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On invariant combinations of Q_{ij} coefficients and a novel invariant I_Q

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ABSTRACT

Invariants of the plane-stress elasticity stiffness matrix $[Q]$ play a key role in composite design, as they simplify design processes. Multiple invariants have been proposed independently in the past, such as the 'trace' $Tr = trace([Q])$ by Tsai and Melo or the invariants U_1, U_4 and U_5, which are often used in context of lamination parameters.

The present paper presents a parametric description of rotation-independent (invariant) linear combinations of the coefficients Q_{ij} of $[Q]$. It is shown that the aforementioned invariants are special cases of the developed parametric description, in which a novel invariant $I_Q = Q_{11} + Q_{22} + Q_{12} + Q_{66}$ plays a key-role.

1. Introduction

Invariants play a key-role in composite design since decades [1]. In recent publications Tsai [2–5], the 'trace' invariant (Tr) has been proposed, as a scalar material property, defined as $Tr = Q_{11}+Q_{22}+2Q_{66} = \bar{Q}_{11} + \bar{Q}_{22} + 2\bar{Q}_{66}$ in context of UD-reinforced plies and corresponding laminates. In earlier publications the invariants U_1, U_4, U_5 and I_1, I_2 have been proposed.

All those invariants are used in context of the classical laminate theory (CLT), which bases on the plane-stress assumption, on engineering-strain formulation and on matrix notation [6]. The development of the 'trace' invariant is presented in the literature [2, p. 65] with a link to tensorial strains, which appears questionable at first glance, as the 'trace' is used in context of engineering strains with the CLT.

In fact, all the aforementioned invariants are linear combinations of the coefficients Q_{ij} of the plane-stress elasticity-stiffness matrix $[Q]$. In the present paper a generalized description of all $[Q]$-specific invariants is presented. It is shown that all previously presented invariants (Tr, U_1 U_4 U_5 and I_1, I_2) are special cases of the generalized form. A novel invariant I_Q is identified. It is defined as $I_Q = Q_{11} + Q_{22} + Q_{12} + Q_{66} = \bar{Q}_{11} + \bar{Q}_{22} + \bar{Q}_{12} + \bar{Q}_{66}$ and plays a key-role in the presented generalized invariant description. I_Q is compatible engineering-strain formulation of the CLT.

1.1. A comment on the 'trace' development

The coefficient matrices of the stress and strain tensors (see Reddy [7, p. 90]) are

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \quad , \quad [\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}. \tag{1}$$

Both are symmetric. For symmetric second-order tensors, as $[T]$ for example

$$[T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}, \tag{2}$$

the principal invariants I_1, I_2 and I_3 are defined as:

$$\begin{aligned} I_1 &= \text{'trace'} = T_{11} + T_{22} + T_{33} \\ I_2 &= trace(adj([T])) \\ &= T_{22} \cdot T_{33} - T_{23}^2 + T_{11} \cdot T_{33} - T_{13}^2 + T_{11} \cdot T_{22} - T_{12}^2 \\ I_3 &= \text{'determinant'} \\ &= T_{11}T_{22}T_{33} + 2T_{12}T_{13}T_{23} - (T_{22}T_{13}^2 + T_{11}T_{23}^2 + T_{33}T_{12}^2) \quad , \end{aligned}$$

with $adj(...)$ indicating the adjoint (see Appendix), and I_2 being the sum of the main sub-determinants.

In context of composite-laminate design the matrix formulation is used (see Reddy [7, p.91]), with $[C]$ being the 6×6 stiffness matrix, deduced from the fourth-order elasticity tensor

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = [C] \cdot \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = [C] \cdot \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad . \tag{3}$$

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The simplification to the 2D plane-stress case, in the CLT, leads to the expressions

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{bmatrix}=\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix}=[Q]\cdot\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{bmatrix}=[Q]\cdot\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix}=[Q]\cdot\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}=[Q][R]\cdot\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{bmatrix}\quad , \quad (4)$$

as outlined in multiple established textbooks in the field (see [2,6–12]). Note that $[R]$ defines the Reuter matrix. The plane-stress elasticity stiffness matrix in the material ($[Q]$) and the global $[\bar{Q}]$ coordinate system is defined as:

$$[Q]=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\quad , \quad [\bar{Q}]=\begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix} \tag{5}$$

while both matrices refer to engineering strains. For tensorial shear strain ($\varepsilon_{ij}=\frac{1}{2}\gamma_{ij}$) it reads:

$$[Q][R]=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & 2Q_{66}\end{bmatrix}\quad . \tag{6}$$

Tsai and Melo [2] identified that the trace Tr of $[Q][R]$ is invariant under rotation, which can be shown with the following $\bar{Q}_{ij}$ coefficients, which refer to the laminate's global coordinate system. Therein, $m = cos(\alpha)$ and $n = sin(\alpha)$, with α being the plies' rotation angle with respect to the laminate's global x-axis.

$$\begin{aligned}
\bar{Q}_{11} &= m^2(m^2Q_{11}+n^2Q_{12})+n^2(m^2Q_{12}+n^2Q_{22})+4m^2n^2Q_{66}\\
\bar{Q}_{22} &= n^2(n^2Q_{11}+m^2Q_{12})+m^2(n^2Q_{12}+m^2Q_{22})+4m^2n^2Q_{66}\\
\bar{Q}_{66} &= m^2n^2(Q_{11}+Q_{22})-2m^2n^2Q_{12}+(m^4-2m^2n^2+n^4)Q_{66}\\
\bar{Q}_{12} &= m^2n^2(Q_{11}+Q_{22})+(m^4+n^4)Q_{12}-4m^2n^2Q_{66}
\end{aligned} \tag{7}$$

Tsai and Melo defined the 'trace' as: $Tr = \bar{Q}_{11}+\bar{Q}_{22}+2\bar{Q}_{66} = Q_{11}+Q_{22}+2Q_{66} = trace([Q][R])$. It is important to highlight here that $trace([Q])$ is not invariant, even though one finds this statement in some references (see e.g. Tsai and Melo [2, p.280]).

1.2. Use of the trace Tr

Tsai and Melo use Tr for the development of a trace-normalized description of ply and laminate properties, which allows for generalizing the design processes to a group of materials.

$$\begin{array}{lcl}
[\bar{Q}] = Tr\cdot[\bar{Q}^*] = & & \text{trace normalized!}\\
[A]=\sum_{k=1}^{n}[\bar{Q}]_k\cdot t_k & & [A^{Tr}]=\sum_{k=1}^{n}[\bar{Q}^*]_k\cdot t_k\\
[B]=\frac{1}{2}\sum_{k=1}^{n}[\bar{Q}]_k\cdot(h_k^2-h_{k-1}^2) & \rightarrow & [B^{Tr}]=\frac{1}{2}\sum_{k=1}^{n}[\bar{Q}^*]_k\cdot(h_k^2-h_{k-1}^2)\\
[D]=\frac{1}{3}\sum_{k=1}^{n}[\bar{Q}]_k(h_k^3-h_{k-1}^3) & & [D^{Tr}]=\frac{1}{3}\sum_{k=1}^{n}[\bar{Q}^*]_k(h_k^3-h_{k-1}^3)
\end{array} \tag{8}$$

For this concept it is essential that Tr (or another scalar parameter) is invariant from rotation, in order to consider it a scalar pre-factor for the summations in $[A]$, $[B]$ and $[D]$. However, other pre-factors are conceivable as well, as long they are invariant from rotation-

2. A parametric description of all Q_{ij}-related invariants

In the literature other invariants of $[Q]$ are presented for different purposes, as for example the following cases:

- **Case** U_1: $U_1=\frac{3}{8}Q_{11}+\frac{3}{8}Q_{22}+\frac{1}{4}Q_{12}+\frac{1}{2}Q_{66}$ (see [2, p.62])
- **Case** U_4: $U_4=\frac{1}{8}Q_{11}+\frac{1}{8}Q_{22}+\frac{3}{4}Q_{12}-\frac{1}{2}Q_{66}$ (see [2, p.62])
- **Case** U_5: $U_5=\frac{1}{8}Q_{11}+\frac{1}{8}Q_{22}-\frac{1}{4}Q_{12}+\frac{1}{2}Q_{66}$ (see [2, p.62])
- **Case** I_1: $I_1=Q_{11}+Q_{22}+2Q_{12}$ (see [2, p.65])
- **Case** I_2: $I_2=$ 'trace' $Tr=Q_{11}+Q_{22}+2Q_{66}$ (see [2, p.65])
- **Case** I_Q: $2I_Q=I_1+I_2=2(Q_{11}+Q_{22}+Q_{12}+Q_{66})$

In fact, Tr and all other presented invariants are linear combinations of the Q_{ij} coefficients. This leads to the question whether all those invariants can be expressed with a single relation.

The following section shows that this is possible. A simple four-parameter optimization approach has been set up. The objective function was defined as

$$\begin{aligned}
Obj = \Big|\frac{1}{8}\,&[p_1\cdot(\bar{Q}_{11}-Q_{11})+p_2\cdot(\bar{Q}_{22}-Q_{22})\\
&+\ p_3\cdot(\bar{Q}_{12}-Q_{12})+p_4\cdot(\bar{Q}_{66}-Q_{66})]\Big|\quad ,
\end{aligned} \tag{9}$$

with the parameters being defined as integer variables, with $1 \le p_1,\dots,p_4 \le 8$, in order to exclude trivial solutions.

Analyses of feasible solutions revealed that for all cases $p_1 = p_2$, which are hereafter summarized using b. In addition, it was found that all solutions show an integer offset in the parameters p_3 and p_4, with $p_3 = b-i$, $p_4 = b+i$. The observed offset is denoted as a hereafter, leading to the expression

$$\begin{aligned}
&b\cdot\bar{Q}_{11}+(b-a)\cdot\bar{Q}_{12}+b\cdot\bar{Q}_{22}+(b+a)\cdot\bar{Q}_{66}=\\
&\quad b\cdot Q_{11}+(b-a)\cdot Q_{12}+b\cdot Q_{22}+(b+a)\cdot Q_{66}\quad ,
\end{aligned} \tag{10}$$

or

$$\begin{aligned}
&b\cdot\left(\bar{Q}_{11}+\bar{Q}_{12}+\bar{Q}_{22}+\bar{Q}_{66}\right)+a\cdot\left(\bar{Q}_{66}-\bar{Q}_{12}\right)=\\
&b\cdot\left(Q_{11}+Q_{12}+Q_{22}+Q_{66}\right)+a\cdot\left(Q_{66}-Q_{12}\right)\quad .
\end{aligned} \tag{11}$$

in rearranged form. The parameters b and a are defined as: $b \ge 1 \in \mathbb{N}$ and $a \in \mathbb{Z}$. Thus, all invariant combinations can be parameterized using b and a as shown hereafter. With the invariant $I_Q = \bar{Q}_{11}+\bar{Q}_{12}+\bar{Q}_{22}+\bar{Q}_{66} = Q_{11}+Q_{12}+Q_{22}+Q_{66}$ and the coefficient from Equation-set (7) one finds:

$$I_{para} = b\cdot I_Q + a\cdot(\bar{Q}_{66}-\bar{Q}_{12}) \tag{12}$$

$$\begin{aligned}
&= b\cdot I_Q + a\cdot\left[m^2n^2(Q_{11}+Q_{22})-2m^2n^2Q_{12}+(m^4-2m^2n^2+n^4)Q_{66}+\right.\\
&\quad\left.-m^2n^2(Q_{11}+Q_{22})-(m^4+n^4)Q_{12}+4m^2n^2Q_{66}\right]\\
&= b\cdot I_Q + a\left[-(m^4+2m^2n^2+n^4)Q_{12}+(m^4+2m^2n^2+n^4)Q_{66}\right]
\end{aligned}$$

$$= b\cdot I_Q + a\left[Q_{66}-Q_{12}\right] \tag{13}$$

$$= b\cdot\left(Q_{11}+Q_{12}+Q_{22}+Q_{66}\right)+a\cdot(Q_{66}-Q_{12})\quad . \tag{14}$$

It is noted that the $Q_{66}-Q_{12} = \bar{Q}_{66}-\bar{Q}_{12}$ is invariant as well (See Eqs. (12) and (13)). In summary, the following parameterized expression I_{para} is invariant from rotation for all combinations of b and a:

$$I_{para} = b\cdot\left(Q_{11}+Q_{12}+Q_{22}+Q_{66}\right)+a\cdot(Q_{66}-Q_{12}) \tag{15}$$

$$= b\cdot\left(\bar{Q}_{11}+\bar{Q}_{12}+\bar{Q}_{22}+\bar{Q}_{66}\right)+a\cdot(\bar{Q}_{66}-\bar{Q}_{12})$$

$$\text{with}\quad b \ge 1 \in \mathbb{N} \text{ and } a \in \mathbb{Z} \tag{16}$$

3. Application

The presented parametric invariant I_{para} shall capture all the aforementioned invariants, which is demonstrated hereafter for Tr, U_1, U_4, U_5, I_1, I_2 and I_Q.

3.1. Case I_1: $b = 1$, $a = -1$

See Tsai and Melo [2, p.65].

$$\begin{aligned}
I_{para} &= b\cdot\left(\bar{Q}_{11}+\bar{Q}_{12}+\bar{Q}_{22}+\bar{Q}_{66}\right)+a\cdot(\bar{Q}_{66}-\bar{Q}_{12})\\
&= \bar{Q}_{11}+\bar{Q}_{22}+2\bar{Q}_{12} = I_1 \text{ in Tsai and Melo}
\end{aligned}$$

Table 1
Case-specific b, a parameters for I_{para} for known invariants. Note that scalar pre factors are disregarded for U_1, U_4, U_5.

Case	Source	Invariant	I_{para} coefficients	
			b	a
I_1	[2, p. 65]	$Q_{11}+Q_{22}+2Q_{12}$	1	1
$I_2 = Tr!$	[2, p. 65]	$Q_{11}+Q_{22}+2Q_{66}$	1	−1
I_Q	present paper	$Q_{11}+Q_{22}+Q_{12}+Q_{66}$	1	0
U_1	[2, p. 62]	$\frac{1}{8}\left(3Q_{11}+3Q_{22}+2Q_{12}+4Q_{66}\right)$	3	1
U_4	[2, p. 62]	$\frac{1}{8}\left(Q_{11}+Q_{22}+6Q_{12}-4Q_{66}\right)$	1	−5
U_5	[2, p. 62]	$\frac{1}{8}\left(Q_{11}+Q_{22}-2Q_{12}+4Q_{66}\right)$	1	3

3.2. Case I_2: $b = 1$, $a = 1$

See Tsai and Melo [2, p.65].

$$I_{para} = b\cdot\left(\bar{Q}_{11}+\bar{Q}_{12}+\bar{Q}_{22}+\bar{Q}_{66}\right)+a\cdot\left(\bar{Q}_{66}-\bar{Q}_{12}\right)$$
$$= \bar{Q}_{11}+\bar{Q}_{22}+2\bar{Q}_{66} = \text{trace} = I_2 \text{ in Tsai and Melo}$$

3.2.1. Case I_Q: $b = 1$, $a = 0$

$$I_{para} = b\cdot\left(\bar{Q}_{11}+\bar{Q}_{12}+\bar{Q}_{22}+\bar{Q}_{66}\right)+0\cdot\left(\bar{Q}_{66}-\bar{Q}_{12}\right)$$
$$= \bar{Q}_{11}+\bar{Q}_{22}+\bar{Q}_{12}+\bar{Q}_{66} = I_Q$$

3.3. Case U_1:

Note that the scalar factor 1/8 is excluded for U_1, U_4 and U_5.

$$U_1 = \frac{3}{8}Q_{11}+\frac{3}{8}Q_{22}+\frac{1}{4}Q_{12}+\frac{1}{2}Q_{66}$$
$$= \frac{1}{8}\left(3Q_{11}+3Q_{22}+2Q_{12}+4Q_{66}\right)$$
$\rightarrow$ inserting $b = 3, a = 1$ in Eq. (13) yields:
$$= \frac{1}{8}(3\cdot(\bar{Q}_{11}+\bar{Q}_{22}+\bar{Q}_{12}+\bar{Q}_{66})+1(\bar{Q}_{66}-\bar{Q}_{12}))$$
$$= \frac{1}{8}(3\cdot\bar{Q}_{11}+3\bar{Q}_{22}+2\bar{Q}_{12}+4\bar{Q}_{66})$$

3.4. Case U_4:

$$U_4 = \frac{1}{8}Q_{11}+\frac{1}{8}Q_{22}+\frac{3}{4}Q_{12}-\frac{1}{2}Q_{66}$$
$$= \frac{1}{8}Q_{11}+\frac{1}{8}Q_{22}+\frac{6}{8}Q_{12}-\frac{4}{8}Q_{66}$$
$$= \frac{1}{8}\left(1Q_{11}+1Q_{22}+6Q_{12}-4Q_{66}\right)$$
$\rightarrow$ inserting $b = 1, a = -5$ in Eq. (13) yields:
$$= \frac{1}{8}(1\cdot(\bar{Q}_{11}+\bar{Q}_{22}+\bar{Q}_{12}+\bar{Q}_{66})-5(\bar{Q}_{66}-\bar{Q}_{12}))$$
$$= \frac{1}{8}(\bar{Q}_{11}+Q_{22}+6\bar{Q}_{12}-4\bar{Q}_{66})$$

3.5. Case U_5:

$$U_5 = \frac{1}{8}Q_{11}+\frac{1}{8}Q_{22}-\frac{1}{4}Q_{12}+\frac{1}{2}Q_{66}$$
$$= \frac{1}{8}\left(Q_{11}+Q_{22}-2Q_{12}+4Q_{66}\right)$$
$\rightarrow$ inserting $b = 1, a = 3$ in Eq. (13) yields:
$$= \frac{1}{8}(1\cdot(\bar{Q}_{11}+\bar{Q}_{22}+\bar{Q}_{12}+\bar{Q}_{66})-3(\bar{Q}_{66}-\bar{Q}_{12}))$$
$$= \frac{1}{8}(\bar{Q}_{11}+Q_{22}-2\bar{Q}_{12}+4\bar{Q}_{66})$$

Table 1 summarizes the previously presented cases.

4. Conclusion

Invariants play a key role in composite design. Different Q_{ij}-dependent invariants of the plane-stress elasticity matrix $[Q]$ were presented in the past individually, such as the ‘trace’ Tr, U_1, U_4, U_5 or I_1, I_2. The present paper presents a generalized parametric invariant description of all rotation independent linear combinations of Q_{11}, Q_{12}, Q_{22} and Q_{66} of $[Q]$ or $[\bar{Q}]$, respectively.

It is demonstrated that previously presented invariants are all covered by the parametric formulation. It is defined as $I_{para} = b\cdot\left(\bar{Q}_{11}+\bar{Q}_{12}+\bar{Q}_{22}+\bar{Q}_{66}\right)+a\cdot(\bar{Q}_{66}-\bar{Q}_{12})$, with the parameter b and a being limited to: $b \geq 1 \in \mathbb{N}$ and $a \in \mathbb{Z}$. Note that Tsai’s trace Tr is determined for $b = a = 1$. The invariant I_Q plays a key-role in the parametric description. It is defined as $I_Q = Q_{11}+Q_{22}+Q_{12}+Q_{66}$. Being the sum of all Q_{ij} coefficients of $[Q]$, makes it an easy to use invariant, compatible with the CLT.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix

The inverse of matrix $[T]$, can be written as

$$[T]^{-1} = \frac{1}{det([T])}\cdot adj([T]) \quad . \tag{17}$$

The adjoint of the symmetric square matrix T,

$$[T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}, \tag{18}$$

denoted as adj(T), is defined as:

$$adj([T]) = \begin{bmatrix} +(T_{22}T_{33}-T_{23}^2) & -(T_{12}T_{33}-T_{13}T_{23}) & +(T_{12}T_{23}-T_{13}T_{22}) \\ -(T_{12}T_{33}-T_{13}T_{23}) & +(T_{11}T_{33}-T_{13}^2) & -(T_{11}T_{22}-T_{12}^2) \\ +(T_{12}T_{23}-T_{13}T_{22}) & -(T_{11}T_{22}-T_{12}^2) & +(T_{11}T_{22}-T_{12}^2) \end{bmatrix}^T \tag{19}$$

Thus, $trace(adj[T])$ is defined as

$$trace(adj([T])) = T_{22} \cdot T_{33} - T_{23}^2 + T_{11} \cdot T_{33} - T_{13}^2 + T_{11} \cdot T_{22} - T_{12}^2 \quad . \quad (20)$$

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