$ codes. We fix two reference polar code designs: A $(64,32)$ CRC-concatenated polar code based on a 6-bit CRC code (i.e., $m=6$) with polynomial 0x3, and a $(128,64)$ CRC-concatenated polar code based on a 7-bit CRC code (i.e., $m=7$) with polynomial 0x9. The inner polar codes were designed by means of DE analysis with Gaussian approximation [24], [25] setting as target SNR $E_b/N_0 = 7$ dB. Two list sizes are used, $L=8$ and $L=32$. For each of the three strategies of Section IV an optimization was carried out with the objective of minimizing the SNR for which the decoder achieves simultaneously $\epsilon_{\mathsf{E}} \leq \epsilon_{\mathsf{E}}^\star$ and $\epsilon_{\mathsf{U}} \leq \epsilon_{\mathsf{U}}^\star$. The optimization was over the threshold T for the SCL/ASCL decoding with threshold test (Section IV-A and Section IV-B), and over the split CRC parameter m_1 for the strategy of Section IV-C. The optimum values of the threshold T and of the parameter m_1 are reported in Table I, and they were obtained by estimating the decoder performance through Monte Carlo simulations. Figure 2 compares the E_b/N_0 values for which the different strategies achieve the target error probabilities with the SNR threshold of Definition 1.

Observe that SCL decoding with threshold test performs well for both codes. In particular, when the information length is small ($K=32$), SCL decoding with additional threshold test performs notably better than the other strategies for both $L=8$ and $L=32$. In the former case, the gain over the reference SCL decoding scheme is around 0.5 dB, while in the latter it is approximately 0.8 dB. At $K=32$, the best strategy allows to attain an E_b/N_0 for $\epsilon_{\mathsf{E}}^\star = 10^{-3}$ and $\epsilon_{\mathsf{U}}^\star =$

[1] The case where some coordinates of $\boldsymbol{u}$ in $[N-m+1, N]$ are allocated to frozen bits can be accommodated with minor, notationally tedious modifications.

TABLE I

VALUES OF OPTIMIZED PARAMETERS. $\epsilon_{\mathsf{E}}^{\star} = 10^{-3}$ AND $\epsilon_{\mathsf{U}}^{\star} = 10^{-5}$. TT STANDS FOR THRESHOLD TEST.

(N, K)	L	SCL with TT	ASCL with TT	Split CRC
$(64, 32)$	8	$T = 0.1442$	$T = 0.1948$	$m_1 = 1$
	32	$T = 0.1039$	$T = 0.1197$	$m_1 = 3$
$(128, 64)$	8	$T = 0.1904$	$T = 0.2597$	$m_1 = 3$
	32	$T = 0.1096$	$T = 0.1211$	$m_1 = 3$

10^{-5} that is visibly lower than the SNR threshold (6). The result can be explained by a lack of tightness of (4)-(5) at very short blocklengths. A similar behavior can be observed also for other bounds based on error exponents. For example, in the short blocklength regime Gallager's RCB can be improved over various DMCs by employing tighter bounding techniques [26]. Still, the performance achieved by SCL decoding with threshold test is remarkable. The result is confirmed for the case of $K = 64$, although here no strategy allows to operate at an E_b/N_0 lower than the SNR threshold. The best performance is also here achieved by SCL decoding with threshold test, when $L = 32$. For $K = 32$ and $L = 32$, ASCL decoding with threshold test shows to be competitive, too. The result is nevertheless not replicated in the other settings (smaller list size and/or longer blocks). The SCL decoding with split CRC performs poorly at very short blocklength, while for $K = 64$ and $L = 32$ the performance is remarkably close to the bound. With $K = 64$ and $L = 8$, the split CRC approach yields the best performance, closely approached by other methods.

The analysis presented so far aims at gaining insights on the performance achievable, in terms of FER and UER, by a CRC-concatenated polar code under various decoding strategies. The study relied on fixing a CRC-concatenated polar code, and testing its performance under the algorithms outlined in Section IV. An interesting question arises on whether concatenated polar code designs tailored to provide good FER vs UER tradeoffs may yield further gains. While deriving a universal criterion to construct concatenated polar codes for a given target $(\epsilon_{\mathsf{E}}^{\star}, \epsilon_{\mathsf{U}}^{\star})$ would provide a general answer to the question, we focus here on a special case and show that a careful design may yield visible gains.

We consider a $(64, 32)$ CRC-concatenated polar code under SCL with $L = 8$, targeting again $\epsilon_{\mathsf{U}}^{\star} = 10^{-3}$ and $\epsilon_{\mathsf{U}}^{\star} = 10^{-5}$. We performed a wide search based on Monte Carlo simulations for the outer CRC code parameters, and for the design SNR to be used by the DE analysis to fix the inner polar code frozen bit positions. The search returned a 9-bit CRC code with polynomial 0x33 and frozen bits selected by DE analysis setting as target SNR $E_b/N_0 = 4$ dB.

The UER and FER versus E_b/N_0 are depicted in Figure 3. Under SCL decoding (reference approach) the code attains the target error probabilities at $E_b/N_0 \approx 4.65$ dB, with about 0.4 dB gain over the analogous decoding strategy for the $(64, 32)$ code reported in Figure 2. By introducing the threshold test the required E_b/N_0 drops to ≈ 4.4 dB, with a slight gain over the corresponding performance reported for the $(64, 32)$ code based on a 6-bit CRC in Figure 2.

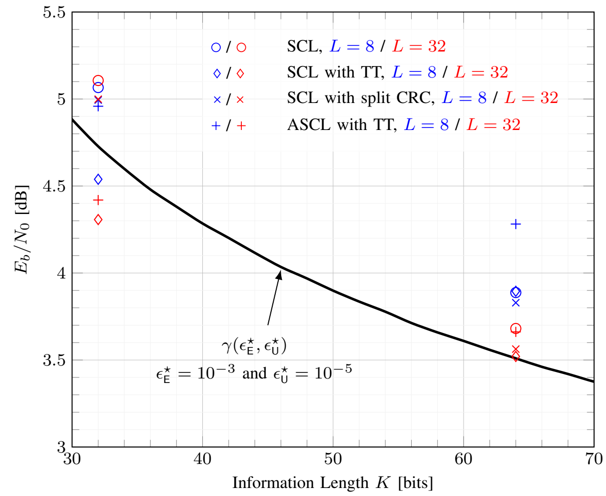

Fig. 2. E_b/N_0 versus information length K required to reach a target overall error probability $\epsilon_{\mathsf{E}}^{\star} = 10^{-3}$ and a target undetected error probability $\epsilon_{\mathsf{U}}^{\star} = 10^{-5}$ (TT stands for threshold test).

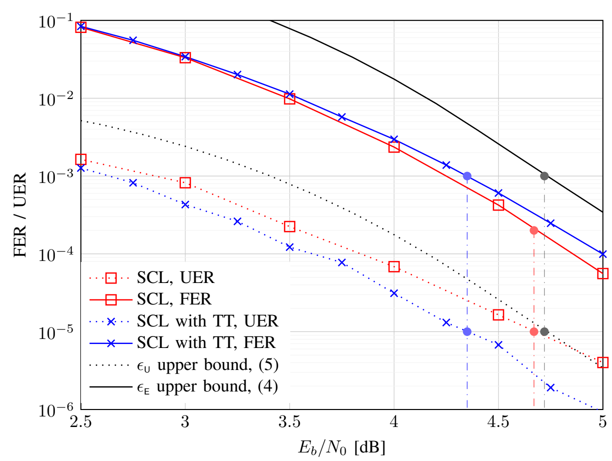

Fig. 3. UER and FER versus SNR for $(64, 32)$ CRC-concatenated polar codes under SCL decoding ($L = 8$) with and without threshold test, optimized for $\epsilon_{\mathsf{E}}^{\star} = 10^{-3}$, $\epsilon_{\mathsf{U}}^{\star} = 10^{-5}$. As a reference, the bounds on ϵ_{E} and ϵ_{U} from (4) and (5) are provided.

VI. CONCLUSIONS

We presented different error detection strategies for CRC-concatenated polar codes and evaluated their performance in terms of undetected error rate (UER) and (overall) frame error rate (FER). For some polar code designs we observed that successive cancellation list decoding with a subsequent threshold test can significantly improve the performance. This holds in particular when the blocklength is small and the list size is sufficiently large. With growing block size, the error detection capability obtained by splitting the CRC bits in two parts, where a part is dedicated to error detection only, becomes increasingly competitive. An interesting open problem is to derive code design guidelines that, under one of the discussed error detection strategies, allow to optimize the performance for a target UER/FER pair.

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