On the Theory of Discrete, Adaptive Space Filling Curves

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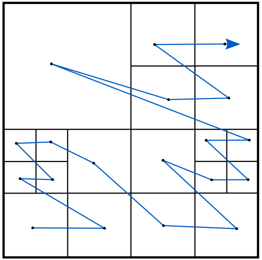
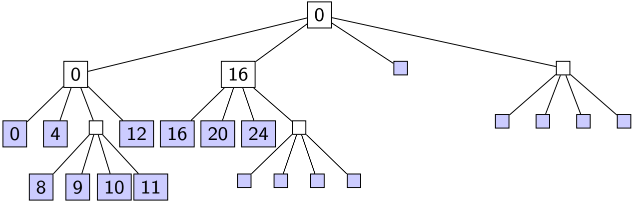
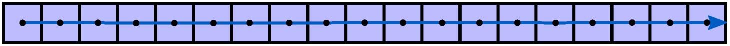
SIAM Conference on Parallel Processing 2020

Intro







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Yes, but...

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Most definitions of SFCs have at least one of these issues:

- Come from an analytical point of view
- Are not explicit
- Only on uniform subdivisions
- Actually allow stuff that should be illegal
- Restrict us to $1 : 2^d$ refinement

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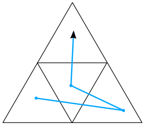

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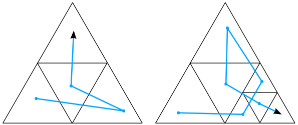

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Theorem

Let X be an SFC, then Y holds.

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We need a <u>rigorous</u> definition of X to correctly proof the theorem.

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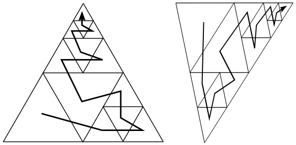

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- Allows for 'crazy' constructions
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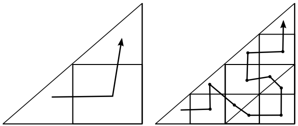

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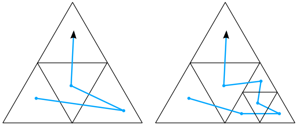

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- Allows us to construct new SFCs. See cross product
- Allows for Code generation

Disclaimer

There exist lots of approaches where discrete SFCs are explicitely or implicitely described.
Not everything we present here is entirely new.
But maybe its the first time, we do it rigorously.

Idea

What is an SFC?

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Some ordering of our elements.

$$\text{AMR mesh} \xrightarrow{\text{SFC}} \{0, 1, 2, 3, \ldots\} \tag{1}$$

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Some ordering of our elements.

$$\text{AMR mesh} \xrightarrow{\text{SFC}} \{0, 1, 2, 3, \dots\} \tag{1}$$

Adaptive meshes are not alone, they come with a refinement hierarchy!

$$
\begin{tikzcd}
\text{AMR hierarchy} & \\
\text{AMR mesh} \arrow[u, "\subseteq"] \arrow[r, "\text{SFC}"] & \{0, 1, 2, 3, \ldots\}
\end{tikzcd}
\tag{2}
$$

$$
\begin{tikzcd}
\text{AMR hierarchy} \arrow[drr, "\text{SFC index}"] & & \\
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\end{tikzcd}
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$$
\begin{tikzcd}
\text{Refinement space} \arrow[dr, "\text{SFC index}"] & \\
\text{Refinement} \arrow[u, "\cup|"] \arrow[r, "\text{SFC}"] & \{\, 0, 1, 2, 3, \dots \,\}
\end{tikzcd}
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$$

Refinement spaces

Definition (refinement space)

A **refinement space** consists of:

- A set $\mathcal{S}$, the **elements**
- A map $\ell\colon \mathcal{S} \to \mathbb{N}_0$, the **level**
- Maps $R^l\colon \ell^{-1}(l) =: \mathcal{S}^l \to \mathcal{P}(S^{l+1})$, the **refinement maps**

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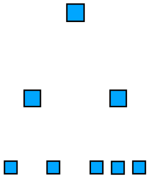

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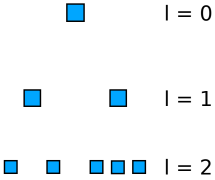

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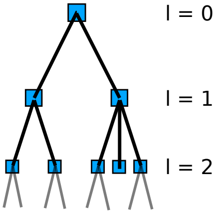

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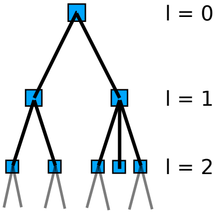
such that

- Every element of level $l > 0$ has a **unique parent**:

$$\bigcup_{E\in\mathcal{S}^l} R^l(E) = \mathcal{S}^{l+1}. \tag{3}$$

$$R^l(E) \cap R^l(E') = \emptyset \text{ for } E \neq E' \in \mathcal{S}^l \tag{4}$$

- $\mathcal{S}^0 = \{\, \mathcal{E}_0 \,\}$

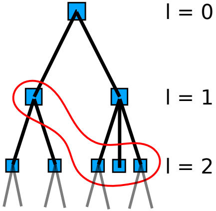


Refinements

Definition (refinement)

A **refinement** of a refinement space $\mathcal{S}$ is a subset $\mathscr{S}$ constructed via successivly refining level 0, thus:

- $\mathscr{S} = \mathcal{S}^0$ is a refinement.
- $\mathscr{S} \backslash E \cup R^l(E)$ is a refinement for each $E \in \mathscr{S}$.



Definition (SFC index)

An **SFC index** $\mathcal{I}$ is a map

$$\mathcal{I}\colon \mathcal{S} \to \mathbb{N}_0 \tag{5}$$

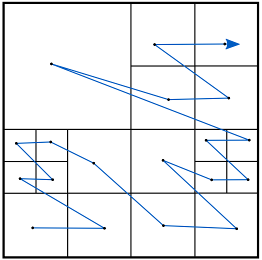
- $\mathcal{I} \times \ell\colon \mathcal{S} \to \mathbb{N}_0 \times \mathbb{N}_0$ is injective. **Restricted to a level, $\mathcal{I}$ is unique.**
- E ancestor of $E' \Rightarrow \mathcal{I}(E) \leq \mathcal{I}(E')$. **Refining does not decrease the index.**
- $\mathcal{I}(E) < \mathcal{I}(\hat{E})$ and $\hat{E}$ not a descendant of $E \Rightarrow \mathcal{I}(E) \leq \mathcal{I}(E') < \mathcal{I}(\hat{E})$ for all descendants E' of E. **Refining is 'local'.**

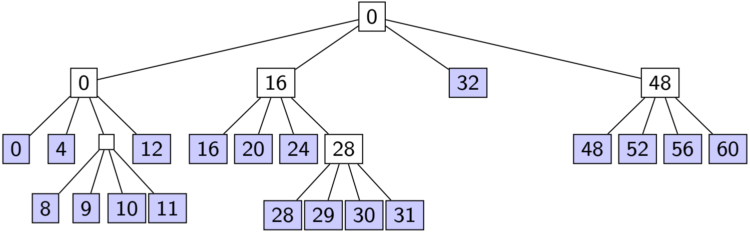
Definition (Space-filling curve)

A (discrete) **Space-filling curve** is an SFC index restricted to a refinement.

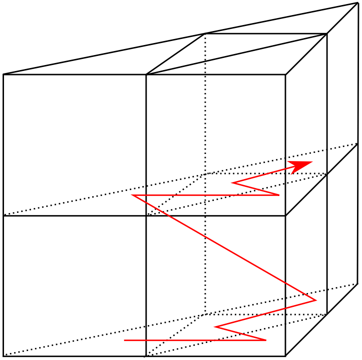
$$\mathcal{I}_{|\mathscr{S}} : \mathscr{S} \to \mathbb{N}_0 \tag{6}$$

Example





Cross product



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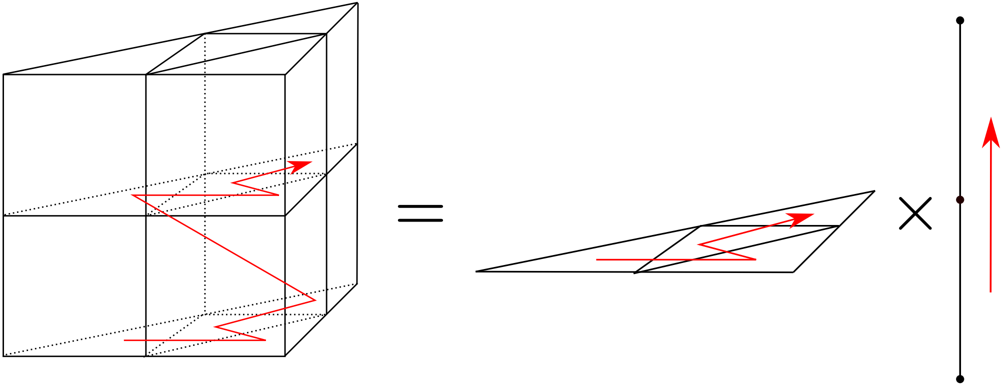

Definition (Cross product of refinement spaces)

Let $(\mathcal{S}_1, R_1, \ell_1)$ and $(\mathcal{S}_2, R_2, \ell_2)$ be refinement spaces. The cross product $\mathcal{S}_\times = \mathcal{S}_1 \times \mathcal{S}_2$ is defined via

- $\mathcal{S}^l_\times := \mathcal{S}^l_1 \times \mathcal{S}^l_2$
- $\ell_\times(E_1, E_2) := \ell(E_1) = \ell(E_2)$
- $R^l_\times(E_1, E_2) := R^l_1(E_1) \times R^l_2(E_2) \subset \mathcal{S}^{l+1}_1 \times \mathcal{S}^{l+1}_2 = \mathcal{S}^{l+1}_\times$

Proposition

The cross product is a refinement space.

$\square$

Cross product

Definition (Cross product of SFC indices)

The cross product index $\mathcal{I}_\times = \mathcal{I}_1 \times \mathcal{I}_2$ on $\mathcal{S}_1 \times S_2$:

$$\mathcal{I}_\times : \mathcal{S}_1 \times \mathcal{S}_2 \to \mathbb{N}_0 \tag{7}$$

defined recursively by

$$\mathcal{I}_\times(E_1, E_2) := \mathcal{I}_x(P_1, P_2) + m_1^{>l} m_2^{>l} \left(\mathrm{sibid}_1(E_1) * \#\mathrm{siblings}(E_2) + \mathrm{sibid}(E_2)\right) \tag{8}$$

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where

$$m^{>l} := \prod_{i>l} \max_E |R^i(E)| \tag{9}$$

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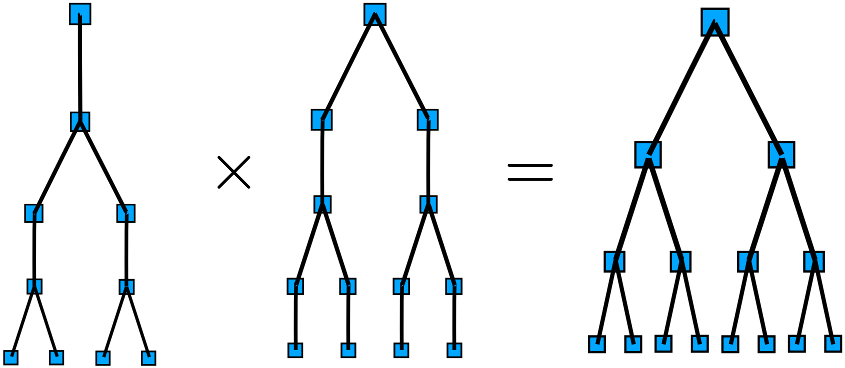
Proposition

The cross product index is an SFC index. □

An interesting example



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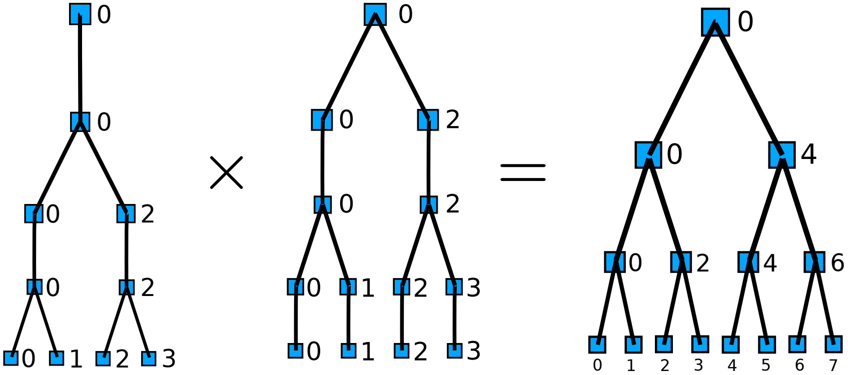

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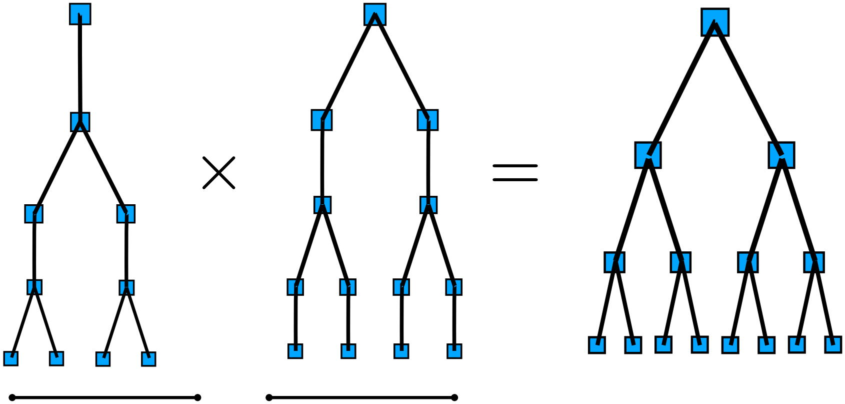
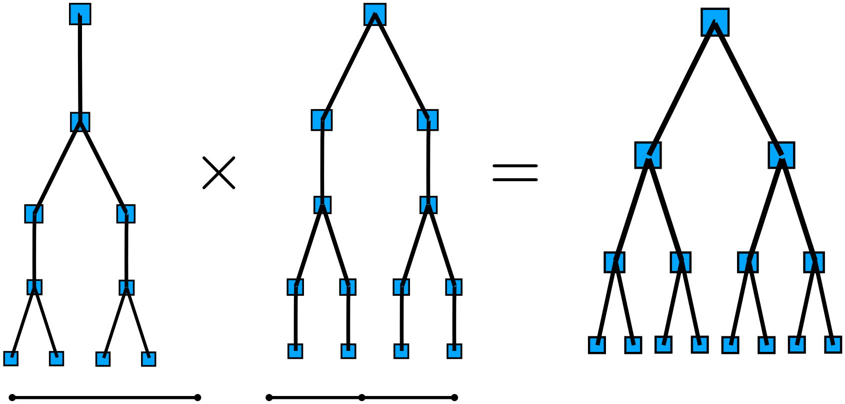


$\ell = 0$

An interesting example



$\ell = 1$

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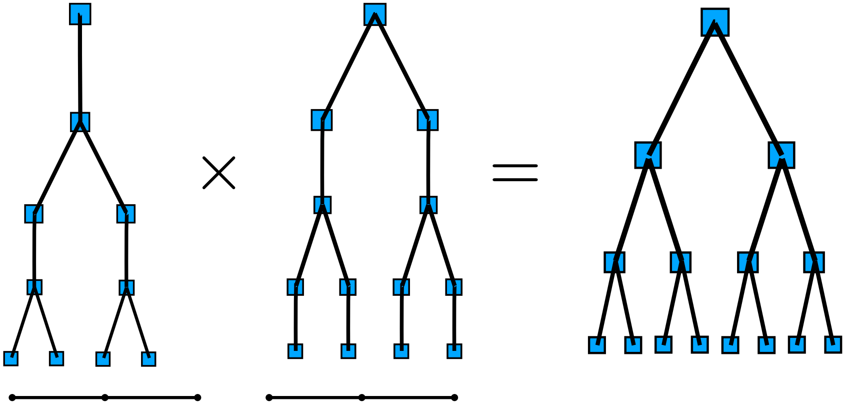

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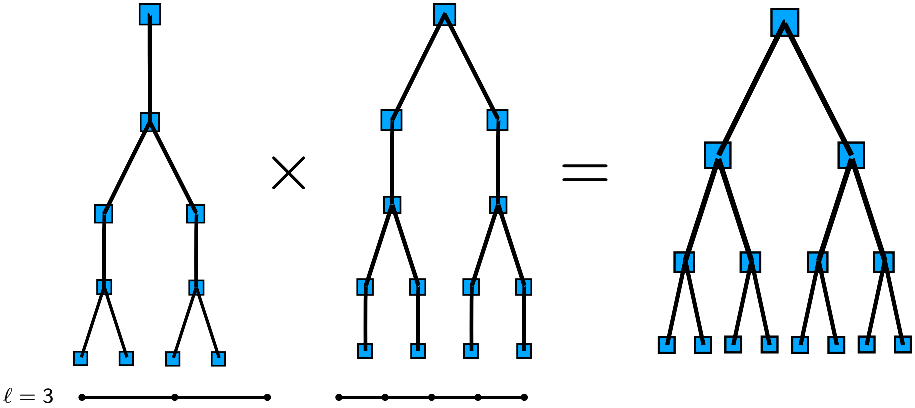

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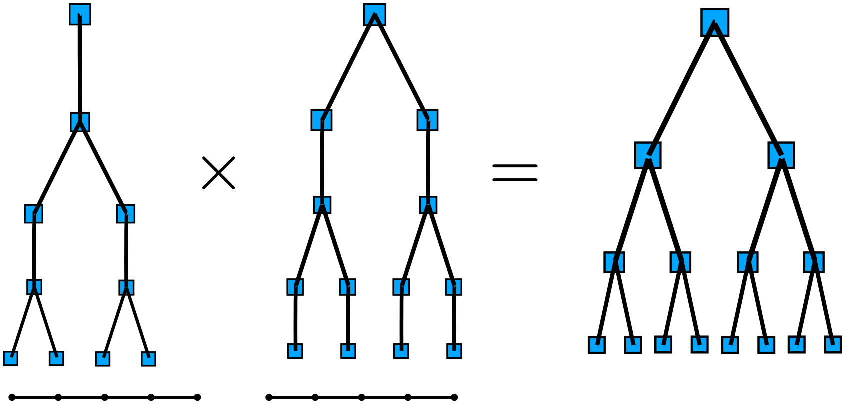

$\ell = 4$

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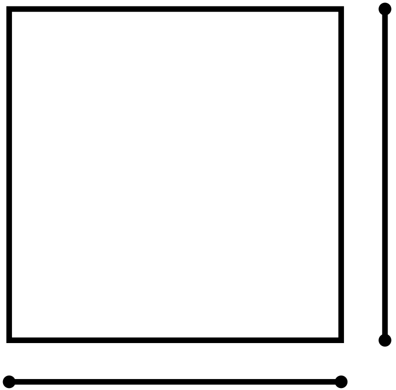

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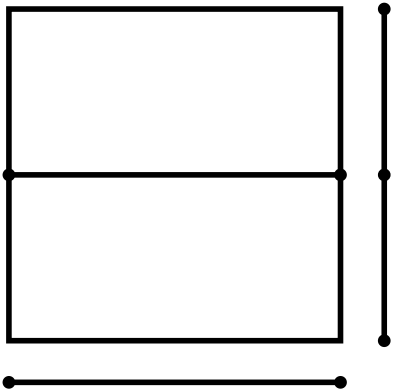

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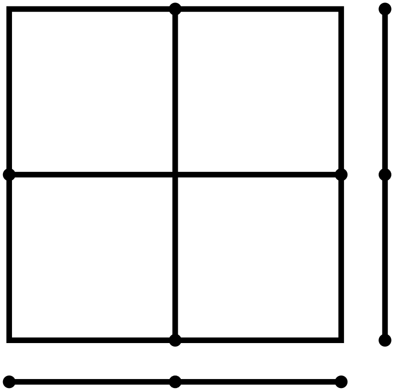

$$\ell = 2$$

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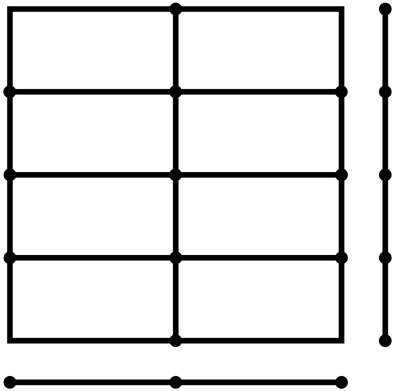
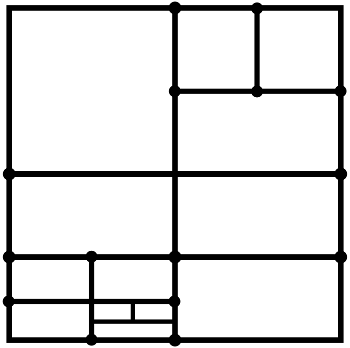


$\ell = 3$

An interesting example



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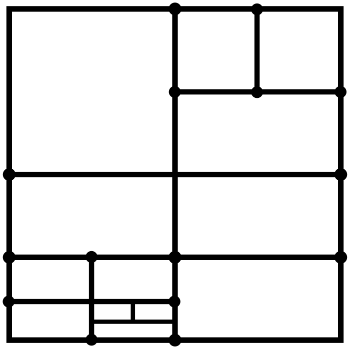

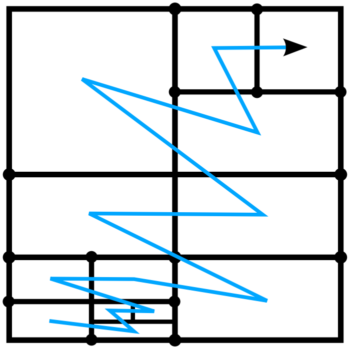

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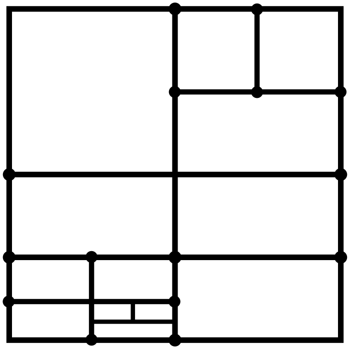

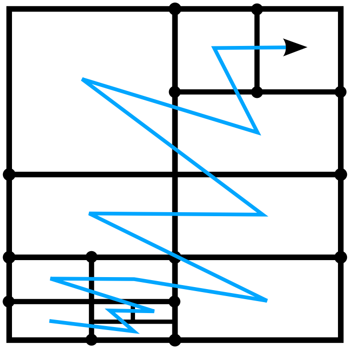

$\Rightarrow$ An SFC for dyadic refinement.

Furthermore

- Can be straightforwardly extended to forest
- Can define partitions
- TODO: How to encode (face-)neighbors?
- TODO: Can we find other cross products?

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Thank you.

Questions?

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