On the Role of Coupled Damping and Gyroscopic Forces in the Stability and Performance of Mechanical Systems

Pablo Borja , *Member, IEEE*, Cosimo Della Santina , *Member, IEEE*, and Azita Dabiri

***Abstract*—Damping injection is a well-studied tool in nonlinear control theory to stabilize and shape the transient of mechanical systems. Interestingly, the injection of coupled damping yielding gyroscopic forces has received far less attention. This letter aims to fill this gap for gyroscopic forces that couple actuated and unactuated coordinates. First, we establish sufficient conditions for the stability of the closed loop. Then, we provide analytic results proving that injecting coupled damping may improve the closed-loop performance. We illustrate the results via the stabilization of three mechanical systems.**

***Index Terms*—Stability of nonlinear systems, Lyapunov methods, nonlinear output feedback.**

I. Introduction

STABILIZATION of admissible equilibria in underactuated mechanical systems is a long-lasting challenge in control theory [1], [2], where passivity-based control (PBC) has imposed itself as one of the main strategies to achieve this goal [3]. Damping injection on actuated variables is a key component of PBC, which guarantees the convergence of the closed-loop system and it allows to shape its transient [4], [5], [6], [7], [8]. Some previous work has investigated the advantage of going beyond that by considering velocity couplings in the form of gyroscopic terms. For instance, in interconnection and damping assignment passivity-based control (IDA-PBC), such terms are used to solve the partial differential equations (PDEs) involved in the control design process, having a direct impact on the stabilization problem [9], [10], [11], [12], [13], [14]. Furthermore, [15] remarks that IDA-PBC including gyroscopic forces guarantees robustness against matched disturbances in fully actuated systems. Finally, experimental anecdotal evidences, e.g., [16], suggest that damping injection from the unactuated coordinates through the actuated channel may improve the performance of the closed-loop system in PBC approaches. In this regard, in [17], the authors discuss the effect of gyroscopic terms on the transitory response of the closed-loop system without formal proof. Nevertheless, analytical results that elucidate the impact of these terms on the performance of the closed-loop system are still lacking.

The goal of this letter is to fill this gap by providing analytical results quantifying the effect that gyroscopic forces and coupled damping have on the stability and performance of underactuated mechanical systems. To this end, we adopt a PBC approach that preserves the mechanical structure and does not require solving PDEs, contrasting with IDA-PBC. Hence, we can transparently analyze the effect of the coupled damping terms and gyroscopic elements. More precisely, we contribute to the state of the art in PBC with:

(i) A passivity-based regulator for underactuated mechanical systems that injects coupled damping, resulting in gyroscopic forces.
(ii) A method to tune this injection to obtain a bound for the $\mathcal{L}_2$-norm of the velocities and improve it.
(iii) A preliminary stability analysis for non-negative closed-loop dissipation matrices.

Interestingly, (iii) cannot be achieved by following energetic arguments. We show the effectiveness of the proposed technique and analysis with three examples of underactuated mechanical systems.

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Pablo Borja is with the Department of Cognitive Robotics, Delft University of Technology, 2628 CD Delft, The Netherlands (e-mail: l.p.borjarosales@tudelft.nl).

Cosimo Della Santina is with the Department of Cognitive Robotics, Delft University of Technology, 2628 CD Delft, The Netherlands and also with the Institute of Robotics and Mechatronics, German Aerospace Center (DLR), 82234 Weβling, Germany (e-mail: c.dellasantina@tudelft.nl).

Azita Dabiri is with the Delft Center for Systems and Control, Delft University of Technology, 2628 CD Delft, The Netherlands (e-mail: a.dabiri@tudelft.nl).

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II. Preliminaries

A. Notation

I_n is the $n \times n$ identity matrix, $\mathbf{0}$ is a vector or matrix whose entries are zeros, A_{ij} is the element (i, j) of $A \in \mathbb{R}^{n\times m}$, $\partial f(x)/\partial x = \left[\partial f(x)/\partial x_1 \cdots \partial f(x)/\partial x_n\right]^\top$, $\|x\| := \sqrt{x^\top x}$, and $\|x\|_D := \sqrt{x^\top D x}$ - with $D \in \mathbb{R}^{n\times n}$. Furthermore, we represent mappings of the form $F : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ as matrices—i.e., $F(x) \in \mathbb{R}^{n\times m}$. Given the distinguished vector $x_\star \in \mathbb{R}^n$, we define $f(x_\star) = f_\star$ and $(\partial f(x)/\partial x)_\star := (\partial f(x)/\partial x)|_{x=x_\star}$. When clear from the context, we omit the arguments of functions.

B. Underactuated Mechanical System

We consider mechanical systems of the following form

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & I_n\\ -I_n & -D(q,p)\end{bmatrix}\begin{bmatrix}\frac{\partial H(q,p)}{\partial q}\\ \frac{\partial H(q,p)}{\partial p}\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ B\end{bmatrix}u$$

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q), \quad (1)$$

where $q, p \in \mathbb{R}^n$ are the generalized positions and momenta, respectively, $u \in \mathbb{R}^m$ denotes the input vector, $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the Hamiltonian of the system (total energy), $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix; $V : \mathbb{R}^n \to \mathbb{R}$ is the potential energy, which, without loss of generality, is assumed to be bounded from below; $D(q,p) \in \mathbb{R}^{n\times n}$ is the dissipation matrix, which is positive semi-definite; and the input matrix is $B = \begin{bmatrix} I_m & \mathbf{0}\end{bmatrix}^\top$. We can split the coordinates into actuated and unactuated as $q_{\mathtt{u}} := B^\perp q, \quad q_{\mathtt{a}} := B^\top q; \quad p_{\mathtt{u}} := B^\perp p, \quad p_{\mathtt{a}} := B^\top p$, where $B^\perp := [\mathbf{0} \;\; I_s]$, with $s := n - m$. Similarly, we can rewrite $M(q)$ and $D(q,p)$ as

$$M(q) = \begin{bmatrix} M_{\mathtt{aa}}(q) & M_{\mathtt{au}}(q) \\ M_{\mathtt{au}}^\top(q) & M_{\mathtt{uu}}(q) \end{bmatrix}; \quad \begin{array}{l} M_{\mathtt{aa}}(q) \in \mathbb{R}^{m\times m}; \\ M_{\mathtt{au}}(q) \in \mathbb{R}^{m\times s}; \\ M_{\mathtt{uu}}(q) \in \mathbb{R}^{s\times s}. \end{array}$$

$$D(q,p) = \begin{bmatrix} D_{\mathtt{aa}}(q,p) & D_{\mathtt{au}}(q,p) \\ D_{\mathtt{au}}^\top(q,p) & D_{\mathtt{uu}}(q,p) \end{bmatrix}; \quad \begin{array}{l} D_{\mathtt{aa}}(q,p) \in \mathbb{R}^{m\times m}; \\ D_{\mathtt{au}}(q,p) \in \mathbb{R}^{m\times s}; \\ D_{\mathtt{uu}}(q,p) \in \mathbb{R}^{s\times s}. \end{array}$$

Henceforth, we refer to $D_{\mathtt{au}}(q,p)$ as coupled damping because this term couples the actuated coordinates and the unactuated ones. Moreover, throughout this letter, several functions are expressed in terms of velocities through the equality $\dot{q} = M^{-1}(q)p$. The set of assignable equilibria for (1) is given by $\mathcal{E} := \{q \in \mathbb{R}^n | \; \partial V(q)/\partial q_{\mathtt{a}} = \mathbf{0}\}$. Equivalently, if $q_\star \in \mathcal{E}$, then there exists $u_\star \in \mathbb{R}^m$ such that $u_\star = (\partial V/\partial q_{\mathtt{a}})_\star$, implying that $(q_\star, \mathbf{0})$ is an equilibrium for (1).

C. Assumptions

The following assumptions characterize the unactuated dynamics of the mechanical systems studied in this letter.

Assumption 1: $(\partial^2 V/\partial q_{\mathtt{u}}^2)_\star \succ 0$.

Assumption 2: $D_{\mathtt{uu}}(q,p) \succ 0, \forall\, q, p \in \mathbb{R}^n$.

Assumption 1 ensures that the effect of gravity on the unactuated coordinates does not prevent the system from being stabilized at the desired equilibrium. Notably, a broad range of underactuated mechanical systems satisfy this assumption, e.g., marine craft, robots with flexible joints, and a wide range of soft robots, and wheeled robots, to mention some. Furthermore, Assumption 2 concerns the natural damping in the unactuated coordinates. We underscore that dissipation is inherent in mechanical systems. Thus, this assumption is not restrictive from a physical point of view.

III. Control Design

In this section, we develop stabilizing controllers consisting of potential energy shaping, standard damping injection, and coupled damping injection. As a consequence of the last control component, the closed-loop system exhibits gyroscopic terms.

The potential energy of the system can be shaped through a twice differentiable function $\Phi : \mathbb{R}^m \to \mathbb{R}$ that depends on the actuated positions. Let $V_{\mathtt{d}} : \mathbb{R}^n \to \mathbb{R}$ be the desired potential energy, given by

$$V_{\mathtt{d}}(q) := V(q) + \Phi(q_{\mathtt{a}}). \quad (2)$$

Accordingly,

$$H_{\mathtt{d}}(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q) + \Phi(q_{\mathtt{a}}), \quad (3)$$

such that $H_{\mathtt{d}_\star} = 0$. The control input that yields the closed-loop energy (3) is given by

$$u_{\mathtt{es}}(q_{\mathtt{a}}) = -\frac{\partial \Phi(q_{\mathtt{a}})}{\partial q_{\mathtt{a}}}. \quad (4)$$

Additionally, damping can be injected into the closed-loop system through a control input of the form

$$u_{\mathtt{di}}(\dot{q}_{\mathtt{a}}) = -D_\psi \dot{q}_{\mathtt{a}}, \quad (5)$$

where the matrix $D_\psi \in \mathbb{R}^{m\times m}$ is positive definite. Furthermore, coupled damping can be injected via the control input

$$u_{\mathtt{gy}}(q,p) = -2\Gamma^\top(q,p)\dot{q}, \quad (6)$$

where $\Gamma(q,p) \in \mathbb{R}^{n\times m}$ and

$$\dot{q} = M^{-1}(q)p. \quad (7)$$

Moreover, to simplify the notation, we consider

$$\Gamma(q,p) = \begin{bmatrix} \Gamma_{\mathtt{a}}(q,p) \\ \Gamma_{\mathtt{u}}(q,p) \end{bmatrix}; \quad \begin{array}{l} \Gamma_{\mathtt{a}}(q,p) \in \mathbb{R}^{m\times m}; \\ \Gamma_{\mathtt{u}}(q,p) \in \mathbb{R}^{s\times m}; \end{array}$$

and define the following matrix

$$D_{\mathtt{d_a}}(q,p) := D_{\mathtt{aa}}(q,p) + D_\psi + \Gamma_{\mathtt{a}}(q,p) + \Gamma_{\mathtt{a}}^\top(q,p). \quad (8)$$

Theorem 1 illustrates how to use $u_{\mathtt{es}}(q_{\mathtt{a}})$, $u_{\mathtt{di}}(\dot{q}_{\mathtt{a}})$, and $u_{\mathtt{gy}}(q,p)$ to solve the stabilization problem for a class of mechanical systems represented by (1).

Theorem 1: Consider the system (1) and the desired configuration $q_\star \in \mathcal{E}$ such that Assumptions 1 and 2 hold. If $\Phi(q_{\mathtt{a}})$ in (2) satisfies

$$\begin{aligned} &\left(\frac{\partial \Phi}{\partial q_{\mathtt{a}}}\right)_\star = -\left(\frac{\partial V}{\partial q_{\mathtt{a}}}\right)_\star \\ &\left(\frac{\partial^2 \Phi}{\partial q_{\mathtt{a}}^2}\right)_\star + \left(\frac{\partial^2 V}{\partial q_{\mathtt{a}}^2}\right)_\star \succ \left(\frac{\partial^2 V}{\partial q_{\mathtt{u}} q_{\mathtt{a}}}\right)_\star \left(\frac{\partial^2 V}{\partial q_{\mathtt{u}}^2}\right)_\star^{-1} \left(\frac{\partial^2 V}{\partial q_{\mathtt{a}} q_{\mathtt{u}}}\right)_\star \end{aligned} \quad (9)$$

and D_ψ and $\Gamma(q,p)$ are chosen such that[1]

$$D_{\mathtt{d_a}} - \left(\Gamma_{\mathtt{u}} + D_{\mathtt{au}}^\top\right)^\top D_{\mathtt{uu}}^{-1}\left(\Gamma_{\mathtt{u}} + D_{\mathtt{au}}^\top\right) \succ 0, \quad (10)$$

then the control law

$$u = u_{\mathtt{es}}(q_{\mathtt{a}}) + u_{\mathtt{di}}(\dot{q}_{\mathtt{a}}) + u_{\mathtt{gy}}(q,p) \quad (11)$$

guarantees that the closed-loop system has a locally asymptotically stable equilibrium at $(q_\star, \mathbf{0})$. Moreover, the equilibrium is globally asymptotically stable if $H_{\mathtt{d}}(q,p)$, defined in (3), is radially unbounded and no other solution than $(q_\star, \mathbf{0})$ remains in the set $\mathcal{S} := \{q, p \in \mathbb{R}^n \mid \dot{H}_{\mathtt{d}} = 0\}$.

Proof: Note that the closed-loop takes the form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_n \\ -I_n & -D_{\mathtt{d}} + J_{\mathtt{d}} \end{bmatrix} \begin{bmatrix} \frac{\partial H_{\mathtt{d}}}{\partial q} \\ \frac{\partial H_{\mathtt{d}}}{\partial p} \end{bmatrix}, \quad (12)$$

with $H_{\mathtt{d}}(q,p)$ defined in (3), and

$$\begin{aligned} D_{\mathtt{d}} &= D + BD_\psi B^\top + \Gamma B^\top + B\Gamma^\top \\ J_{\mathtt{d}} &= \Gamma B^\top - B\Gamma^\top. \end{aligned} \quad (13)$$

In particular, $D_{\mathtt{d}}(q,p)$ can be rewritten as

$$D_{\mathtt{d}} = \begin{bmatrix} D_{\mathtt{d_a}} & \Gamma_{\mathtt{u}}^\top + D_{\mathtt{au}} \\ \Gamma_{\mathtt{u}} + D_{\mathtt{au}}^\top & D_{\mathtt{uu}} \end{bmatrix}. \quad (14)$$

[1] We omit the arguments for the sake of readability.

Hence, a Schur complement analysis shows that (10) implies that $D_{\mathtt{d}}(q,p)$ is positive definite. Thus,

$$\dot{H}_{\mathtt{d}} = -\|M^{-1}p\|^2_{D_{\mathtt{d}}} = -\|\dot{q}\|^2_{D_{\mathtt{d}}} \leq 0, \tag{15}$$

where (7) is used to obtain the second equality in (15). Now, note that

$$\left(\frac{\partial V_{\mathtt{d}}}{\partial q}\right)_{\star} = \left[\left(\frac{\partial V}{\partial q_{\mathtt{a}}}\right)^{\top}_{\star} + \left(\frac{\partial \Phi}{\partial q_{\mathtt{a}}}\right)^{\top}_{\star} \quad \left(\frac{\partial V}{\partial q_{\mathtt{u}}}\right)^{\top}_{\star}\right]^{\top}$$
$$\left(\frac{\partial^2 V_{\mathtt{d}}}{\partial q^2}\right)_{\star} = \begin{bmatrix} \left(\frac{\partial^2 V}{\partial q_{\mathtt{a}}^2}\right)_{\star} + \left(\frac{\partial^2 \Phi}{\partial q_{\mathtt{a}}^2}\right)_{\star} & \left(\frac{\partial^2 V}{\partial q_{\mathtt{u}} q_{\mathtt{a}}}\right)_{\star} \\ \left(\frac{\partial^2 V}{\partial q_{\mathtt{a}} q_{\mathtt{u}}}\right)_{\star} & \left(\frac{\partial^2 V}{\partial q_{\mathtt{u}}^2}\right)_{\star} \end{bmatrix}.$$

Therefore, (9) implies

$$\left(\frac{\partial V_{\mathtt{d}}}{\partial q}\right)_{\star} = \mathbf{0}, \qquad \left(\frac{\partial^2 V_{\mathtt{d}}}{\partial q^2}\right)_{\star} \succ 0,$$

where the inequality is obtained via a Schur complement analysis. Hence, the desired equilibrium is a strict local minimum of $H_{\mathtt{d}}(q,p)$, i.e., this function is positive definite with respect to $(q_{\star}, \mathbf{0})$, which together with (15) implies that $H_{\mathtt{d}}(q,p)$ qualifies as a Lyapunov function and the desired equilibrium is locally stable. Consider $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that the trajectories starting in Ω are bounded and the only equilibrium contained in the mentioned set is the desired one—note that Lyapunov stability guarantees the existence of Ω. Moreover,

$$\dot{H}_{\mathtt{d}} \equiv 0 \implies \dot{q} = \mathbf{0} \implies p = \mathbf{0} \implies \dot{p} = \mathbf{0}.$$

However, since $(q_{\star}, \mathbf{0})$ is a strict local minimum of $H_{\mathtt{d}}(q,p)$, in Ω, we have that

$$\dot{p} = \mathbf{0} \implies V_{\mathtt{d}} q = \mathbf{0} \implies q = q_{\star}.$$

Thus, it follows from LaSalle's invariance principle (see, for instance, [18]) that the trajectories of the closed-loop system converge to the desired equilibrium. Note that if $H_{\mathtt{d}}(q,p)$ is radially unbounded, then $\Omega \equiv \mathbb{R}^n \times \mathbb{R}^n$. Moreover, since only $(q_{\star}, \mathbf{0})$ can stay in $\mathcal{S}$, there is only one equilibrium for the closed-loop. Consequently, the desired equilibrium is globally asymptotically stable. ∎

Each element of the control law (11) has a physical interpretation. In particular, $u_{\mathtt{es}}(q_{\mathtt{a}})$ shapes the potential energy assigning the desired equilibrium to the closed-loop system while ensuring that it is a strict minimum of the closed-loop energy $H_{\mathtt{d}}(q,p)$; $u_{\mathtt{di}}(\dot{q}_{\mathtt{a}})$ injects damping through the actuated coordinates of the systems, guaranteeing that $D_{\mathtt{d}}(q,p)$ is positive definite; $u_{\mathtt{gy}}(q,p)$ injects coupled damping, which results in gyroscopic terms in the closed-loop system. Notably, the last term is not required to ensure stability. However, it has an important role in the performance of the closed-loop system as discussed in Section IV.

Remark 1: For systems of the form (1) satisfying Assumption 1, there always exists a smooth function $\Phi(q_{\mathtt{a}})$ such that (9) holds. See [19] for further details.

Remark 2: The injection of coupled damping $u_{\mathtt{gy}}(q,p)$ can be combined with other PBC techniques that do not require Assumption 1 but preserve the mechanical structure of the system, e.g., IDA-PBC. For further details, see the example in Section VI-C.

IV. Performance Assessment

The terms $u_{\mathtt{di}}(\dot{q}_{\mathtt{a}})$ and $u_{\mathtt{gy}}(q,p)$ defined in (5) and (6), respectively, affect the damping of the closed-loop system. Consequently, these terms play a relevant role in the performance of the closed-loop system. In particular, the bound of the $\mathcal{L}_2$-norm of the velocities can be improved by choosing appropriate values for the mentioned control terms. To show this, we revisit the following linear algebra theorem, whose proof can be found in [20].

Theorem 2 [20]: For a complex matrix $A \in \mathbb{C}^{n\times m}$, $n \leq m$, we have

$$\sigma_n(A) \geq \min_{1\leq k\leq n} \left\{ |A_{kk}| - \frac{1}{2} \sum_{j=1;j\neq k}^{n} \left(|A_{kj}| + |A_{jk}|\right) \right\},$$

where $\sigma_n(A)$ denotes the smallest singular value of A.

In light of Theorems 1 and 2 we can establish the following result.

Proposition 1: Consider the closed-loop system (12) and suppose that $D_{\mathtt{uu}}$ is constant. An $\mathcal{L}_2$-norm bound for the velocities is given by

$$\|\dot{q}\|_{\mathcal{L}_2} \leq \left(\frac{1}{\lambda_{\mathtt{min}}(D_{\mathtt{d}})} H_{\mathtt{d}}(q_0, p_0)\right)^{\frac{1}{2}}. \tag{16}$$

Moreover, the lowest possible value of this bound is obtained by selecting D_{ψ} and $\Gamma(q,p)$ such that

$$\Gamma_{\mathtt{u}}(q,p) = D^{\top}_{\mathtt{au}}(q,p) \tag{17}$$
$$\lambda_{\mathtt{min}}(D_{\mathtt{uu}}) \leq \lambda_{\mathtt{min}}\big(D_{\mathtt{d_a}}\big), \tag{18}$$

where $\lambda_{\mathtt{min}}(\cdot)$ denotes the smallest eigenvalue of the corresponding matrix and $D_{\mathtt{d_a}}(q,p)$ is defined as in (8).

Proof: Note that

$$\dot{H}_{\mathtt{d}} = -\|\dot{q}\|^2_{D_{\mathtt{d}}} \leq -\lambda_{\mathtt{min}}(D_{\mathtt{d}})\|\dot{q}\|^2. \tag{19}$$

Recall that (12) has an asymptotically stable equilibrium at $(q_{\star}, \mathbf{0})$ (see Theorem 1 and its proof) and $H_{\mathtt{d}_{\star}} = 0$. Therefore, integrating (19) from zero to infinity and noting that $\lim_{t\to\infty} H_{\mathtt{d}}(q,p) = H_{\mathtt{d}_{\star}}$, we obtain

$$H_{\mathtt{d}}(q_0, p_0) \geq \lambda_{\mathtt{min}}(D_{\mathtt{d}}) \int_0^{\infty} \|\dot{q}\|^2 \, \mathrm{d}\, t,$$

where $q_0, p_0 \in \mathbb{R}^n$ denote the initial conditions of (1). Hence, given the definition of $\mathcal{L}_2$-norm [4], we obtain the bound (16). Note that this bound decreases as $\lambda_{\mathtt{min}}(D_{\mathtt{d}})$ increases. Furthermore, from (14) and (17), we have

$$D_{\mathtt{d}} = \begin{bmatrix} D_{\mathtt{d_a}}(q) & \mathbf{0} \\ \mathbf{0} & D_{\mathtt{uu}} \end{bmatrix} \succ 0. \tag{20}$$

Now, consider an arbitrary matrix $\hat{D}_{\mathtt{au}} \in \mathbb{R}^{m\times s}$ different from zero such that

$$\hat{D}_{\mathtt{d}}(q) := \begin{bmatrix} D_{\mathtt{d_a}}(q) & \hat{D}_{\mathtt{au}} \\ \hat{D}^{\top}_{\mathtt{au}} & D_{\mathtt{uu}} \end{bmatrix} \succ 0.$$

Since $D_{\mathtt{d}}$ and $\hat{D}_{\mathtt{d}}(q)$ are positive definite (consequently symmetric), the singular values of these matrices are the same as their eigenvalues. Therefore, from Theorem 2, we conclude

$$\lambda_{\mathtt{min}}(\hat{D}_{\mathtt{d}}) < \lambda_{\mathtt{min}}(D_{\mathtt{d}}).$$

Moreover, D_{d} has a block-diagonal structure as shown in (20). Hence, from (18), we obtain

$$\lambda_{\min}(D_{\mathrm{d}}) = \lambda_{\min}(D_{\mathrm{uu}}).$$

We stress that the eigenvalues of D_{uu} cannot be modified by the controller. Thus, the selection (17)-(18) guarantees the smallest value for the right-hand element of the inequality (16). ■

We recall that the $\mathcal{L}_2$-norm of a signal is closely related to its energy. Thus, a small $\mathcal{L}_2$-norm of the velocities implies that the energy of the transitory behavior is low. Moreover, $u_{\mathrm{di}}(\dot{q}_{\mathrm{a}})$ and $u_{\mathrm{gy}}(q, p)$ depend on $\dot{q}$. Hence, a small $\mathcal{L}_2$-norm of the velocities also implies that these control terms spend a small amount of energy.

Remark 3: The result of Proposition 1 can be extended to non-constant matrices $D_{\mathrm{uu}}(q, p)$ by considering

$$\overline{\lambda} := \max\{\lambda_{\min}(D_{\mathrm{uu}}(q, p))\};$$
$$\underline{\lambda} := \min\{\lambda_{\min}(D_{\mathrm{uu}}(q, p))\}.$$

Then, replacing (18) by $\overline{\lambda} \leq \lambda_{\min}(D_{\mathrm{d_a}})$ and following the rationale of Proposition 1, we obtain the bound

$$\|\dot{q}\|_{\mathcal{L}_2} \leq \left(\frac{1}{\underline{\lambda}} H_{\mathrm{d}}(q_0, p_0)\right)^{\frac{1}{2}}.$$

However, we underscore that this bound is, in general, more conservative than (16).

V. Relaxing the Assumption on D_{d}

The results of Sections III and IV require $D_{\mathrm{d}}(q, p)$ to be positive definite. Remarkably, near the equilibrium point, this condition may be restrictive as is shown by the following proposition.

Proposition 2: Consider the system (12), such that $\Phi(q_{\mathrm{a}})$ satisfies (9) for the desired configuration $q_\star \in \mathcal{E}$, and

$$\mathcal{K} := \left(\frac{\partial^2 V_{\mathrm{d}}}{\partial q^2}\right)_\star, \quad F_{\mathrm{d}} := J_{\mathrm{d}_\star} - D_{\mathrm{d}_\star}.$$

Then, the system has a locally exponentially stable equilibrium at $(q_\star, \mathbf{0})$ if the solutions $\lambda \in \mathbb{C}$ to

$$\det\Big(M_\star \lambda^2 - F_{\mathrm{d}}\lambda + \mathcal{K}\Big) = 0 \tag{21}$$

satisfy $\mathrm{Re}(\lambda_i) < 0$ for all $i \in \{1, \ldots, n\}$.

Proof: The linearization of (12) around the point $(q_\star, \mathbf{0})$ is given by $\dot{z} = \mathcal{A}z$, where

$$z := \begin{bmatrix} q - q_\star \\ p \end{bmatrix}, \quad \mathcal{A} := \begin{bmatrix} \mathbf{0} & M_\star^{-1} \\ -\mathcal{K} & F_{\mathrm{d}} M_\star^{-1} \end{bmatrix}. \tag{22}$$

Note that, according to Lyapunov's indirect method (see [18, Th. 4.7]), the closed-loop system is locally exponentially stable if $\mathcal{A}$ is Hurwitz. However, $\mathcal{A}_{\mathrm{d}} = S_{\mathrm{d}} \mathcal{A} S_{\mathrm{d}}^{-1}$, with

$$S_{\mathrm{d}} := \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & M_\star^{-1} \end{bmatrix}, \quad \mathcal{A}_{\mathrm{d}} := \begin{bmatrix} \mathbf{0} & I_n \\ -M_\star^{-1}\mathcal{K} & M_\star^{-1} F_{\mathrm{d}} \end{bmatrix}.$$

Thus $\mathcal{A}_{\mathrm{d}}$ is similar to $\mathcal{A}$, consequently their eigenvalues are the same. Moreover, $\mathcal{A}_{\mathrm{d}}$ is a companion matrix of

$$L(\lambda) = I_n \lambda^2 - M_\star^{-1} F_{\mathrm{d}} \lambda + M_\star^{-1} \mathcal{K}.$$

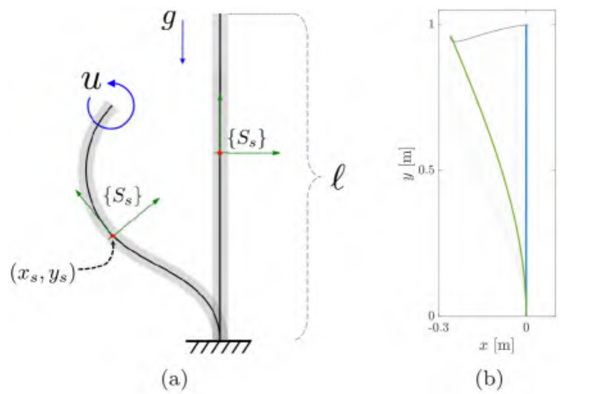

Fig. 1. Soft inverted pendulum with affine curvature [22]. Panel (a) shows the schematic of the system, where a reference frame $\{S_s\}$, attached to the point s, is highlighted. Panel (b) shows the evolution for the tuning (iii). The green and blue lines are the initial and final conditions, respectively. Intermediate shapes and the tip evolution are depicted in light and dark grey, respectively.

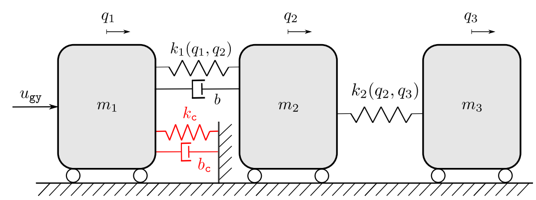

Fig. 2. Mechanical system studied in Subsection VI-B with a physical interpretation of the control terms $u_{\mathrm{es}}(q_{\mathrm{a}})$ and $u_{\mathrm{di}}(\dot{q}_{\mathrm{a}})$ in red. Note that the gyroscopic forces and coupled damping are injected through $u_{\mathrm{gy}}(q, p)$.

Hence, the eigenvalues of $\mathcal{A}_{\mathrm{d}}$ are the values $\lambda \in \mathbb{C}$ such that $\det(L(\lambda)) = 0$ (see [21]). Furthermore, since $M_\star$ is positive definite,

$$\det(L) = 0 \iff \det(M_\star^{-1})\det(M_\star L) = 0$$
$$\iff \det\Big(M_\star \lambda^2 - F_{\mathrm{d}}\lambda + \mathcal{K}\Big) = 0.$$

Thus, if every λ solution to (21) satisfies $\mathrm{Re}(\lambda) < 0$, then $\mathcal{A}$ is Hurwitz and the closed-loop system has a locally exponentially stable equilibrium at the desired point. ■

Remark 4: The result of Proposition 2 indicates that Assumption 2 and (10) are not necessary conditions for the stability of the desired equilibrium. Consequently, near the equilibrium, choosing the energy as the Lyapunov function may be restrictive.

VI. Examples

A. Soft Inverted Pendulum

Consider the soft inverted pendulum depicted in Fig. 1. The curvature function of this system is $\kappa_s = \theta_0(t) + \theta_1(t)s$, where $s \in (0, 1]$ parameterizes the positions along the main axis of the pendulum. For further details on the model see [22]. Considering the coordinates $q_1 = \theta_0 + \frac{1}{2}\theta_1$ $q_2 = \frac{1}{2}\theta_0 + \frac{1}{3}\theta_1$ this system admits a pH representation of the form (1), where the mass matrix and potential energy are obtained following the procedure proposed in [22] and using MATLAB for the corresponding computations. The control objective is to stabilize the system at its upward configuration, i.e., $q_\star = \mathbf{0}$. In

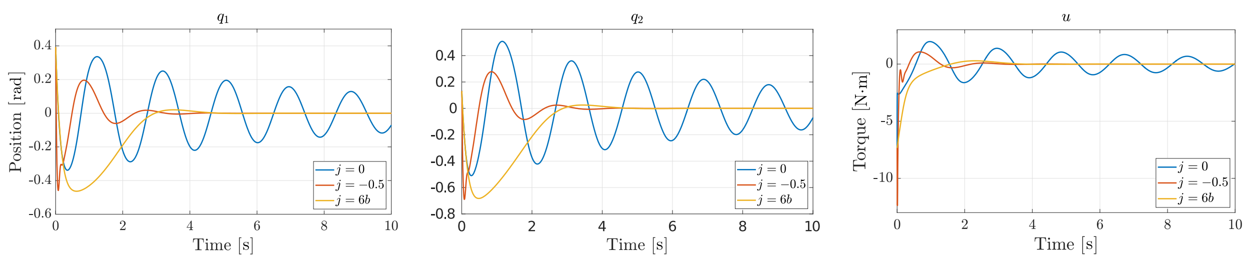

Fig. 3. Evolution of the soft inverted pendulum's configuration variables and control input for different values of the gyroscopic action j. The controllers that inject gyroscopic terms, i.e., $j \neq 0$ (depicted in orange and yellow), exhibit fewer oscillations and faster convergence.

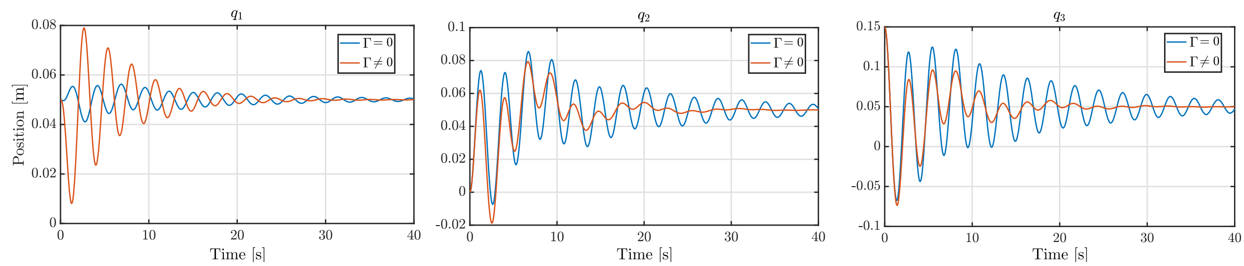

Fig. 4. Evolution of the mass-spring-damper system positions with (orange) and without (blue) coupled damping injection. The former results in a faster stabilization of the unactuated masses at the expense of oscillations with greater amplitude in the actuated mass.

particular, we have

$$\left(\frac{\partial^2 V}{\partial q^2}\right)_\star = k\begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix} + \frac{mg\ell}{30}\begin{bmatrix} -34 & 33 \\ 33 & -36 \end{bmatrix}, \ D = b\begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix},$$

where k and b are positive constant parameters; m and ℓ denote the mass and the length of the rod, respectively; and g is the gravitational acceleration. We consider $u_{\mathtt{di}}$ defined as in (5) and $u_{\mathtt{es}} = -K_{\mathrm{P}} q_{\mathrm{a}} + .(\partial V/\partial q_{\mathrm{a}})|_{q_{\mathrm{u}}=0}$, $\quad u_{\mathtt{gy}} = -2j\dot{q}_{\mathrm{u}}$, where K_P is positive and j is constant. Moreover, we consider the values $m = \ell = k = 1$, $b = 0.1$, $K_{\mathrm{P}} = 15$, $D_\psi = 0.8$. Hence,

$$D_{\mathtt{d}} = \begin{bmatrix} 1.2 & -0.6 + j \\ -0.6 + j & 1.2 \end{bmatrix}, \ \mathcal{K} \approx \begin{bmatrix} 117.1 & 4.79 \\ 4.79 & 0.23 \end{bmatrix}.$$

We consider three cases: (i) no gyroscopic terms, i.e., $j = 0$; (ii) a manually tuned injection of gyroscopic forces and coupled damping; and (iii) the coupled damping injection obtained in Proposition 1, i.e., $j = 6b = 0.6$. The simulation results, under initial conditions $(1/8, 1/24)\pi$, are shown in Fig. 3, where we observe that faster convergence and fewer oscillations are associate with the injection of gyroscopic forces and coupled damping. Furthermore, the norms of the velocities and the torque ranges are

$$\begin{aligned} &\|\dot{q}\|_{\mathcal{L}_2} = 84.87, && u \in [-7.34, 1.97], && \text{for} \quad j = 0 \\ &\|\dot{q}\|_{\mathcal{L}_2} = 137.83, && u \in [-12.38, 1.07], && \text{for} \quad j = -0.5 \\ &\|\dot{q}\|_{\mathcal{L}_2} = 59.16, && u \in [-7.34, 0.29], && \text{for} \quad j = 6b, \end{aligned}$$

which corroborate the result of Proposition 1.

B. Nonlinear Mass-Spring-Damper System

Consider a mass-spring-damper system consisting of three masses connected in series through two nonlinear springs and a damper between the first and second mass. Moreover, suppose a control input corresponding to a force exerted on the first mass. Such a system admits a pH representation of the form (1), with

$$D = \begin{bmatrix} b & -b & 0 \\ -b & b & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and the non-quadratic elastic potential $V(q) = \frac{a_1}{2}\ln(\cosh(q_1 - q_2)) + \frac{a_2}{2}(q_2 - q_3)^2 + \frac{a_3}{4}(q_2 - q_3)^4$, where m_1 is the actuated mass and m_2 and m_3 are the unactuated ones; q_1, q_2, and q_3 represent the positions; a_1, a_2, a_3, and b are positive parameters. The set of assignable equilibria is constrained to $q_1 = q_2 = q_3$, where the control objective is to stabilize the position of the third mass at $q_\star \neq 0$. To this end, we consider a controller of the form (11). In particular, we propose $u_{\mathtt{es}}(q) = -k_{\mathrm{c}}(q_1 - q_\star)$, $u_{\mathtt{di}}(\dot{q}_1) = -b_{\mathrm{c}}\dot{q}_1$, and $u_{\mathtt{gy}}(\dot{q}) = -2(j_2\dot{q}_2 + j_3\dot{q}_3)$, where k_{c} and b_{c} are positive and j_2, j_3 are constant. The physical interpretation of the controller is shown in Fig. 2. Note that

$$D_{\mathtt{d}} = \begin{bmatrix} b_{\mathrm{c}} + b & -b + j_2 & j_3 \\ -b + j_2 & b & 0 \\ j_3 & 0 & 0 \end{bmatrix}$$

has no definite sign if $j_3 \neq 0$. Thus, the stability of the closed-loop system cannot be analyzed with the result of Theorem 1 if j_3 is different from zero. While the control objective is achieved with j_2 and j_3 equal to zero (see Fig. 4), the closed-loop system exhibits poor damping propagation resulting in an oscillatory behavior for the unactuated masses. To overcome this issue, we use the result of Proposition 2 to analyze the stability of the equilibrium for j_2 and j_3 different from zero. In particular, we consider $m_1 = m_2 = 1$, $m_3 = 0.4$, $b = 0.5$, $a_1 = 0.5$, $a_2 = 1.5$, $a_3 = 2.5$, $k_{\mathrm{c}} = 5$, $b_{\mathrm{c}} = 2.5$, and $q_\star = 0.05$. Hence, we have

$$40\det(L) = 16\lambda^6 + 56\lambda^5 + \gamma_4\lambda^4 + \gamma_3\lambda^3 + \gamma_2\lambda^2 + \gamma_1\lambda + 150, \tag{23}$$

with $\gamma_1 = 60(j_2 + j_3) + 225$, $\gamma_2 = 60(j_2 + j_3) + 607$, $\gamma_3 = 16j_2 + 342$, and $\gamma_4 = 16j_2 + 200$. The polynomial (23) can be analyzed via the Routh-Hurwitz criterion. Fig. 4 shows the simulation results for $j_2 = j_3 = 0$ (blue) and $j_2 = b$ and $j_3 = -0.5$ (orange). In particular, the latter values guarantee that (23) is stable. We observe in Fig. 4 that the proposed gyroscopic forces and coupled damping reduce the oscillations in the unactuated masses.

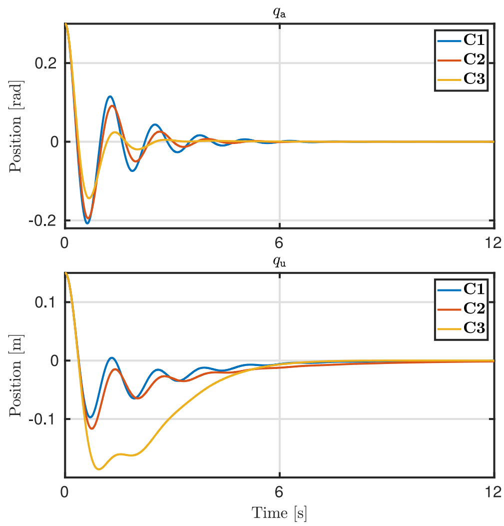

Fig. 5. Evolution of the configuration variables of the ball and beam system for IDA-PBC plus coupled damping injection.

C. Ball and Beam System

The behavior of the ball and beam system can be represented by (1), with

$$D = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix};\ M = \begin{bmatrix} L^2 + q_{\mathrm{u}}^2 & 0 \\ 0 & 1 \end{bmatrix};\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$V(q) = g q_{\mathrm{u}} \sin(q_{\mathrm{a}})$; where L, b, and g are positive constant parameters. The control objective is to stabilize the system at $q_\star = (0, 0)$. Note that this system does not satisfy Assumption 1. In [23], the authors design a controller based on IDA-PBC, neglecting the natural dissipation $b\dot{q}_{\mathrm{u}}$. Considering the natural dissipation, the mentioned controller combined with $u_{\mathrm{gy}}(q, p)$ yields[2]

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & M^{-1}M_{\mathrm{d}} \\ -M_{\mathrm{d}}M^{-1} & \Upsilon \end{bmatrix} \begin{bmatrix} \frac{\partial H_{\mathrm{d}}}{\partial q} \\ \frac{\partial H_{\mathrm{d}}}{\partial p} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ B \end{bmatrix} u_{\mathrm{gy}}$$

$$H_{\mathrm{d}}(q, p) = \frac{1}{2} p^\top M_{\mathrm{d}}^{-1}(q) p - g \cos(q_{\mathrm{a}}) + \frac{1}{2}\Psi^2(q),$$

with $\Psi(q) := q_{\mathrm{a}} + \frac{1}{\sqrt{2}} \operatorname{arcsinh}(\frac{q_{\mathrm{u}}}{L})$ and

$$\Upsilon(q, p) := -DM^{-1}(q)M_{\mathrm{d}}(q) + J_2(q, p) - Bk_{\mathrm{d}}B^\top,$$

where $J_2(q, p)$ is a skew-symmetric matrix and k_{d} is a positive gain.[3] Note that to prove stability considering $H_{\mathrm{d}}(q, p)$ as a Lyapunov candidate, we require the symmetric part $\Upsilon(q, p)$ to be negative semi-definite. Here, we show that the coupled damping injection $u_{\mathrm{gy}}(q, p)$ proposed in (6) can be combined with IDA-PBC to achieve this objective and to modify the transient response of the closed-loop system. To this end, we consider three cases:

C1 $\Gamma = \mathbf{0}$, i.e., no coupled damping injection.

C2 $\Gamma_{\mathrm{a}} = \frac{1}{2} b (L^2 + q_{\mathrm{u}}^2) + \frac{\sqrt{2}}{\sqrt{L^2 + q_{\mathrm{u}}^2}} n$ and $\Gamma_{\mathrm{u}} = -\frac{n}{L^2 + q_{\mathrm{u}}^2}$, which guarantees $\Upsilon(q, p) + \Upsilon^\top(q, p) \prec 0$.

C3 $\Gamma_{\mathrm{a}} = 2k_\gamma$ and $\Gamma_{\mathrm{u}} = k_\gamma$, with k_γ constant, which illustrates the effect of $u_{\mathrm{gy}}(q, p)$ on the response.

[2] We omit the arguments due to space limitations.

[3] See [23] for the expressions of $M_{\mathrm{d}}(q)$ and $J_2(q, p)$.

We consider $g = 9.78$, $L = 0.5$, $k_{\mathrm{d}} = 0.3$, $n = 0.05$, $k_\gamma = 0.1$, $q_0 = (0.3, 0.15)$, and $p_0 = \mathbf{0}$ for simulation purposes. The stability of $(q_\star, \mathbf{0})$ is proven via linearization in **C1** and **C3**, while $H_{\mathrm{d}}(q, p)$ qualifies as a Lypunov function in **C2** because of the proposed $\Gamma(q)$. The simulation results are shown in Fig. 5.

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