DEPARTMENT OF INFORMATICS

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Masters Thesis in Robotics, Cognition, Intelligence

Formulation of Decoupled Dynamics for Efficient Control of Floating Base Robots

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Formulierung der enkoppelten Dynamik von schwebenden Robotern für deren effiziente Steuerung

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Disclaimer

I confirm that this master's thesis is my own work and I have documented all sources and material used.

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Abstract

Floating base robots are a broad class of robotic systems. They consist of a floating base and chains of manipulators mounted on the base. Examples of floating base robots are space robots, where a robotic arm is mounted on a spaceship and legged robots, which contain legs and arms mounted on a torso. Floating base robots are much more complex to analyze and to control than fixed robots. In contrast to industrial robots tied to the floor, any motion of the joints induces a motion of the base. In 3 dimensions the base has 6 degrees of freedom, rendering the configuration space a curved manifold $SE(3) \times \mathbb{R}^n$.

Legged robots are more that an order of magnitude less energy efficient than walking, hopping and running of humans and animals [KAA+18]. This prohibits the widespread use of such systems, given the limited resources on board. The goal of this work was to systematically find decompositions of the system dynamics into external and internal components and to use those decompositions for efficient control strategies. The conjecture is that such control strategies make legged motion more natural and energy efficient, by injecting energy only when needed.

In this work, we first systematically analyze the conditions, under which the base dynamics decouples from the internal motion of the joints. In contrast to the Euler-Netwon methodology usually employed in robotics, concepts from mathematical Physics are used. The Hamel equations on the manifold $SE(3) \times \mathbb{R}^n$ are derived in a coordinate free formulation and the equivalence of the equations of motion to the ones derived from the Euler-Netwon method is proven. The problem of decoupling of the equations of motion is turned upside down. Starting with an unknown decoupling transformation, necessary conditions for the properties of the transform are formulated, the equations of motion are transformed and the resulting equations of motion are solved for transformations, which decouple the system. It turns out that any transformation, which decouples the base wrench from the joints torques and diagonalizes the mass matrix, decouples the dynamics. Employing a transformation, which results in a constant of motion of the transformed system, leads to an invariance structure. These results hold only for the Bolzmann-Hamel equations, which have a non-passive Coriolis matrix. Therefore, a passive formulation is derived and the

conditions for decoupling are stated. The system does not decouple under any transform and only shows invariance, when the transformation leads to a constant of motion.

The results of decoupling and invariance are applied to the decomposition of the dynamics into external and internal velocity. For control, two different types of decomposition are used. The first decomposition uses an inertially aligned center of mass frame, which leads to a constant of motion for the dynamics of the external velocity. The second decomposition assumes that the center of mass frame is aligned to the locked velocity. In this frame the total momentum is not conserved and the angular velocity is not necessarily integrable, which could lead to a path dependent orientation of the CoM frame. The problem of integrability of the CoM frame is addressed by deriving necessary and sufficient conditions for intergrability.

The results of decoupling are applied to hopping robots. Monopods with one prismatic or one revolute joint are considered. Impedance control is used to control the system. It consists of virtual radial springs between the foot and the center of mass and polar springs at the center of mass.

For the prismatic hopper, stable fully actuated hopping in place is demonstrated. Underactuated hopping in place is unstable by design. Fully actuated hopping forward was not considered and is subject to future work.

For the revolute hopper, stable fully actuated hopping forward is demonstrated by using radial damping during flight. The gaits look natural with an almost linear horizontal momentum. The occurrence of stable gaits is rather insensitive to initial conditions, all measures are fully stationary and the control actions are confined to desirable values. It turns out that the radial controller acts as the stabilizing element allowing the polar controller to converge to stable gaits.

Contents

1. Introduction

The thesis is structured in four parts: Introduction Theory, and Dynamics and Conserved Quantities, Application to Hopping Robots and Conclusion.

Chapter 1 gives a brief introduction to floating base robots and their control problem. It further motivates the use of energy based methods for analyzing free floating systems and defines decoupling transforms. The research tasks of this work are summarized as problem statements and related to existing research.

Chapter 2 contains the theory needed for the subsequent modeling. A big part is devoted to Lee groups and Lee algebra. After introducing body frames, relations between Lee group elements and Lee algebra elements and link twist Jacobians are formulated generally for $SE(n)$. This allows direct translation to vector relations for $SE(2)$ and $SE(3)$.

Chapter 3 is about dynamics. The chapter first introduces the global formulation of Hamel equations on manifolds using Hamiltonian's variation principle. Then the equations of motion of a floating base robot on the manifold $SE(3) \times \mathbb{R}^n$ are derived. It is proven that the equations of motion are the same as the ones derived from the Euler-Newton method. A section is devoted to constants of motion and the integrability of the center of mass frame aligned with the locked velocity. Section Decoupling and Invariance derives the decoupling transform and applies it to the equations of motion. Subsequently a passive formulation is presented. It is shown that the dynamics decouple only, when the momentum map is part of the transform. The Chapter concludes with an axiomatic approach to external/internal decomposition. The conditions necessary for decomposition for an arbitrarily aligned center of mass frame are derived.

The chapter 4 applies the results of the previous chapters to hopping robots in the plane. The first part develops the relations of the Lee algebra on $SE(2)$, defines the link Jacobians and the Lagrangian. It also explains models specific to hopping and presents the control strategy employed. The subsequent sections show the results for a prismatic and a revolute hopper in the plane.

Chapter 5 gives a discussion and conclusion.

1.1. Motivation

The topic of this thesis is to use so called Energy based methods from mathematical Physics to systematically decouple the equations of motion of floating base robots for efficient control. Those methods include a Lagrangian formalism on manifolds to derive the equations of motion and constants of motion (conserved quantities along the trajectory). They are subsequently applied to the decoupling transformations.

Floating base robots are a broad class of robotic systems, where chains of robotic actuators are mounted on a movable base. Examples of floating base robots are space robots and legged robots. In the case of space robots a robot arm is mounted on a spaceship, while in the case of legged systems actuated legs are mounted on a torso. The mathematics of floating base robots is much more involved than the one of fixed robots, due to the nature of rigid body motion and due to Newton's third law:

- The moving base has 3 linear and 3 angular coordinates which are non-Cartesian.
- The reaction force to a joint motor torque alters the movement of the base in the case of a floating base robot, while it is absorbed by the ground in the case of a robotic arm bolted to the ground.

The first observation implies that the usual Lagrange formalism cannot be applied, since the base coordinates constitute the non-Cartesian manifold $SE(3) \times \mathbb{R}^n$. Therefore, the Lagrange formulation has to be generalized to manifolds. We will derive the Hamel equations from variation calculus in a global (coordinate free) framework in Chapter 3.1.3.

The second observation implies that the mass matrix M and the Coriolis matrix C couple the motion of the joints and of the base. This behavior is not desirable for control applications. e.g. Given a space robot doing repair work, it is very hard to control the end effector, if every movement of a joint changes the pose of the base. It would be much easier, if the motion of the base and of the arm were decoupled. A similar problem occurs in legged systems. The goal is to move the center of mass on a desired trajectory. However, this is only possible indirectly through contact with the ground of the legs. If a transformation of the equations of motion to an allocation space can be found, such that the resulting equations of motion are decoupled[Gio20], a controller can be defined on the allocation space and be applied to the equations of motion using the Jacobian transpose.

The decoupling transform is usually found by intuition. However in mathematical Physics there is a strong link between decoupling and constants of motion, which can be used to find the required transformation. A constant of motion is a conserved quantity along the solution of the equations of motion. It can be found by exploiting Nöther's theorem [Noe18]: If the Lagrangian of the system is invariant under a continuous symmetry, then there exists a constant of motion. Transforming the system variables to the constants of motion decouples the equations of motion for the conserved quantities. We are not only interested in transformations that fully decouple the base of the robot from the joints, but also in the weaker case, where joint movements do not influence the base. To this end, the question of finding suitable transformations is turned upside down in Chapter 3.3.2Decoupling Transform, by answering the question, which transforms give the desired result.

Another important topic for control is the passive formulation of the Coriolis matrix. While the equations of motion are uniquely defined by the Lagrangian, the shape of the Coriolis matrix is arbitrary as long as the equations of motion are fulfilled. The Coriolis matrix is said to be passive, if the following condition is met:

$$\dot{\boldsymbol{M}} = \boldsymbol{C} + \boldsymbol{C}^T \tag{1.1}$$

A passive formulation of the floating base robot system derived from energy based methods is defined in Section 3.3.3 and the decoupling properties are analyzed. This topic is the second item in Problem Statement.

Legged robots can be viewed as hybrid systems with two sets of dynamics. During flight, the robot is a free floating system, while at stance it is a fixed robot, which can topple over. If the robot dynamics can be decoupled during flight, the arms and legs can be separately controlled from movement of the center of mass. If the system dynamics can also be decoupled during the stance phase, it should be possible to design impedance controllers, which lead to an efficient movement of the center of mass.

1.2. Related Work

This work builds upon the thesis of Giordano [Gio20], which used internal and external decomposition for the control of space robots. In this work the question, when the dynamics decouple is formulated generally and the conditions necessary are stated.

Lagrangian based methods are not commonly used in Robotics. Murray [MLS94] develops the dynamics for the Cartesian case. The Hamel equations for free floating robots are used in [STNN17]. A Lee-group formulation of the kinematics and dynamics of constrained multi body systems is developed in [MM03]. This work develops the Hamel equations in a global formulation based on the book of Lee et. al. [LLM17]. Prominent books on geometrical mechanics are [MR13, BB04].

There is a large body of literature on legged locomotion and hopping. A review of hardware and control for single leg robot is presented in [SSS07]. Many control strategies for hopping use a form of Raibert's controller of the spring loaded inverted pendulum [Rai86]. Those strategies do not use a free floating model during flight, but assume that the angle of attack at touch down can be set arbitrarily.

1.3. Problem Statement

This section lists the problems that had to be addressed by the Thesis.

1.3.1. Energy-Based Decoupling for Free-Floating Robot Systems

Given a free floating system on $SE(3) \times \mathbb{R}^n$defined by the Lagrangian:

$$L = \frac{1}{2} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{M}_{bb} & \boldsymbol{M}_{bq} \\ \boldsymbol{M}_{bq}^T & \boldsymbol{M}_{qq} \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix} \tag{1.2}$$

With equations of motion:

$$\begin{bmatrix} \boldsymbol{M}_{bb} & \boldsymbol{M}_{bq} \\ \boldsymbol{M}_{bq}^T & \boldsymbol{M}_{qq} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\nu}}_b \\ \ddot{\boldsymbol{q}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_b & \boldsymbol{C}_{bq} \\ \boldsymbol{C}_{qb} & \boldsymbol{C}_q \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_b \\ \boldsymbol{\tau} \end{bmatrix} \tag{1.3}$$

Find all transforms $\boldsymbol{T}$:

$$\begin{bmatrix} \boldsymbol{v}_x \\ \boldsymbol{v}_y \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix} \qquad \boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_x & \boldsymbol{T}_{xy} \\ \boldsymbol{T}_{yx} & \boldsymbol{T}_y \end{bmatrix} \tag{1.4}$$

such that for the transformed equations of motion:

$$\begin{bmatrix} \boldsymbol{M}_x & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_y \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\nu}}_b \\ \ddot{\boldsymbol{q}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_x & \boldsymbol{C}_{xy} \\ \boldsymbol{C}_{yx} & \boldsymbol{C}_y \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_b \\ \boldsymbol{\tau} \end{bmatrix} \tag{1.5}$$

the blocks $\boldsymbol{C}_x$ and $\boldsymbol{C}_{xy}$ of the Coriolis matrix have the properties:

1. $\boldsymbol{C}_x \boldsymbol{\nu}_b + \boldsymbol{C}_{xy} \dot{\boldsymbol{q}} = 0$ vanish (invariance structure).
2. Or only $\boldsymbol{C}_{xy} \dot{\boldsymbol{q}} = 0$ vanishes (decoupling).

1.3.2. Passive Formulation

Find a passive formulation of the Coriolis $\boldsymbol{C}$ matrix and find all transformations that decouple the system. The system is passive if the following relation holds

$$\dot{\boldsymbol{M}} = \boldsymbol{C} + \boldsymbol{C}^T \tag{1.6}$$

1.3.3. Application to Hopping Robots

Apply the decomposition of the system dynamics into internal and external velocity to hopping legs in the plane. The fist system is a prismatic leg with a point mass for the base and the foot, respectively. The task consists of calculating the system dynamics and the Jacobians for control symbolically, writing a numeric simulation program and defining a control strategy based on decoupled dynamics. Tune the system for hopping in place and hopping forward, if applicable.

2. Theory

This Part contains the theory needed for solving problems for free floating robots. Mathematical concepts, such as Lee algebra, variation on manifolds and Hamel equations on manifolds are very abstract. In contrast to many other fields in mathematics, it is difficult for an engineer to understand the ideas behind those concepts. But it is even harder to apply them. One has to think hard how those concepts can be translated to the level, where actual calculations can be performed.

While there are many books on geometry, such as [MR13] or [BB04], it requires a huge amount of time to make practical use out of them, which an engineer in robotics normally does not have.

To this end, the Theory Part in this work takes the known facts about these topics as a starting point and translates them to practical frameworks in a handy notation optimized to perform actual calculations. In the Modeling Part the frameworks are applied to free floating robots.

To the best of our knowledge there is no reference in the literature that provides such a practical approach for engineers in robots on those topics.

2.1. Lee Groups and Lee Algebras

All relevant manifolds in robotics, such as the Euclidean space $\mathbb{R}^n$, the rotation groups $SO(2)$ and $SO(3)$, the special Euclidean groups $SE(2)$ and $SE(3)$ and the product groups $SE(2) \times \mathbb{R}^n$ and $SE(3) \times \mathbb{R}^n$ are Lee groups.

The theory of Lee groups and Lee algebras can be viewed from two perspectives. The Lee group formalism can pragmatically be seen as a toolbox for deriving relations between Lee group elements and Lee algebras elements. In our context the Lee group elements are rigid body transformations, while the Lee algebra elements correspond to velocities. On the other hand, the theory of Lee groups and Lee algebras has a geometric meaning. In this work we concentrate on the pragmatic view in Chapter 2.1.2.

2.1.1. Body Frames

This section defines rigid body transformations and body frames. The notation used is aligned with [Gio20, Chaper 2.1.1] and [MLS94, Chapter 2].

The coordinate transformation from an orthonormal frame Y to an orthonormal frame X is given by a rotation and a shift. The coordinates of a vector $\boldsymbol{y}$ expressed in frame Y are translated to coordinates $\boldsymbol{x}$ expressed in frame X by:

$$\boldsymbol{x} = \boldsymbol{R}_{xy}\boldsymbol{y} + \boldsymbol{o}_{xy} \tag{2.1}$$

The vector $\boldsymbol{o}_{xy}$ goes from the origin of frame X to the origin of frame Y. Is is expressed in frame X and is a $n \times 1$ vector. The rotation matrix $\boldsymbol{R}_{xy}$ transforms the from frame Y to frame X. It is a $n \times n$ matrix with $n(n-1)/2$ degrees of freedom. The rotation matrix $\boldsymbol{R}_{xy}$ has the following properties:

$$\boldsymbol{R}_{xy}^{-1} = \boldsymbol{R}_{xy}^{T} \tag{2.2a}$$

$$\boldsymbol{R}_{yx} = \boldsymbol{R}_{xy}^{T} \tag{2.2b}$$

$$\boldsymbol{R}_{xz} = \boldsymbol{R}_{xy}\boldsymbol{R}_{yz} \tag{2.2c}$$

$$\dot{\boldsymbol{R}}_{xy}^{T}\boldsymbol{R}_{xy} = -\boldsymbol{R}_{xy}^{T}\dot{\boldsymbol{R}}_{xy} \tag{2.2d}$$

With regards to robotics, we are not so much interested in coordinate transformations, but in transformations between frames. To this end, for each body y of a robot, a rotating frame Y is defined, such that the mass of body y does not move in frame Y. This frame is called body frame. The big advantage of introducing body frames is that the moment of inertia of body y, $\boldsymbol{I}_y$, is a constant matrix in frame Y. The pose of frame Y expressed in frame X can be compactly written by introducing the homogeneous transformation $\boldsymbol{H}_{xy}$:

$$\boldsymbol{H}_{xy} = \begin{bmatrix} \boldsymbol{R}_{xy} & \boldsymbol{o}_{xy} \\ \boldsymbol{0} & 1 \end{bmatrix} \qquad \boldsymbol{H}_{xy}^{-1} = \begin{bmatrix} \boldsymbol{R}_{xy}^{T} & -\boldsymbol{R}_{xy}^{T}\boldsymbol{o}_{xy} \\ \boldsymbol{0} & 1 \end{bmatrix} \tag{2.3}$$

The matrix $\boldsymbol{H}_{xy}$ is called rigid body transformation. It is a $(n+1) \times (n+1)$ matrix with $n(n+1)/2$ degrees of freedom.

The rigid body transformation $\boldsymbol{H}_{xy}$ has the following properties:

$$\boldsymbol{H}_{yx} = \boldsymbol{H}_{xy}^{-1} \tag{2.4a}$$

$$\boldsymbol{H}_{xz} = \boldsymbol{H}_{xy}\boldsymbol{H}_{yz} \tag{2.4b}$$

$$\dot{\boldsymbol{H}}_{xy}^{-1}\boldsymbol{H}_{xy} = -\boldsymbol{H}_{xy}^{-1}\dot{\boldsymbol{H}}_{xy} \tag{2.4c}$$

2.1.2. Lee Algebra for SE(n)

All the definitions and relations in this Chapter hold for rigid body transformations in any dimension. The general approach is to first derive relations between group elements $\boldsymbol{H}_{xy} \in SE(n)$ and Lee algebra elements $\hat{\boldsymbol{\omega}}_{xy} \in se(n)$ and at the very end specify the relations in terms of the Lee group and Lee algebra under consideration.

The linear body velocity $\boldsymbol{v}_{xy}$ expressed in frame X and is defined as:

$$\boldsymbol{v}_{xy} = \boldsymbol{R}_{xy}^{T}\dot{\boldsymbol{o}}_{xy} \tag{2.5}$$

The angular body velocity matrix $\hat{\boldsymbol{\omega}}_{xy}$ is also expressed in frame X and is defined as:

$$\hat{\boldsymbol{\omega}}_{xy} = \boldsymbol{R}_{xy}^T \dot{\boldsymbol{R}}_{xy} \tag{2.6}$$

The rotation matrix is a Lee group element $\boldsymbol{R}_{xy} \in SO(n)$. The matrix $\hat{\boldsymbol{\omega}}_{xy}$ is an element of the Lee algebra $\hat{\boldsymbol{\omega}}_{xy} \in so(n)$. It is skew symmetric: $\hat{\boldsymbol{\omega}}_{xy}^T = -\hat{\boldsymbol{\omega}}_{xy}$. This can be easily seen by using the identity (2.2d):

$$\begin{aligned}\hat{\boldsymbol{\omega}}_{xy}^T &= (\boldsymbol{R}_{xy}^T \dot{\boldsymbol{R}}_{xy})^T = \dot{\boldsymbol{R}}_{xy}^T \boldsymbol{R}_{xy} \\ &= -\boldsymbol{R}_{xy}^T \dot{\boldsymbol{R}}_{xy}^T = -\hat{\boldsymbol{\omega}}_{xy}\end{aligned}$$

Similar to the pure rotational case (2.6), a body velocity matrix $\hat{\boldsymbol{\nu}}_{xy} \in se(n)$ derived from the group element $\boldsymbol{H}_{xy} \in SE(n)$ is defined as:

$$\begin{aligned}\hat{\boldsymbol{\nu}}_{xy} &= \boldsymbol{H}_{xy}^{-1} \dot{\boldsymbol{H}}_{xy} \\ \begin{bmatrix}\hat{\boldsymbol{\omega}}_{xy} & \boldsymbol{v}_{xy} \\ \boldsymbol{0} & 0\end{bmatrix} &= \begin{bmatrix}\hat{\boldsymbol{\omega}}_{xy} & \boldsymbol{R}_{xy}^T \dot{\boldsymbol{o}}_{xy} \\ \boldsymbol{0} & 0\end{bmatrix}\end{aligned} \tag{2.7}$$

with the time derivative of $\boldsymbol{H}_{xy} \in SE(n)$ given by

$$\dot{\boldsymbol{H}}_{xy} = \begin{bmatrix}\boldsymbol{R}_{xy}\hat{\boldsymbol{\omega}}_{xy} & \dot{\boldsymbol{o}}_{xy} \\ \boldsymbol{0} & 0\end{bmatrix} \tag{2.8}$$

The matrix $\hat{\boldsymbol{\nu}}_{xy}$ is an element of the Lee algebra $se(n)$, $\hat{\boldsymbol{\nu}}_{xy} \in se(n)$, and contains the linear body velocity vector (2.5) and the angular body velocity (2.6) matrix. It is a $(n+1) \times (n+1)$ matrix with $n(n+1)/2$ degrees of freedom.

The $n(n-1)/2$ components of $\hat{\boldsymbol{\omega}}_{xy}$ can be put into a $n(n-1)/2 \times 1$ vector $\boldsymbol{\omega}_{xy}$. Apart from being practical to define the components of the skew matrix $\hat{\boldsymbol{\omega}}_{xy}$ as the vector $\boldsymbol{\omega}_{xy}$, the vector $\boldsymbol{\omega}_{xy}$ also has a physical interpretation: $\boldsymbol{\omega}_{xy}$ is the instantaneous direction of rotation defined in the frame Y.

The $n(n+1)/2$ elements of the $se(n)$ body velocity matrix $\hat{\boldsymbol{\nu}}_{xy}$ can be put into the $n(n+1)/2 \times 1$ vector, containing the linear and angular velocity:

$$\boldsymbol{\nu}_{xy} = \begin{bmatrix}\boldsymbol{v}_{xy} \\ \boldsymbol{\omega}_{xy}\end{bmatrix} \tag{2.9}$$

It is important to note that although the vector $\boldsymbol{\nu}_{xy}$ looks like a regular Cartesian vector, it is not an element of the Cartesian vector space $\mathbb{R}^{n(n+1)/2}$, since for $\boldsymbol{\nu}_{xy}$ no inner product can be defined [MLS94, Appendix 3.2].

The following operations are defined for transforming between $se(n)$ matrices and vectors: The hat operator transforms a vector to a skew matrix: $(\boldsymbol{x})\hat{} = \hat{\boldsymbol{x}}$. The reverse operator, the hatchek operator, transforms skew matrices to vectors $(\hat{\boldsymbol{x}})\check{} = \boldsymbol{x}$.

Two important operations can be defined on the Lee algebra.

Definition 2.1. *For any Lee algebra element $\hat{\boldsymbol{\nu}}_1, \hat{\boldsymbol{\nu}}_3 \in se(n)$, any Lee group element $\boldsymbol{H}_2 \in SE(n)$, with the corresponding rotation matrix $\boldsymbol{R}_2 \in SO(n)$, the big adjoint transformation is defined as [MLS94]:*

$$\begin{aligned} \hat{\boldsymbol{\nu}}_3 = \; & \boldsymbol{Ad}_2 \hat{\boldsymbol{\nu}}_1 = \boldsymbol{H}_2 \hat{\boldsymbol{\nu}}_1 \boldsymbol{H}_2^{-1} \\ \begin{bmatrix} \hat{\boldsymbol{\omega}}_3 & \boldsymbol{v}_3 \\ \boldsymbol{0} & 0 \end{bmatrix} = & \begin{bmatrix} \boldsymbol{R}_2 \hat{\boldsymbol{\omega}}_1 \boldsymbol{R}_2^T & \boldsymbol{R}_2 \boldsymbol{v}_1 - \boldsymbol{R}_2 \hat{\boldsymbol{\omega}}_1 \boldsymbol{R}_2^T \boldsymbol{o}_2 \\ \boldsymbol{0} & 0 \end{bmatrix} \end{aligned} \tag{2.10}$$

Since $\boldsymbol{Ad}_2$ is a function of $\boldsymbol{H}_2$, it is sometimes written as $\boldsymbol{Ad}_{H_2}$.

Definition 2.2. *For any Lee algebra element $\hat{\boldsymbol{\nu}}_1, \hat{\boldsymbol{\nu}}_2, \hat{\boldsymbol{\nu}}_3 \in se(n)$ the little adjoint transformation is defined as [MLS94]:*

$$\begin{aligned} \hat{\boldsymbol{\nu}}_3 = \; & \boldsymbol{ad}_{\nu_2} \hat{\boldsymbol{\nu}}_1 = [\hat{\boldsymbol{\nu}}_2, \hat{\boldsymbol{\nu}}_1] = \hat{\boldsymbol{\nu}}_2 \hat{\boldsymbol{\nu}}_1 - \hat{\boldsymbol{\nu}}_1, \hat{\boldsymbol{\nu}}_2 \\ \begin{bmatrix} \hat{\boldsymbol{\omega}}_3 & \boldsymbol{v}_3 \\ \boldsymbol{0} & 0 \end{bmatrix} = & \begin{bmatrix} \hat{\boldsymbol{\omega}}_2 \hat{\boldsymbol{\omega}}_1 - \hat{\boldsymbol{\omega}}_1 \hat{\boldsymbol{\omega}}_2 & \hat{\boldsymbol{\omega}}_2 \boldsymbol{v}_1 - \hat{\boldsymbol{\omega}}_1 \boldsymbol{v}_2 \\ \boldsymbol{0} & 0 \end{bmatrix} \end{aligned} \tag{2.11}$$

It is a function of a Lee algebra element $\hat{\boldsymbol{\nu}}_2$. Therefore the algebra element $\hat{\boldsymbol{\nu}}_2$ appears in the name of the little adjoint. It is important to note that both adjoint transformations hold for any Lee group and Lee algebra elements, regardless whether they define a transformation between frames and a body velocity or not. They can take an arbitrary Lee algebra element as input and always provide a Lee algebra element as output. Therefore it is ensured that *any* nested sequence of big and little adjoint operations yield an $se(3)$ element.

With the above transformations useful relations between body frames and body velocities can be derived.

Lemma 2.3. *The big adjoint between 3 frames X, Y and Z is given by:*

$$\boldsymbol{Ad}_{xz} = \boldsymbol{Ad}_{xy} \, \boldsymbol{Ad}_{yz} \tag{2.12}$$

Proof. Let $\boldsymbol{H}_{xz} = \boldsymbol{H}_{xy} \boldsymbol{H}_{yz}$.

$$\begin{aligned} \boldsymbol{Ad}_{xy} \, \boldsymbol{Ad}_{yz} \hat{\boldsymbol{\nu}}_1 = & \boldsymbol{H}_{xy} (\boldsymbol{H}_{yz} \hat{\boldsymbol{\nu}}_1 \boldsymbol{H}_{yz}^{-1}) \boldsymbol{H}_{xy}^{-1} \\ = & \boldsymbol{H}_{xz} \hat{\boldsymbol{\nu}}_1 \boldsymbol{H}_{xz}^{-1} \\ = & \boldsymbol{Ad}_{xz} \end{aligned}$$

Lemma 2.4. *The body velocity $\hat{\boldsymbol{\nu}}_{xz}$ between 3 frames X, Y and Z can be expressed as:*

$$\hat{\boldsymbol{\nu}}_{xz} = \boldsymbol{Ad}_{zy} \hat{\boldsymbol{\nu}}_{xy} + \hat{\boldsymbol{\nu}}_{yz} \tag{2.13}$$

Proof. Let $\boldsymbol{H}_{xz} = \boldsymbol{H}_{xy}\boldsymbol{H}_{yz}$. The body velocity $\hat{\boldsymbol{\nu}}_{xz}$ is given by:

$$
\begin{aligned}
\hat{\boldsymbol{\nu}}_{xz} &= \boldsymbol{H}_{xz}^{-1}\dot{\boldsymbol{H}}_{xz} \\
&= \boldsymbol{H}_{yz}^{-1}\boldsymbol{H}_{xy}^{-1}(\dot{\boldsymbol{H}}_{xy}\boldsymbol{H}_{yz} + \boldsymbol{H}_{xy}\dot{\boldsymbol{H}}_{yz}) \\
&= \boldsymbol{H}_{zy}\hat{\boldsymbol{\nu}}_{xy}\boldsymbol{H}_{zy}^{-1} + \hat{\boldsymbol{\nu}}_{yz} \\
&= \boldsymbol{Ad}_{zy}\hat{\boldsymbol{\nu}}_{xy} + \hat{\boldsymbol{\nu}}_{yz}
\end{aligned}
$$

where the identities (2.4a) were used.

Lemma 2.5. *The following relation holds for the time derivative of $\boldsymbol{Ad}_{xy}$:*

$$
\dot{\boldsymbol{Ad}}_{xy} = \boldsymbol{Ad}_{xy}\,\boldsymbol{ad}_{\nu_{xy}} \tag{2.14}
$$

Proof.

$$
\begin{aligned}
\dot{\boldsymbol{Ad}}_{xy}\hat{\boldsymbol{\nu}}_1 &= \dot{\boldsymbol{H}}_{xy}\hat{\boldsymbol{\nu}}_1\boldsymbol{H}_{xy}^{-1} + \boldsymbol{H}_{xy}\hat{\boldsymbol{\nu}}_1\dot{\boldsymbol{H}}_{xy}^{-1} \\
&= \boldsymbol{H}_{xy}\boldsymbol{H}_{xy}^{-1}\dot{\boldsymbol{H}}_{xy}\hat{\boldsymbol{\nu}}_1\boldsymbol{H}_{xy}^{-1} - \boldsymbol{H}_{xy}\hat{\boldsymbol{\nu}}_1\boldsymbol{H}_{xy}^{-1}\dot{\boldsymbol{H}}_{xy}\boldsymbol{H}_{xy}^{-1} \\
&= \boldsymbol{H}_{xy}\hat{\boldsymbol{\nu}}_{xy}\hat{\boldsymbol{\nu}}_1\boldsymbol{H}_{xy}^{-1} - \boldsymbol{H}_{xy}\hat{\boldsymbol{\nu}}_1\hat{\boldsymbol{\nu}}_{xy}\boldsymbol{H}_{xy}^{-1} \\
&= \boldsymbol{H}_{xy}[\hat{\boldsymbol{\nu}}_{xy}, \hat{\boldsymbol{\nu}}_1]\boldsymbol{H}_{xy}^{-1}
\end{aligned}
$$

Lemma 2.6. *The inverse of $\boldsymbol{Ad}_{xy}$ is:*

$$
\boldsymbol{Ad}_{xy}^{-1} = \boldsymbol{Ad}_{yx} \tag{2.15}
$$

Proof.

$$
\begin{aligned}
\boldsymbol{Ad}_{xy}^{-1}\hat{\boldsymbol{\nu}}_1 &= \boldsymbol{H}_{xy}^{-1}\hat{\boldsymbol{\nu}}_1\boldsymbol{H}_{xy} \\
&= \boldsymbol{H}_{yx}\hat{\boldsymbol{\nu}}_1\boldsymbol{H}_{yx}^{-1} \\
&= \boldsymbol{Ad}_{yx}
\end{aligned}
$$

Lemma 2.7. *The little adjoint transformation is anti-symmetric:*

$$
\boldsymbol{ad}_{\nu_{xy}} = -\,\boldsymbol{ad}_{\nu_{yx}} \tag{2.16}
$$

Proof.

$$
\begin{aligned}
\boldsymbol{ad}_{\nu_{xy}}\hat{\boldsymbol{\nu}}_1 &= \boldsymbol{ad}_{\nu_{xy}}\hat{\boldsymbol{\nu}}_1 - \boldsymbol{ad}_{\nu_1}\hat{\boldsymbol{\nu}}_{xy} \\
&= -[\,\boldsymbol{ad}_{\nu_1}, \hat{\boldsymbol{\nu}}_{xy}] \\
&= -\,\boldsymbol{ad}_{\nu_{yx}}
\end{aligned}
$$

Lemma 2.8. *The little adjoint transformation applied to the same algebra element is zero:*

$$
\boldsymbol{ad}_{\nu_{xy}}\boldsymbol{\nu}_{xy} = 0 \tag{2.17}
$$

Proof.

$$\begin{aligned} \boldsymbol{ad}_{\nu_{xy}} \hat{\boldsymbol{\nu}}_{xy} &= \hat{\boldsymbol{\nu}}_{xy} \hat{\boldsymbol{\nu}}_{xy} - \hat{\boldsymbol{\nu}}_{xy}, \hat{\boldsymbol{\nu}}_{xy} \\ &= 0 \end{aligned}$$

A relation between little adjoint transformations between different frames is often used in the modeling section.

Lemma 2.9. *The little adjoint transformation between 3 frames X, Y and Z is given by:*

$$\boldsymbol{ad}_{\nu_{xz}} = \boldsymbol{Ad}_{zy}\, \boldsymbol{ad}_{\nu_{xy}}\, \boldsymbol{Ad}_{zy}^{-1} + \boldsymbol{ad}_{\nu_{yz}} \tag{2.18}$$

Proof. Using $\hat{\boldsymbol{\nu}}_{xz} = \boldsymbol{Ad}_{zy} \hat{\boldsymbol{\nu}}_{xy} + \hat{\boldsymbol{\nu}}_{yz}$ (2.13)

$$\begin{aligned} \boldsymbol{ad}_{\nu_{xz}} \hat{\boldsymbol{\nu}}_1 &= [\hat{\boldsymbol{\nu}}_{xy}, \hat{\boldsymbol{\nu}}_1] = [\boldsymbol{H}_{zy} \hat{\boldsymbol{\nu}}_{xy} \boldsymbol{H}_{zy}^{-1}, \hat{\boldsymbol{\nu}}_1] + [\hat{\boldsymbol{\nu}}_{yz}, \hat{\boldsymbol{\nu}}_1] \\ &= \boldsymbol{H}_{zy} \hat{\boldsymbol{\nu}}_{xy} \boldsymbol{H}_{zy}^{-1} \hat{\boldsymbol{\nu}}_1 - \hat{\boldsymbol{\nu}}_1 \boldsymbol{H}_{zy} \hat{\boldsymbol{\nu}}_{xy} \boldsymbol{H}_{zy}^{-1} + \boldsymbol{ad}_{\nu_{yz}} \hat{\boldsymbol{\nu}}_1 \\ &= \boldsymbol{H}_{zy} (\hat{\boldsymbol{\nu}}_{xy} \boldsymbol{H}_{zy}^{-1} \hat{\boldsymbol{\nu}}_1 \boldsymbol{H}_{zy} - \boldsymbol{H}_{zy}^{-1} \hat{\boldsymbol{\nu}}_1 \boldsymbol{H}_{zy} \hat{\boldsymbol{\nu}}_{xy}) \boldsymbol{H}_{zy}^{-1} + \boldsymbol{ad}_{\nu_{yz}} \hat{\boldsymbol{\nu}}_1 \\ &= \boldsymbol{H}_{zy} [\hat{\boldsymbol{\nu}}_{xy}, \boldsymbol{H}_{zy}^{-1} \hat{\boldsymbol{\nu}}_1 \boldsymbol{H}_{zy}] \boldsymbol{H}_{zy}^{-1} + \boldsymbol{ad}_{\nu_{yz}} \hat{\boldsymbol{\nu}}_1 \\ &= \boldsymbol{Ad}_{zy}\, \boldsymbol{ad}_{\nu_{xy}}\, \boldsymbol{Ad}_{zy}^{-1} + \boldsymbol{ad}_{\nu_{yz}} \end{aligned}$$

To simplify notion only one subscript is used, if the quantities $\hat{\boldsymbol{\omega}}$,$\hat{\boldsymbol{\nu}}$, $\boldsymbol{H}$, $\boldsymbol{A}$, $\boldsymbol{A}$ are relative to the inertial frame I, e.g $\hat{\boldsymbol{\nu}}_{Ix} = \hat{\boldsymbol{\nu}}_x$.

2.1.3. Link Twist Jacobian for Free Floating Robots on SE(n)

In this section the results from section 2.1 are applied to free floating robots. It is a generalization of [MLS94, Chapter 3.4] to free floating robots and to $SE(n)$ [1].

The body velocity $\hat{\boldsymbol{\nu}}_j$ of joint j is (2.13) is:

$$\hat{\boldsymbol{\nu}}_j = \boldsymbol{Ad}_{jb} \hat{\boldsymbol{\nu}}_b + \hat{\boldsymbol{\nu}}_{bj}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{2.19}$$

$$\hat{\boldsymbol{\nu}}_{bj}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{H}_{bj}^{-1} \dot{\boldsymbol{H}}_{bj} \tag{2.20}$$

[1] Using the term Jacobi matrix or Jacobian for $SE(n)$ can be misleading and is technically wrong, since a Jacobi matrix is defined as the matrix of all partial derivatives of a function with respect to its arguments. For Cartesian variables $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ the position $\boldsymbol{y}$ with respect to $\boldsymbol{x}$ is $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$. The velocity of $\boldsymbol{x}$ and $\boldsymbol{y}$ is $\dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{y}}$, respectively. The velocity $\dot{\boldsymbol{y}}$ with respect $\dot{\boldsymbol{x}}$ is given by the Jacobian: $\dot{\boldsymbol{y}} = \boldsymbol{J}_{\dot{\boldsymbol{y}}}\, \dot{\boldsymbol{x}}$. For non-Cartesian variables the velocity relation neither holds for the function argument $\boldsymbol{x}$, nor for the function output $\boldsymbol{y}$. For $SE(n)$ the corresponding relations are $\boldsymbol{H}_{xy} = \boldsymbol{F}(\boldsymbol{H}_{ib}, \boldsymbol{q})$ and $\boldsymbol{\nu}_{xy} = \boldsymbol{J}_{\boldsymbol{\nu}_{xy}} \boldsymbol{v}_b$ with $\boldsymbol{H}_{xy}, \boldsymbol{H}_{ib} \in SE(n)$, $\boldsymbol{v}_b = [\boldsymbol{\nu}_{ib}; \dot{\boldsymbol{q}}]$, $\boldsymbol{\nu}_{xy}, \boldsymbol{\nu}_{ib} \in se(n)$ and $\dot{\boldsymbol{q}} \in \mathbb{R}^n$. The functional form of the "Jabobian" $\boldsymbol{J}_{\boldsymbol{\nu}_{xy}}$ is not a Jacobi matrix.

For a free floating robot the group element $\boldsymbol{H}_{bj}$ depends only on the joint variables $\boldsymbol{q}$. Therefore, term $\dot{\boldsymbol{H}}_{bj}$ can be expanded with regards to $\boldsymbol{q}$ using the chain rule like in [MLS94, eq. 3.41]. Using Einstein notation for index i: [2]

$$\hat{\boldsymbol{\nu}}_{bj}(\boldsymbol{q}) = \boldsymbol{H}_{bj}^{-1}(\boldsymbol{q})\dot{\boldsymbol{H}}_{bj}(\boldsymbol{q}) \tag{2.21}$$

$$= \boldsymbol{H}_{bj}^{-1}(\boldsymbol{q})\frac{\partial \boldsymbol{H}_{bj}(\boldsymbol{q})}{\partial q_i}\dot{\boldsymbol{q}}_i \in se(n) \tag{2.22}$$

This can be more compactly written by defining the link twist Jacobian $\hat{\boldsymbol{J}}_{jq}$, whose ith element is: [3]

$$\begin{aligned}\hat{\boldsymbol{J}}^i_{jq} &= \boldsymbol{H}_{bj}^{-1}(\boldsymbol{q})\frac{\partial}{\partial q_i}\boldsymbol{H}_{bj}(\boldsymbol{q})\\ &= \begin{bmatrix}\boldsymbol{R}_{bj}^T\frac{\partial \boldsymbol{R}_{bj}}{\partial \boldsymbol{q}_i} & \boldsymbol{R}_{bj}^T\frac{\partial \boldsymbol{o}_{bj}}{\partial \boldsymbol{q}_i}\\ \boldsymbol{0} & 0\end{bmatrix} \in se(n)\end{aligned} \tag{2.23}$$

yielding

$$\hat{\boldsymbol{\nu}}_j = \boldsymbol{Ad}_{jb}\hat{\boldsymbol{\nu}}_b + \hat{\boldsymbol{J}}^i_{jq}\dot{\boldsymbol{q}}_i \tag{2.24}$$

The formulation on the Lee algebra has the big advantage that the Jacobian $\boldsymbol{J}_{jq}$ is directly derived from the Lee group element $\boldsymbol{H}_{bj}$. This fact will be later used to derive complex relations in Chapter 3.1.3 and in Chapter 3.3.3.

The link twist Jacobian $\boldsymbol{J}_{jq}$ can be simplified for robots with one degree of freedom per joint. The transformation $\boldsymbol{H}_{bj}$ from the base frame to the frame of the joint j is given by the product of the transformations $\boldsymbol{H}_{i-1,i}(q_i)$ from i to joint $i-1$:

$$\boldsymbol{H}_{bj} = \boldsymbol{H}_{01}(q_1)\boldsymbol{H}_{12}(q_2),\dots,\boldsymbol{H}_{j-1,j}(q_j) \tag{2.25}$$

where the index 0 corresponds to b. The derivative of $\boldsymbol{H}_{bj}$ with respect to q_i depends only on the term $\boldsymbol{H}_{i-1,i}$

For a rotational joint the rotation $\boldsymbol{R}_{i-1,i}$ is time dependent, while the distance $\boldsymbol{o}_{i-1,i}$ from

[2] The $se(n)$ element $\hat{\boldsymbol{\nu}}_{bj}$ is a linear combination of the matrices $\boldsymbol{H}_{bj}^{-1}\frac{\partial \boldsymbol{H}_{bj}}{\partial q_i}$ for arbitrary values of the scalars $\dot{\boldsymbol{q}}_i$. Therefore, $\boldsymbol{H}_{bj}^{-1}\frac{\partial \boldsymbol{H}_{bj}}{\partial q_i} \in se(n)$

[3] $\boldsymbol{R}_{bj}^T\frac{\partial \boldsymbol{R}_{bj}}{\partial \boldsymbol{q}_i}$ is skew symmetric.

frame $i-1$to frame i is constant [4]:

$$
\begin{aligned}
\boldsymbol{H}^{rot}_{i-1,i}(q_i) &= \begin{bmatrix} \boldsymbol{R}_{i-1,i}(q_i) & \boldsymbol{o}_{i-1,i} \\ \mathbf{0} & 1 \end{bmatrix} \\
\frac{\partial}{\partial q_i}\boldsymbol{H}^{rot}_{i-1,i}(q_i) &= \begin{bmatrix} \frac{\partial}{\partial q_i}\boldsymbol{R}_{i-1,i}(q_i) & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \\
(\boldsymbol{H}^{-1,rot}_{i-1,i}(q_i)\frac{\partial}{\partial q_i}\boldsymbol{H}^{rot}_{i-1,i}(q_i)\check{)} &= \begin{bmatrix} \mathbf{0}_{3\times 1} \\ (\boldsymbol{R}^T_{i-1,i}\frac{\partial}{\partial q_i}\boldsymbol{R}_{i-1,i}(q_i)\check{)} \end{bmatrix}
\end{aligned}
\tag{2.26}
$$

For a prismatic joint the distance $\boldsymbol{o}_{i-1,i}$ from frame $i-1$ to frame i is time dependent, while the rotation $\boldsymbol{R}_{i-1,i}$ is constant:

$$
\begin{aligned}
\boldsymbol{H}^{pris}_{i-1,i}(q_i) &= \begin{bmatrix} \boldsymbol{R}_{i-1,i} & \boldsymbol{o}_{i-1,i}(q_i) \\ \mathbf{0} & 1 \end{bmatrix} \\
\frac{\partial}{\partial q_i}\boldsymbol{H}^{pris}_{i-1,i}(q_i) &= \begin{bmatrix} \mathbf{0}_{3\times 3} & \frac{\partial}{\partial q_i}o_{i-1,i}(q_i) \\ \mathbf{0} & 0 \end{bmatrix} \\
(\boldsymbol{H}^{-1,prism}_{i-1,i}(q_i)\frac{\partial}{\partial q_i}\boldsymbol{H}^{prism}_{i-1,i}(q_i)\check{)} &= \begin{bmatrix} \boldsymbol{R}^T_{i-1,i}\frac{\partial}{\partial q_i}o_{i-1,i}(q_i) \\ \mathbf{0}_{3\times 1} \end{bmatrix}
\end{aligned}
\tag{2.27}
$$

Using the relations:

$$
\begin{aligned}
\boldsymbol{H}^{-1}_{bj}(\boldsymbol{q}) &= \boldsymbol{H}^{-1}_{j-1,j},\ldots,\boldsymbol{H}^{-1}_{i,i+1}\boldsymbol{H}^{-1}_{i-1,i}\boldsymbol{H}^{-1}_{i-2,i-1},\ldots,\boldsymbol{H}^{-1}_{0,1} \\
\frac{\partial}{\partial q_i}\boldsymbol{H}_{bj}(\boldsymbol{q}) &= \boldsymbol{H}_{01},\ldots,\boldsymbol{H}_{i-2,i-1},\frac{\partial}{\partial q_i}\boldsymbol{H}_{i-1,i}(q_i)\boldsymbol{H}_{i,i+1},\ldots,\boldsymbol{H}_{j-1,j}
\end{aligned}
$$

the ith column of the Jacobian (2.23) is given by:

$$
\begin{aligned}
\hat{\boldsymbol{J}}^i_{jq} &= \boldsymbol{Ad}_{ji}(\boldsymbol{H}^{-1}_{i-1,i}\frac{\partial}{\partial q_i}\boldsymbol{H}_{i-1,i}(q_i)) \\
\boldsymbol{J}^i_{jq} &= \boldsymbol{A}_{ji}(\boldsymbol{H}^{-1}_{i-1,i}\frac{\partial}{\partial q_i}\boldsymbol{H}_{i-1,i}(q_i)\check{)}
\end{aligned}
\tag{2.28}
$$

Finally, using the transformations for the rotational (2.26) and the prismatic (2.27) joints, respectively, and (2.28) the ith column of the Jacobian $\boldsymbol{J}^i_{jq}$ can be explicitly calculated $SE(3)$.

[4]For $SE(2)$ the term $(\boldsymbol{R}^T_{i-1,i}\frac{\partial}{\partial q_i}\boldsymbol{R}_{i-1,i}(q_i)\check{)} = 1$, due to relation (4.4) $\boldsymbol{R}^T_{xy}\frac{\partial \boldsymbol{R}_{xy}(\theta_{xy})}{\partial \theta_{xy}} = \boldsymbol{S}$, where $\boldsymbol{S}$ is the basis vector of $se(2)$. For $SE(3)$ the result depends on the parameterization of the rotation matrix.

If exponential coordinates are used, $\boldsymbol{H}_{bj} = e^{\sum_{i=1}^{j}\hat{\boldsymbol{\xi}}_i(0)\boldsymbol{q}_i}\boldsymbol{H}_{bj}(0)$, The ith column of the link twist Jacobian is then: $\boldsymbol{J}^i_{jq} = \boldsymbol{A}_{jb}\boldsymbol{\xi}_i(0)$

$$\boldsymbol{J}_{jq}^{i,rot} = \begin{bmatrix} \hat{\boldsymbol{o}}_{ji}\boldsymbol{R}_{ji}(\boldsymbol{R}_{i-1,i}^T \frac{\partial \boldsymbol{R}_{i-1,i}(q_i)}{\partial q_i})^\vee \\ \boldsymbol{R}_{ji}(\boldsymbol{R}_{i-1,i}^T \frac{\partial \boldsymbol{R}_{i-1,i}(q_i)}{\partial q_i})^\vee \end{bmatrix} \tag{2.29}$$

$$\boldsymbol{J}_{jq}^{i,prism} = \begin{bmatrix} \boldsymbol{R}_{ji}\boldsymbol{R}_{i-1,i}^T(\frac{\partial}{\partial q_i}\boldsymbol{o}_{i-1,i}(q_i)) \\ \boldsymbol{0}_{3\times 1} \end{bmatrix} \tag{2.30}$$

For $SE(2)$ most of these relations have been implemented in the Symbolic Lee Algebra Toolbox for $SE(2) \times \mathbb{R}^n$ A.2. For $SE(3)$ it would be interesting to explore what benefits a Lee algebra approach could bring for symbolic and numeric computations of system matrices, Jacobians, momentum maps, etc. It would be beneficial to use exponential coordinates[4].

2.1.4. The Exponential Map on SE(n)

The definition of the body velocity (2.7) can be viewed as a differential equation for trajectories on the group:

$$\dot{\boldsymbol{H}}(t) = \boldsymbol{H}(t)\hat{\boldsymbol{\nu}}(t)$$

A trajectory on the manifold $SE(n)$ is the time evolution of the group element $\boldsymbol{H}(t)$. The differential equation can be solved for constant body velocities $\hat{\boldsymbol{\nu}}$, which is fulfilled for trajectories starting at the unit element of the group and only considering infinitesimal small values of t:

$$\boldsymbol{H}(\delta t) = e^{\delta t\, \hat{\boldsymbol{\nu}}} \tag{2.31}$$

This relation is called exponential map [MLS94].

All possible infinitesimal variations of a group element, can be generated by an algebra element through a screw motion. This means that for any infinitesimal variation a group element $\boldsymbol{H}$ an algebra element $\hat{\boldsymbol{\nu}}$ can be found that generates the variation by the amount ϵ:

$$\boldsymbol{H}^\epsilon = \boldsymbol{H}e^{\epsilon\, \hat{\boldsymbol{\nu}}} \tag{2.32}$$

2.2. Relations on SE(3)

2.2.1. Adjoint Transformations

In this section the relations derived for $SE(n)$ are now specified for $SE(3)$. The corresponding relations for $SE(2)$ are derived in section 4.1.1.

The angular body velocity matrix is a 3×3 skew matrix containing 3 different elements:

$$\hat{\boldsymbol{\omega}}_{xy} = \begin{bmatrix} 0 & -\omega_{xy}^3 & \omega_{xy}^2 \\ \omega_{xy}^3 & 0 & -\omega_{xy}^1 \\ -\omega_{xy}^2 & \omega_{xy}^1 & 0 \end{bmatrix} \tag{2.33}$$

The body velocity matrix $\hat{\boldsymbol{\nu}}_{xy} \in se(3)$ (2.7) is:

$$\hat{\boldsymbol{\nu}}_{xy} = \begin{bmatrix} \hat{\boldsymbol{\omega}}_{xy} & \boldsymbol{v}_{xy} \\ \boldsymbol{0} & 0 \end{bmatrix}$$

with $\boldsymbol{v}_{xy} = \boldsymbol{R}_{xy}^T \dot{\boldsymbol{o}}_{xy}$. It has 6 different components.

The adjoint transformations in (2.10) and in (2.11) can be defined on twists.

Lemma 2.10. *The transformation $\boldsymbol{\nu}_2 = \boldsymbol{A}_{xy}\boldsymbol{\nu}_1$. transforms any twist $\boldsymbol{\nu}_1$ corresponding to $\hat{\boldsymbol{\nu}}_1 \in se(3)$ to a twist $\boldsymbol{\nu}_2$ corresponding to $\hat{\boldsymbol{\nu}}_2 \in se(3)$. The 6×6 matrix $\boldsymbol{A}_{xy}$ is given by:*

$$\boldsymbol{A}_{xy} = \begin{bmatrix} \boldsymbol{R}_{xy} & \hat{\boldsymbol{o}}_{xy}\boldsymbol{R}_{xy} \\ \boldsymbol{0} & \boldsymbol{R}_{xy} \end{bmatrix} \tag{2.34}$$

The result of using $\boldsymbol{A}_{xy}\boldsymbol{\nu}_1$ is the same as applying the big adjoint transformation (2.10) $\boldsymbol{Ad}_{xy}\hat{\boldsymbol{\nu}}_1$ and converting the result to a vector. i.e $\boldsymbol{A}_{xy}\boldsymbol{\nu}_1 = (\boldsymbol{Ad}_{xy}\hat{\boldsymbol{\nu}}_1)\check{}$.

Proof. Comparing (2.34) and (2.10), we need to prove that:

$$\begin{aligned} \boldsymbol{R}_{xy}\boldsymbol{\omega}_1 &= (\boldsymbol{R}_{xy}\hat{\boldsymbol{\omega}}_1\boldsymbol{R}_{xy}^T)\check{} \\ \boldsymbol{R}_{xy}\boldsymbol{v}_1 + \hat{\boldsymbol{o}}_{xy}\boldsymbol{R}_{xy}\boldsymbol{\omega}_1 &= \boldsymbol{R}_{xy}\boldsymbol{v}_1 - \boldsymbol{R}_{xy}\hat{\boldsymbol{\omega}}_1\boldsymbol{R}_{xy}^T\boldsymbol{o}_{xy} \end{aligned}$$

Using (A.6f) $\boldsymbol{R}_{xy}\boldsymbol{\omega}_1 = (\boldsymbol{R}_{xy}\hat{\boldsymbol{\omega}}_1\boldsymbol{R}_{xy}^T)\check{}$ the first part is proven. The first terms in the second equation are the same. The second term of the second equation $-\boldsymbol{R}_{xy}\hat{\boldsymbol{\omega}}_1\boldsymbol{R}_{xy}^T\boldsymbol{o}_{xy} = -(\boldsymbol{R}_{xy}\boldsymbol{\omega}_1)\hat{}\boldsymbol{o}_{xy} = \hat{\boldsymbol{o}}_{xy}\boldsymbol{R}_{xy}\boldsymbol{\omega}_1$ due to the properties of the cross product (A.6b).

Lemma 2.11. *The transformation $\boldsymbol{\nu}_2 = \boldsymbol{a}_{xy}\boldsymbol{\nu}_1$ transforms a twist $\boldsymbol{\nu}_1$ corresponding to $\hat{\boldsymbol{\nu}}_1 \in se(3)$ to a twist $\boldsymbol{\nu}_2$ corresponding to $\hat{\boldsymbol{\nu}}_2 \in se(3)$. The 6×6 matrix $\boldsymbol{a}_{xy}$ is given by:*

$$\boldsymbol{a}_{xy} = \begin{bmatrix} \hat{\boldsymbol{\omega}}_{xy} & \hat{\boldsymbol{v}}_{xy} \\ \boldsymbol{0} & \hat{\boldsymbol{\omega}}_{xy} \end{bmatrix} \tag{2.35}$$

The result is the same as transforming $\hat{\boldsymbol{\nu}}_1$ with little adjoint transformation (2.11) and converting the result to a twist, i.e $\boldsymbol{a}_{xy}\boldsymbol{\nu}_1 = (\boldsymbol{ad}_{\nu_{xy}}\hat{\boldsymbol{\nu}}_1)\check{}$.

Proof. Comparing (2.11) with (2.35) we need to prove:

$$\begin{aligned} \hat{\boldsymbol{\omega}}_{xy}\boldsymbol{\omega}_1 &= (\hat{\boldsymbol{\omega}}_{xy}\hat{\boldsymbol{\omega}}_1 - \hat{\boldsymbol{\omega}}_1\hat{\boldsymbol{\omega}}_{xy})\check{} \\ \hat{\boldsymbol{\omega}}_{xy}\boldsymbol{v}_1 + \hat{\boldsymbol{v}}_{xy}\boldsymbol{\omega}_1 &= \hat{\boldsymbol{\omega}}_{xy}\boldsymbol{v}_1 - \hat{\boldsymbol{\omega}}_1\boldsymbol{v}_{xy} \end{aligned}$$

The first part can be proven using the identity (A.6c): $(\hat{\boldsymbol{\omega}}_{xy}\hat{\boldsymbol{\omega}}_1 - \hat{\boldsymbol{\omega}}_1\hat{\boldsymbol{\omega}}_{xy})\check{} = \boldsymbol{\omega}_{xy} \times \boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_{xy} \times \boldsymbol{\omega}_1 = \hat{\boldsymbol{\omega}}_{xy}\boldsymbol{\omega}_1$. The second part is the same due to the cross product property (A.6b): $\hat{\boldsymbol{v}}_{xy}\boldsymbol{\omega}_1 = -\hat{\boldsymbol{\omega}}_1\boldsymbol{v}_{xy}$.

Using Lemma 2.10 and Lemma 2.11 all relation previously defined on the Lee algebra can be formulated for twists

$$\boldsymbol{A}_{xy}^{-1} = \boldsymbol{A}_{yx} \tag{2.36a}$$
$$\dot{\boldsymbol{A}}_{xy} = \boldsymbol{A}_{xy}\boldsymbol{a}_{xy} \tag{2.36b}$$
$$\boldsymbol{a}_{xy}\boldsymbol{\nu}_{xy} = \boldsymbol{0} \tag{2.36c}$$
$$\boldsymbol{a}_{xy}\boldsymbol{\nu}_{zw} = -\boldsymbol{a}_{zw}\boldsymbol{\nu}_{xy} \tag{2.36d}$$

The little adjoint transformation for twists is often written as $\boldsymbol{a}_{xy}$ instead of $\boldsymbol{a}_{\nu_{xy}}$.

The transformation rules between frame x, y and z and properties are:

$$\boldsymbol{A}_{xz} = \boldsymbol{A}_{xy}\boldsymbol{A}_{yz} \tag{2.37a}$$
$$\boldsymbol{\nu}_{xz} = \boldsymbol{A}_{zy}\,\boldsymbol{\nu}_{xy} + \boldsymbol{\nu}_{yz} \tag{2.37b}$$
$$\boldsymbol{a}_{xz} = \boldsymbol{A}_{zy}\boldsymbol{a}_{xy}\boldsymbol{A}_{zy}^{-1} + \boldsymbol{a}_{yz} \tag{2.37c}$$

2.2.2. Link Twist Jacobian on SE(3)

The ith column of the link twist Jacobian for $SE(3)$ is (2.23):

$$\boldsymbol{J}_{jq}^{i} = \begin{bmatrix} \boldsymbol{R}_{bj}^{T}\frac{\partial \boldsymbol{o}_{bj}}{\partial q_i} \\ (\boldsymbol{R}_{bj}^{T}\frac{\partial \boldsymbol{R}_{bj}}{\partial q_i})^{\vee} \end{bmatrix} \tag{2.38}$$

The body velocity $\boldsymbol{\nu}_j$ in terms of the link twist Jacobian is then (2.19):

$$\boldsymbol{\nu}_j = \boldsymbol{A}_{jb}\boldsymbol{\nu}_b + \boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \tag{2.39}$$
$$\boldsymbol{J}_{jq} = [\boldsymbol{J}_{jq}^{1}\dots\boldsymbol{J}_{jq}^{n}] \tag{2.40}$$

2.2.3. Lagrange Function

The Lagrangian of a floating robot system is given by the kinetic energy $T(\boldsymbol{\nu}_b, \dot{\boldsymbol{q}})$ minus the potential energy $U_g(\boldsymbol{o}_{ib}, \boldsymbol{\theta}_{ib}, \boldsymbol{q})$:

$$L(\boldsymbol{o}_{ib}, \boldsymbol{\theta}_{ib}, \boldsymbol{q}, \boldsymbol{\nu}_b, \dot{\boldsymbol{q}}) = T(\boldsymbol{\nu}_b, \dot{\boldsymbol{q}}) - U_g(\boldsymbol{o}_{ib}, \boldsymbol{\theta}_{ib}, \boldsymbol{q}) \tag{2.41}$$

We derive an expression for the kinetic energy. The kinetic energy is the sum of all kinetic energies indexed by j_{ci} around the centers of mass:

$$T = \frac{1}{2}\sum_{j=0}^{n}\sum_{i=1}^{n_j}\boldsymbol{\nu}_{j_{ci}}^{T}\begin{bmatrix} m_{j_{ci}}\mathbb{I}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{j_{ci}} \end{bmatrix}\boldsymbol{\nu}_{j_{ci}} \tag{2.42}$$

It is assumed that in frame j there are n_j bodies. A body numbered by j_{ci} has a body velocity $\boldsymbol{\nu}_{j_{ci}}$, a mass $m_{j_{ci}}$ and an inertia $I_{j_{ci}}$, which is aligned with the principal axis of the body. The center of mass of body j_{ci} is located at $\boldsymbol{o}_{j,ci}$. Since $\boldsymbol{o}_{j,ci}$ is constant in frame j, the velocity of body j_{ci} transforms as: $\boldsymbol{\nu}_{j_{ci}} = \boldsymbol{A}_{j_{ci},j}\boldsymbol{\nu}_j$.

The mass matrix of frame j around the origin of frame j is given by [5]:

$$\boldsymbol{\Lambda}_j = \sum_{i=1}^{n_j} \boldsymbol{A}_{j,j_{ci}}^T \begin{bmatrix} m_{j_{ci}}\mathbb{I}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{j_{ci}} \end{bmatrix} \boldsymbol{A}_{j,j_{ci}} \tag{2.43}$$

For $SE(3)$ the constant adjoint matrix $\boldsymbol{A}_{j_{ci},j}$ is:

$$\boldsymbol{A}_{j_{ci},j} = \begin{bmatrix} \boldsymbol{R}_{j,j_{ci}} & \hat{\boldsymbol{o}}_{j,j_{ci}} \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix} \tag{2.44}$$

and the mass matrix of frame j is

$$\boldsymbol{\Lambda}_j = \sum_{i=1}^{n_j} \begin{bmatrix} m_{j_{ci}}\mathbb{I}_{3\times3} & -m_{j_{ci}}\hat{\boldsymbol{o}}_{j,j_{ci}} \\ -m_{j_{ci}}\hat{\boldsymbol{o}}_{j,j_{ci}} & \boldsymbol{I}_{j_{ci}} - m_{j_{ci}}\hat{\boldsymbol{o}}_{j,j_{ci}}\hat{\boldsymbol{o}}_{j,j_{ci}} \end{bmatrix} \tag{2.45}$$

It is the same as in [GOA13, eq. 2.23]

The kinetic energy in terms of $\boldsymbol{\Lambda}_j$ is

$$T = \frac{1}{2}\sum_{i=0}^{n} \boldsymbol{\nu}_j^T \boldsymbol{\Lambda}_j \boldsymbol{\nu}_j \tag{2.46}$$

The body velocity of the jth frame $\boldsymbol{\nu}_j$ (2.39) can be rewritten by defining the Jacobian $\boldsymbol{J}_{jb}$, which is a $6 \times (6+n)$ matrix:

$$\begin{aligned} \boldsymbol{J}_{0b} &= \begin{bmatrix} \mathbb{I}_{6\times6} & \boldsymbol{0}_{6\times n} \end{bmatrix} \\ \boldsymbol{J}_{jb} &= \begin{bmatrix} \boldsymbol{A}_{jb} & \boldsymbol{J}_{jq} \end{bmatrix} \end{aligned} \tag{2.47}$$

$$\boldsymbol{\nu}_j = \boldsymbol{J}_{jb} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix} \tag{2.48}$$

The Lagrangian (2.41) in terms of $\boldsymbol{J}_{jb}$ is

$$L = \frac{1}{2} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix}^T \boldsymbol{M}_b \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix} - U_g(\boldsymbol{o}_{ib}, \boldsymbol{\theta}_{ib}, \boldsymbol{q}) \tag{2.49}$$

with the mass matrix $\boldsymbol{M}_b$ given by

$$\begin{bmatrix} \boldsymbol{M}_{bb} & \boldsymbol{M}_{bq} \\ \boldsymbol{M}_{bq}^T & \boldsymbol{M}_{qq} \end{bmatrix} = \sum_{i=0}^{n} \boldsymbol{J}_{jb}^T \boldsymbol{\Lambda}_j \boldsymbol{J}_{jb} \tag{2.50}$$

[5] Interestingly, this relation holds for $SE(n)$ as long as the adjoint matrix $\boldsymbol{A}_{j,j_{ci}}$ is specified for $SE(n)$.

The block matrix elements are given by

$$\boldsymbol{M}_{bb} = \sum_{j=0}^{n} \boldsymbol{A}_{jb}^T \boldsymbol{\Lambda}_j \boldsymbol{A}_{jb} \tag{2.51a}$$

$$\boldsymbol{M}_{bq} = \sum_{j=0}^{n} \boldsymbol{A}_{jb}^T \boldsymbol{\Lambda}_j \boldsymbol{J}_{jq} \tag{2.51b}$$

$$\boldsymbol{M}_{qq} = \sum_{j=0}^{n} \boldsymbol{J}_{jq}^T \boldsymbol{\Lambda}_j \boldsymbol{J}_{jq} \tag{2.51c}$$

We assume that gravity points downwards on the y-axis. The gravity potential is given by the sum of the contributions of all masses:

$$U_g(\boldsymbol{o}_{ib}, \boldsymbol{\theta}_{ib}, \boldsymbol{q}) = m\, g \boldsymbol{o}_{ib} + g\, \boldsymbol{r}_2^T \sum_{j=0}^{n} \sum_{i=1}^{n_j} m_{j_{ci}} \boldsymbol{o}_{b,j_{ci}}(\boldsymbol{q}) \tag{2.52}$$

where m is the total mass of the system and $\boldsymbol{r}_2$ is the second row of $\boldsymbol{R}_{ib}$. The gravity system matrix $\boldsymbol{G}$ can be derived from the potential using (3.32a):

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{R}_{ib}^T \frac{\partial U_g}{\partial \boldsymbol{o}_{ib}} \\ \sum_{i=1}^{3} (\boldsymbol{r}_i \times \frac{\partial U}{\partial \boldsymbol{r}_i}) \\ \frac{\partial U_g}{\partial \boldsymbol{q}} \end{bmatrix} \tag{2.53}$$

where the term $\frac{\partial U_g}{\partial \boldsymbol{o}_{ib}} = \begin{bmatrix} 0 \\ mg \end{bmatrix}$.

3. Dynamics and Conserved Quantities

3.1. Global Formulation of Hamel Equations on Manifolds

In this section the machinery to derive Hamel equations on manifolds is set up. We restrict ourselves to the Lee groups relevant for robotics, which are the Euclidean space $\mathbb{R}^n$, the rotation group $SO(3)$, the special Euclidean group $SE(3)$ and the product group $SE(3) \times \mathbb{R}^n$. In this work a global formulation for the elements of the manifold is used [LLM17]. Global formulation in this context means that the formulation does not require local maps and local coordinates. This can be achieved under the assumption that the group elements can be represented globally by matrices embedded in a higher dimensional space [LLM17]. This assumption is certainly fulfilled for all robotics application, since rotation matrices and rigid body transforms can be embedded in the general linear group of invertible matrices $GL_n(\mathbb{R})$.

This chapter is much aligned with Lee's book [LLM17]. However the notation used here is more streamlined and optimized for doing pen and paper calculations. In the sequel the Hamel equations are developed for the Lee groups specified above.

In this section there is an ambiguity in the notation for the variables $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. The variables $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ used in this section are a placeholder for configuration variables and velocities, receptively. They have nothing to do with the joint variables $\boldsymbol{q}$ and velocities $\dot{\boldsymbol{q}}$ used outside this section.

In the case of mechanics, the Lagrange function is given by the kinetic energy $T(\boldsymbol{q}, \dot{\boldsymbol{q}})$ minus the potential energy $U(\boldsymbol{q})$: $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U(\boldsymbol{q})$. The variables $\boldsymbol{q}$ represent the configuration of the system, while $\dot{\boldsymbol{q}}$ are the velocities. The variables $\boldsymbol{q}$ are elements of the configuration manifold, while the velocities $\dot{\boldsymbol{q}}$ are elements of the corresponding tangent space. It is important to note that the velocities $\dot{\boldsymbol{q}}$ are not necessarily the time derivatives of the configuration variables. This only holds for Cartesian variables. The shape of the configuration variables and of the velocities depends on the manifold and its tangent space, which is specific for the problem at hand.

The Hamel equations can be derived from Hamiltonian's variation principle, which states that the infinitesimal variation of the action integral:

$$A^\epsilon = \int_{t_0}^{t_1} L(\boldsymbol{q}^\epsilon, \dot{\boldsymbol{q}}^\epsilon) dt \tag{3.1}$$

$$0 \stackrel{!}{=} \delta A = \frac{d}{d\epsilon} A^\epsilon \Big|_{\epsilon=0} \tag{3.2}$$

vanishes for all possible variations $\boldsymbol{q}^{\epsilon}$ and $\dot{\boldsymbol{q}}^{\epsilon}$ with fixed boundaries [LLM17, chapter-3.2]:

$$\begin{aligned} \boldsymbol{q}^{\epsilon}(t_0) &= \boldsymbol{q}(t_0) \qquad & \boldsymbol{q}^{\epsilon}(t_1) &= \boldsymbol{q}(t_1) \\ \dot{\boldsymbol{q}}^{\epsilon}(t_0) &= \dot{\boldsymbol{q}}(t_0) \qquad & \dot{\boldsymbol{q}}^{\epsilon}(t_1) &= \dot{\boldsymbol{q}}(t_1) \end{aligned} \tag{3.3}$$

The differentiation with respect to ϵ can be pulled under the integral:

$$0 \stackrel{!}{=} \int_{t_0}^{t_1} (\frac{d}{d\epsilon} L(\boldsymbol{q}^{\epsilon}, \dot{\boldsymbol{q}}^{\epsilon})\Big|_{\epsilon=0})\, dt = \int_{t_0}^{t_1} (\frac{\partial L}{\boldsymbol{q}} \delta\boldsymbol{q} + \frac{\partial L}{\dot{\boldsymbol{q}}} \delta\dot{\boldsymbol{q}})\, dt \tag{3.4}$$

with the variation of the variables given by:

$$\begin{aligned} \delta\boldsymbol{q} &= \frac{d}{d\epsilon} \boldsymbol{q}^{\epsilon}\Big|_{\epsilon=0} \\ \delta\dot{\boldsymbol{q}} &= \frac{d}{d\epsilon} \dot{\boldsymbol{q}}^{\epsilon}\Big|_{\epsilon=0} \end{aligned} \tag{3.5}$$

It is important to note that the variation of the action integral (3.4) holds for any configuration manifold and its tangent space. However, the variations of the variables (3.5) vary with the manifold.

3.1.1. Hamel Equations on $\mathbf{R}^n$

The Lagrangian $L(\boldsymbol{x}, \dot{\boldsymbol{x}})$ depends on Cartesian variables denoted by $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$. The configuration manifold and the tangent space are the Cartesian vector space $\mathbb{R}^n$. All possible variations are pure translations:

$$\begin{aligned} \boldsymbol{x}^{\epsilon} &= \boldsymbol{x} + \epsilon\delta\boldsymbol{x} \\ \dot{\boldsymbol{x}}^{\epsilon} &= \dot{\boldsymbol{x}} + \epsilon\delta\dot{\boldsymbol{x}} \end{aligned} \tag{3.6}$$

Therefore, the variation of the configuration variables and the velocities (3.5) is simply $\delta\boldsymbol{x}$ and $\delta\boldsymbol{x}$. Plugging $\delta\boldsymbol{x}$ and $\delta\dot{\boldsymbol{x}}$ into equation (3.4) yields:

$$0 \stackrel{!}{=} \delta A = \int_{t_0}^{t_1} (\frac{\partial L}{\delta\boldsymbol{x}} \delta\boldsymbol{x} + \frac{\partial L}{\delta\dot{\boldsymbol{x}}} \delta\dot{\boldsymbol{x}})\, dt \tag{3.7}$$

Integration by parts of the second term gives:

$$\int_{t_0}^{t_1} (\frac{\partial L}{\delta\dot{\boldsymbol{x}}} \delta\dot{\boldsymbol{x}})\, dt = \frac{\partial L}{\delta\dot{\boldsymbol{x}}} \delta\boldsymbol{x}\Big|_{t0}^{t1} - \int_{t_0}^{t_1} (\frac{d}{dt} \frac{\partial L}{\delta\dot{\boldsymbol{x}}} \delta\boldsymbol{x})\, dt \tag{3.8}$$

Because of (3.3), the variations of the configuration variables $\delta\boldsymbol{x}$ vanish at the boundaries, i.e., $\delta\boldsymbol{x}(t_0) = \delta\boldsymbol{x}(t_1) = 0$. Therefore the term at the boundary in (3.8) is zero. Plugging (3.8) into equation (3.7) gives:

$$0 \stackrel{!}{=} \delta A = \int_{t_0}^{t_1} (\frac{\partial L}{\delta\boldsymbol{x}} - \frac{d}{dt} \frac{\partial L}{\delta\dot{\boldsymbol{x}}}) \delta\boldsymbol{x}\, dt$$

Since the integral must be zero for all possible variations $\delta \boldsymbol{x}$, the term in the bracket must vanish. The result is the well known Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\delta \dot{\boldsymbol{x}}} - \frac{\partial L}{\delta \boldsymbol{x}} = \mathbf{0} \tag{3.9}$$

The conjugate momentum is defined as:

$$\boldsymbol{p} = \frac{\partial L}{\delta \dot{\boldsymbol{x}}}$$

The Hamiltonian is a function of the configuration variables and the momentum. It yields the total mechanical energy on the trajectory of the system. The trajectory $(\boldsymbol{x}, \dot{\boldsymbol{x}})$ is the solution of the Lagrange equations (3.9). The Hamiltonian is given by the Legendre Transformation [LLM17, eq. 3.10]:

$$H(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{p}^T \dot{\boldsymbol{x}}(\boldsymbol{p}, \boldsymbol{x}) - L(\boldsymbol{x}, \dot{\boldsymbol{x}}(\boldsymbol{p}, \boldsymbol{x})) \tag{3.10}$$

It is assumed that $\dot{\boldsymbol{x}}$ can be expressed in terms of $\boldsymbol{p}$ and $\boldsymbol{x}$. This mathematically requires that the mass matrix is invertible, which holds for any physical system. The Hamiltonian H is a constant of motion along the trajectory of the system. Therefore the total mechanical energy is conserved:

$$\begin{aligned}
\dot{H} &= \frac{d}{dt}(\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} - L(\boldsymbol{x}, \dot{\boldsymbol{x}})) \\
&= (\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{x}}})\dot{\boldsymbol{x}} + \frac{\partial L}{\partial \dot{\boldsymbol{x}}}\ddot{\boldsymbol{x}} - \frac{\partial L}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} - \frac{\partial L}{\partial \dot{\boldsymbol{x}}}\ddot{\boldsymbol{x}} \\
&= (\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} - \frac{\partial L}{\partial \boldsymbol{x}})\dot{\boldsymbol{x}} \\
&= 0
\end{aligned}$$

3.1.2. Hamel Equations on SO(3)

We want to derive the Euler equation for the rigid body with a potential $U(\boldsymbol{R})$. In this case the configuration variable is the rotation matrix $\boldsymbol{R}$ and its configuration manifold is $SO(3)$. The velocities $\hat{\boldsymbol{\omega}}$ are elements the Lee algebra $se(3)$. The matrix elements of the skew matrix $\hat{\boldsymbol{\omega}}$ can be collected in the vector $\boldsymbol{\omega}$. The Lagrangian is given by:

$$\begin{aligned}
L(\boldsymbol{\omega}) &= T(\boldsymbol{w}) - U(\boldsymbol{R}) \\
T(\boldsymbol{w}) &= \boldsymbol{\omega}^T \boldsymbol{I} \boldsymbol{\omega}
\end{aligned} \tag{3.11}$$

where $\boldsymbol{\omega}$ is the left trivialized body velocity $\boldsymbol{\omega} = \boldsymbol{R}^T \dot{\boldsymbol{R}}$ and $\boldsymbol{I}$ is the inertia matrix. Hamiltonian's principle (3.4) is given by:

$$0 \stackrel{!}{=} \delta A = \int_{t_0}^{t_1} (\frac{\partial L}{\partial \boldsymbol{\omega}}\delta \boldsymbol{\omega} + \frac{\partial L}{\partial \boldsymbol{R}}\delta \boldsymbol{R})dt \tag{3.12}$$

The variation of the body velocity $\delta\boldsymbol{\omega}$ has to be derived from the variation of the rotation $\boldsymbol{R}^\epsilon$, which is defined by an infinitesimal rotation generated by a twist $\delta\hat{\boldsymbol{\eta}}(t) \in so(3)$.

$$\boldsymbol{R}^\epsilon = \boldsymbol{R}e^{\epsilon\delta\hat{\boldsymbol{\eta}}}$$

$$\delta\boldsymbol{R} = \frac{d}{d\epsilon}\boldsymbol{R}^\epsilon\Big|_{\epsilon=0} = \boldsymbol{R}\delta\hat{\boldsymbol{\eta}} \tag{3.13}$$

$$\delta\dot{\boldsymbol{R}} = \frac{d}{dt}\delta\boldsymbol{R} = \dot{\boldsymbol{R}}\delta\hat{\boldsymbol{\eta}} + \boldsymbol{R}\dot{\hat{\boldsymbol{\eta}}} \tag{3.14}$$

We define the column vector $\boldsymbol{r}_i$ as the ith row vector of $\boldsymbol{R}$ transposed and the column vector $\delta\boldsymbol{r}_i$ as the ith row vector of $\delta\boldsymbol{R}$ transposed. For the column vectors, it holds:

$$\delta\boldsymbol{r}_i = -\delta\hat{\boldsymbol{\eta}}\,\boldsymbol{r}_i = \boldsymbol{r}_i \times \delta\boldsymbol{\eta} \qquad\qquad i \in (1,2,3) \tag{3.15}$$

The variation of the body velocity can be calculated as:

$$\begin{aligned}
\delta\hat{\boldsymbol{\omega}} = \delta(\boldsymbol{R}^T\dot{\boldsymbol{R}}) &= \delta\boldsymbol{R}^T\dot{\boldsymbol{R}} + \boldsymbol{R}^T\delta\dot{\boldsymbol{R}} \\
&= -\delta\hat{\boldsymbol{\eta}}\boldsymbol{R}^T\dot{\boldsymbol{R}} + \boldsymbol{R}^T\dot{\boldsymbol{R}}\delta\hat{\boldsymbol{\eta}} + \delta\dot{\hat{\boldsymbol{\eta}}} \\
&= -\delta\hat{\boldsymbol{\eta}}\,\hat{\boldsymbol{\omega}} + \hat{\boldsymbol{\omega}}\,\delta\hat{\boldsymbol{\eta}} + \delta\dot{\hat{\boldsymbol{\eta}}} \\
&= (\delta\dot{\boldsymbol{\eta}} + \boldsymbol{\omega} \times \delta\boldsymbol{\eta})\hat{} \\
\delta\boldsymbol{\omega} &= \delta\dot{\boldsymbol{\eta}} + \boldsymbol{\omega} \times \delta\boldsymbol{\eta}
\end{aligned} \tag{3.16}$$

where the identity (A.6c) was used. The action integral (3.12) contains the variation of the Lagrangian with respect to the rotation matrix. It is defined through the components of the matrix. Using (3.15) and (A.6d) gives:

$$\begin{aligned}
\frac{\partial L}{\partial \boldsymbol{R}}\delta\boldsymbol{R} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial L}{\partial \boldsymbol{R}_{ij}}\delta\boldsymbol{R}_{ij} = \sum_{i=1}^{3}\frac{\partial L}{\partial \boldsymbol{r}_i}\delta\boldsymbol{r}_i \\
&= \sum_{i=1}^{3}\frac{\partial L}{\partial \boldsymbol{r}_i}(\boldsymbol{r}_i \times \delta\boldsymbol{\eta}) = \sum_{i=1}^{3}(\frac{\partial L}{\partial \boldsymbol{r}_i} \times \boldsymbol{r}_i)\delta\boldsymbol{\eta}
\end{aligned} \tag{3.17}$$

Now the variations (3.16), (3.15) and (3.17) can be plugged into the action integral (3.12):

$$\begin{aligned}
0 \overset{!}{=} \delta A &= \int_{t_0}^{t_1}[\frac{\partial L}{\partial \boldsymbol{\omega}}\delta\boldsymbol{\omega} + \sum_{i=1}^{3}\frac{\partial L}{\partial \boldsymbol{r}_i}\delta\boldsymbol{r}_i]\,dt \\
&= \int_{t_0}^{t_1}[\frac{\partial L}{\delta \boldsymbol{\omega}}\delta\dot{\boldsymbol{\eta}} + \frac{\partial L}{\delta \boldsymbol{\omega}}\,\boldsymbol{\omega} \times \delta\boldsymbol{\eta}\sum_{i=1}^{3}(\frac{\partial L}{\partial \boldsymbol{r}_i} \times \boldsymbol{r}_i)\delta\boldsymbol{\eta}]\,dt
\end{aligned}$$

The first term is integrated by parts:

$$\begin{aligned}
\int_{t_0}^{t_1}(\frac{\partial L}{\delta \boldsymbol{\omega}}\delta\dot{\boldsymbol{\eta}})\,dt &= \frac{\partial L}{\delta \boldsymbol{\omega}}\delta\boldsymbol{\eta}\Big|_{t0}^{t1} - \int_{t_0}^{t_1}(\frac{d}{dt}\frac{\partial L}{\delta \boldsymbol{\omega}}\delta\boldsymbol{\eta})\,dt \\
&= -\int_{t_0}^{t_1}(\frac{d}{dt}\frac{\partial L}{\delta \boldsymbol{\omega}}\delta\boldsymbol{\eta})\,dt
\end{aligned} \tag{3.18}$$

The boundary term vanishes, since the variations at the boundaries are 0: $\delta\boldsymbol{\omega}(t_0) = \delta\boldsymbol{\omega}(t_1) = \mathbf{0}$. The action integral becomes:

$$0 \overset{!}{=} \delta A = -\int_{t_0}^{t_1} \left(\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}} + \boldsymbol{\omega} \times \frac{\partial L}{\partial \boldsymbol{\omega}} + \boldsymbol{r}_i \times \frac{\partial L}{\partial \boldsymbol{r}_i}\right)\delta\boldsymbol{\eta}\, dt$$

According to Hamilton's variation principle δA has to vanish for all variations $\delta\boldsymbol{\eta}$. Therefore, the terms in brackets have to sum up to zero. Using the Lagrangian (3.11) of the rigid body yields the Euler equations:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right) + \boldsymbol{\omega} \times \frac{\partial L}{\partial \boldsymbol{\omega}} + \sum_{i=1}^{3}\left(\boldsymbol{r}_i \times \frac{\partial L}{\partial \boldsymbol{r}_i}\right) &= 0 \\ \boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} + \sum_{i=1}^{3}\left(\boldsymbol{r}_i \times \frac{\partial L}{\partial \boldsymbol{r}_i}\right) &= 0 \end{aligned} \tag{3.19}$$

3.1.3. Hamel Equations on SE(3) x $\mathbf{R}^\mathbf{n}$

In the section 3.1.3 the Hamel equations for the Lee group manifold $SE(3) \times \mathbb{R}^n$are derived from variation principles (3.4). Subsequently the equations of motion are calculated in section 3.1.3.

Derivation of the Hamel Equations

With the machinery developed in 3.1 we can now derive the Hamel equations for the floating base robot system. First, we develop the Hamel equations for a free floating system using the Lagrangian (2.49). The derivation with potentials is derived subsequently. The configuration manifold is defined by the configuration variables $(\boldsymbol{R}_{ib}, \boldsymbol{O}_{ib}, \boldsymbol{q})$. In the free floating case, the Lagrangian depends only on the velocities $(\boldsymbol{v}_b, \boldsymbol{\omega}_b, \dot{\boldsymbol{q}})$ and on $\boldsymbol{q}$. Using the Hamilton's action principle (3.4), we take the derivatives of the Langrangian with respect to the velocities and with respect to $\boldsymbol{q}$:

$$0 \overset{!}{=} \delta A = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \boldsymbol{v}_b}\delta\boldsymbol{v}_b + \frac{\partial L}{\partial \boldsymbol{\omega}_b}\delta\boldsymbol{\omega}_b + \frac{\partial L}{\partial \boldsymbol{q}}\delta\boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\delta\dot{\boldsymbol{q}}\right)dt \tag{3.20}$$

We need to derive the variations of the linear body velocity $\delta\boldsymbol{v}_b$, the angular body velocity $\delta\boldsymbol{\omega}_b$, the joint position $\delta\boldsymbol{q}$ and the joint velocity $\delta\dot{\boldsymbol{q}}$. Given a point $\boldsymbol{H}_{ib}$ on the manifold $SE(3)$, all possible infinitesimal variations (2.32) are given by (3.5):

$$\delta\boldsymbol{H}_{ib} = \left.\frac{\partial \boldsymbol{H}_{ib}^{\epsilon}}{\partial \epsilon}\right|_{\epsilon=0} = \boldsymbol{H}_{ib}\delta\hat{\boldsymbol{\Gamma}} \tag{3.21}$$

The variation of the group element $\boldsymbol{H}_{ib}$ is generated by the $se(3)$ algebra element $\delta\hat{\boldsymbol{\Gamma}}$:

$$\delta\hat{\boldsymbol{\Gamma}} = \begin{bmatrix} \delta\hat{\boldsymbol{\eta}} & \delta\boldsymbol{\chi} \\ \mathbf{0} & 0 \end{bmatrix} \tag{3.22}$$

The components of $\delta \boldsymbol{H_{ib}}$ (3.21) are:

$$\delta \boldsymbol{H_{ib}} = \begin{bmatrix} \delta \boldsymbol{R}_{ib} & \delta \boldsymbol{o}_{ib} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{ib} & \boldsymbol{o}_{ib} \\ \boldsymbol{o}_{ib} & 1 \end{bmatrix} \begin{bmatrix} \delta \hat{\boldsymbol{\eta}} & \delta \boldsymbol{\chi} \\ \boldsymbol{0} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \boldsymbol{R}_{ib} \delta \hat{\boldsymbol{\eta}} & \boldsymbol{R}_{ib} \delta \boldsymbol{\chi} \\ \boldsymbol{0} & 0 \end{bmatrix}$$

From this we can read off the components of $\delta \boldsymbol{H_{ib}}$ and derive their derivatives:

$$\delta \boldsymbol{R}_{ib} = \boldsymbol{R}_{ib} \delta \hat{\boldsymbol{\eta}} \tag{3.23a}$$
$$\delta \boldsymbol{o}_{ib} = \boldsymbol{R}_{ib} \delta \boldsymbol{\chi} \tag{3.23b}$$
$$\delta \dot{\boldsymbol{R}}_{ib} = \dot{\boldsymbol{R}}_{ib} \delta \hat{\boldsymbol{\eta}} + \boldsymbol{R}_{ib} \delta \dot{\hat{\boldsymbol{\eta}}} \tag{3.23c}$$
$$\delta \dot{\boldsymbol{o}}_{ib} = \dot{\boldsymbol{R}}_{ib} \delta \boldsymbol{\chi} + \boldsymbol{R}_{ib} \delta \dot{\boldsymbol{\chi}} \tag{3.23d}$$

The linear body velocity $\boldsymbol{v}_b$ and the angular body velocity $\boldsymbol{\omega}_b$ are given by:

$$\boldsymbol{v}_b = \boldsymbol{R}_{ib}^T \dot{\boldsymbol{o}}_{ib}$$
$$\hat{\boldsymbol{\omega}}_b = \boldsymbol{R}_{ib}^T \dot{\boldsymbol{R}}_{ib}$$

Using these definitions, the variations of the angular body velocity

$$\begin{aligned} \delta \hat{\boldsymbol{\omega}}_{\boldsymbol{b}} &= \delta(\boldsymbol{R}_{ib}^T \dot{\boldsymbol{R}}_{ib}) \\ &= \delta \boldsymbol{R}_{ib}^T \dot{\boldsymbol{R}}_{ib} + \boldsymbol{R}_{ib}^T \delta \dot{\boldsymbol{R}}_{ib} \\ &= -\delta \hat{\boldsymbol{\eta}} \boldsymbol{R}_{ib}^T \dot{\boldsymbol{R}}_{ib} + \boldsymbol{R}_{ib}^T \dot{\boldsymbol{R}}_{ib} \delta \hat{\boldsymbol{\eta}} + \boldsymbol{R}_{ib} \delta \dot{\hat{\boldsymbol{\eta}}} \\ &= -\delta \hat{\boldsymbol{\eta}} \hat{\boldsymbol{\omega}}_{\boldsymbol{b}} + \hat{\boldsymbol{\omega}}_{\boldsymbol{b}} \delta \hat{\boldsymbol{\eta}} + \delta \dot{\hat{\boldsymbol{\eta}}} \\ &= (\delta \dot{\boldsymbol{\eta}} + \boldsymbol{\omega}_b \times \delta \boldsymbol{\eta})\hat{} \end{aligned}$$
$$\delta \boldsymbol{\omega}_b = \delta \dot{\boldsymbol{\eta}} + \boldsymbol{\omega}_b \times \delta \boldsymbol{\eta} \tag{3.24}$$

and variations of the linear body velocity

$$\begin{aligned} \delta \hat{\boldsymbol{v}}_{\boldsymbol{b}} &= \delta(\boldsymbol{R}_{ib}^T \dot{\boldsymbol{o}}_{ib}) \\ &= \delta \boldsymbol{R}_{ib}^T \dot{\boldsymbol{o}}_{ib} + \boldsymbol{R}_{ib}^T \delta \dot{\boldsymbol{o}}_{ib} \\ &= -\delta \hat{\boldsymbol{\eta}} \boldsymbol{R}_{ib}^T \dot{\boldsymbol{o}}_{ib} + \boldsymbol{R}_{ib}^T (\dot{\boldsymbol{R}}_{ib} \delta \boldsymbol{\chi} + \boldsymbol{R}_{ib} \delta \dot{\boldsymbol{\chi}}) \\ &= -\delta \hat{\boldsymbol{\eta}} \boldsymbol{v}_b + \hat{\boldsymbol{\omega}}_{\boldsymbol{b}} \delta \boldsymbol{\chi} + \delta \dot{\boldsymbol{\chi}} \end{aligned}$$
$$\delta \boldsymbol{v}_b = \delta \dot{\boldsymbol{\chi}} + \boldsymbol{\omega}_b \times \delta \boldsymbol{\chi} + \boldsymbol{v}_b \times \delta \boldsymbol{\eta} \tag{3.25}$$

can be computed. The variations of the Cartesian joint variables $\delta \boldsymbol{q}$ and the $\delta \dot{\boldsymbol{q}}$ are just translations. The variations defined in Hamilton's variation principle (3.20) are:

$$0 \overset{!}{=} \delta A = \int_{t_0}^{t_1} [\frac{\partial L}{\partial \boldsymbol{v}_b}(\delta \dot{\boldsymbol{\chi}} + \boldsymbol{\omega}_b \times \delta \boldsymbol{\chi} + \boldsymbol{v}_b \times \delta \boldsymbol{\eta}) + \frac{\partial L}{\partial \boldsymbol{\omega}_b}(\delta \dot{\boldsymbol{\eta}} + \boldsymbol{\omega}_b \times \delta \boldsymbol{\eta}) + \frac{\partial L}{\partial \boldsymbol{q}} \delta \boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \delta \dot{\boldsymbol{q}}] dt \tag{3.26}$$

The terms containing $\delta\dot{\boldsymbol{\chi}}$, $\delta\dot{\boldsymbol{\eta}}$ and $\dot{\boldsymbol{q}}$ can be integrated by parts:

$$\begin{aligned}
\int_{t_0}^{t_1}(\frac{\partial L}{\partial \boldsymbol{v}_b}\delta\dot{\boldsymbol{\chi}})\,dt &= \frac{\partial L}{\partial \boldsymbol{v}_b}\delta\boldsymbol{\chi}\Big|_{t0}^{t1} - \int_{t_0}^{t_1}(\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{v}_b}\delta\boldsymbol{\chi})\,dt = -\int_{t_0}^{t_1}(\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{v}_b}\delta\boldsymbol{\chi})\,dt \\
\int_{t_0}^{t_1}(\frac{\partial L}{\partial \boldsymbol{\omega}_b}\delta\dot{\boldsymbol{\eta}})\,dt &= \frac{\partial L}{\partial \boldsymbol{\omega}_b}\delta\boldsymbol{\eta}\Big|_{t0}^{t1} - \int_{t_0}^{t_1}(\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}_b}\delta\boldsymbol{\eta})\,dt = -\int_{t_0}^{t_1}(\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}_b}\delta\boldsymbol{\eta})\,dt \\
\int_{t_0}^{t_1}(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\delta\dot{\boldsymbol{q}})\,dt &= \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\delta\boldsymbol{q}\Big|_{t0}^{t1} - \int_{t_0}^{t_1}(\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\delta\boldsymbol{q})\,dt = -\int_{t_0}^{t_1}(\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\delta\boldsymbol{q})\,dt
\end{aligned} \tag{3.27}$$

The terms at the boundary vanish, since the variations at the boundaries are zero (3.3). Using (3.27) and the cyclical property of the cross product with the scalar product (A.6d) and putting them in (3.26) gives the final result:

$$0 \stackrel{!}{=} -\int_{t_0}^{t_1}[(\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{v}_b} + \boldsymbol{\omega}_b \times \frac{\partial L}{\partial \boldsymbol{v}_b})\delta\boldsymbol{\chi} + (\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}_b} + \boldsymbol{\omega}_b \times \frac{\partial L}{\partial \boldsymbol{\omega}_b} + \boldsymbol{v}_b \times \frac{\partial L}{\partial \boldsymbol{v}_b})\delta\boldsymbol{\eta} + (\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}})\delta\boldsymbol{q}]dt \tag{3.28}$$

Since the action integral has to vanish for all variations, the terms in the bracket must be zero. Therefore the Hamel equations for $SE(3) \times \mathbb{R}^n$are:

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix}\frac{\partial L}{\partial \boldsymbol{v}_b}\\ \frac{\partial L}{\partial \boldsymbol{\omega}_b}\end{bmatrix} - \boldsymbol{a}_{ib}^T\begin{bmatrix}\frac{\partial L}{\partial \boldsymbol{v}_b}\\ \frac{\partial L}{\partial \boldsymbol{\omega}_b}\end{bmatrix} &= \boldsymbol{F_b} \\
\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} &= \boldsymbol{\tau}
\end{aligned}$$

where $\boldsymbol{a}_{ib}$ is the little adjoint matrix. The equations of motion can be compactly written in terms of the body velocity $\boldsymbol{\nu}_b$:

$$\frac{d}{dt}\frac{\partial L}{\boldsymbol{\nu}_b} - \boldsymbol{a}_b^T\frac{\partial L}{\boldsymbol{\nu}_b} = \boldsymbol{F_b} \tag{3.30a}$$

$$\frac{d}{dt}\frac{\partial L}{\dot{\boldsymbol{q}}} - \frac{\partial L}{\boldsymbol{q}} = \boldsymbol{\tau} \tag{3.30b}$$

If the Lagrangian contains in addition to velocities also configuration variables $(\boldsymbol{R}_{ib}, \boldsymbol{O}_{ib}, \boldsymbol{q})$, which is the case, if it contains a potential $U(\boldsymbol{o}_{ib}, \boldsymbol{R}_{ib}, \boldsymbol{q})$, the action integral (3.28) contains 2 additional terms. These are the derivatives with respect to the configuration variables. Using (3.23b) for $\delta\boldsymbol{o}_{ib}$ and (3.17) for $\delta\boldsymbol{R}$, the additional variations are:

$$\begin{aligned}
\frac{\partial L}{\partial \boldsymbol{o}_{ib}}\delta\boldsymbol{o}_{ib} &= \frac{\partial L}{\partial \boldsymbol{o}_{ib}}\boldsymbol{R}_{ib}\,\delta\boldsymbol{\chi} = \boldsymbol{R}_{ib}^T\frac{\partial L}{\partial \boldsymbol{o}_{ib}}\,\delta\boldsymbol{\chi} \\
\frac{\partial L}{\partial \boldsymbol{R}_{ib}}\delta\boldsymbol{R}_{ib} &= -(\boldsymbol{r}_i \times \frac{\partial L}{\partial \boldsymbol{r}_i})\,\delta\boldsymbol{\eta}
\end{aligned}$$

The action integral (3.28) becomes:

$$
\begin{aligned}
0 \overset{!}{=} -\int_{t_0}^{t_1} [(\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{v}_b} + \boldsymbol{\omega}_b \times \frac{\partial L}{\partial \boldsymbol{v}_b} - \boldsymbol{R}_{ib}^T \frac{\partial L}{\partial \boldsymbol{o}_{ib}})\delta \boldsymbol{\chi} \\
+(\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\omega}_b} + \boldsymbol{\omega}_b \times \frac{\partial L}{\partial \boldsymbol{\omega}_b} + \boldsymbol{v}_b \times \frac{\partial L}{\partial \boldsymbol{v}_b} + \boldsymbol{r}_i \times \frac{\partial L}{\partial \boldsymbol{r}_i})\delta \boldsymbol{\eta} \\
+(\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}}]dt
\end{aligned}
\tag{3.31}
$$

Therefore the Hamel equations are given by:

$$
\frac{d}{dt}\begin{bmatrix} \frac{\partial L}{\partial \boldsymbol{v}_b} \\ \frac{\partial L}{\partial \boldsymbol{\omega}_b} \end{bmatrix} - \boldsymbol{a}_{ib}^T \begin{bmatrix} \frac{\partial L}{\partial \boldsymbol{v}_b} \\ \frac{\partial L}{\partial \boldsymbol{\omega}_b} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{R}_{ib}^T \frac{\partial L}{\partial \boldsymbol{o}_{ib}} \\ \sum_{i=1}^{3}(\boldsymbol{r}_i \times \frac{\partial L}{\partial \boldsymbol{r}_i}) \end{bmatrix} = \boldsymbol{F_b} \tag{3.32a}
$$

$$
\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = \boldsymbol{\tau} \tag{3.32b}
$$

The momenta $\boldsymbol{h}_b$ and $\boldsymbol{h_q}$ are given by:

$$
\boldsymbol{h}_b = \frac{\partial L}{\partial \boldsymbol{\nu}_b} \tag{3.33}
$$

$$
\boldsymbol{h}_{\dot{\boldsymbol{q}}} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \tag{3.34}
$$

The Hamiltonian $\boldsymbol{H}(\boldsymbol{o}_{ib}, \boldsymbol{R}_{ib}, \boldsymbol{q}, \boldsymbol{h}_b, \boldsymbol{h}_{\dot{\boldsymbol{q}}})$ is given by the Legendre transformation:

$$
\begin{aligned}
&\boldsymbol{H}(\boldsymbol{o}_{ib}, \boldsymbol{R}_{ib}, \boldsymbol{q}, \boldsymbol{h}_b, \boldsymbol{h}_{\dot{\boldsymbol{q}}}) = \\
&\qquad \boldsymbol{h}_b^T \boldsymbol{\nu}_b(\boldsymbol{h}_b, \boldsymbol{h}_{\dot{\boldsymbol{q}}}, \boldsymbol{q}) + \boldsymbol{h}_{\dot{\boldsymbol{q}}}^T \dot{\boldsymbol{q}}(\boldsymbol{h}_b, \boldsymbol{h}_{\dot{\boldsymbol{q}}}, \boldsymbol{q}) \\
&\qquad -L(\boldsymbol{o}_{ib}, \boldsymbol{R}_{ib}, \boldsymbol{q}, \boldsymbol{\nu}_b(\boldsymbol{h}_b, \boldsymbol{h}_{\dot{\boldsymbol{q}}}, \boldsymbol{q}), \dot{\boldsymbol{q}}(\boldsymbol{h}_b, \boldsymbol{h}_{\dot{\boldsymbol{q}}}, \boldsymbol{q}), \boldsymbol{q})
\end{aligned}
\tag{3.35}
$$

The derivatives of $\boldsymbol{H}$ are:

$$
\begin{aligned}
\frac{\partial}{\partial \boldsymbol{o}_{ib}}\boldsymbol{H}(\boldsymbol{o}_{ib}, \boldsymbol{R}_{ib}, \boldsymbol{q}, \boldsymbol{h}_b, \boldsymbol{h}_{\dot{\boldsymbol{q}}}) &= -\frac{\partial L}{\partial \boldsymbol{o}_{ib}} \\
\frac{\partial}{\partial \boldsymbol{R}_{ib}}\boldsymbol{H}(\boldsymbol{o}_{ib}, \boldsymbol{R}_{ib}, \boldsymbol{q}, \boldsymbol{h}_b, \boldsymbol{h}_{\dot{\boldsymbol{q}}}) &= -\frac{\partial L}{\partial \boldsymbol{R}_{ib}} \\
\frac{\partial}{\partial \boldsymbol{q}}\boldsymbol{H}(\boldsymbol{o}_{ib}, \boldsymbol{R}_{ib}, \boldsymbol{q}, \boldsymbol{h}_b, \boldsymbol{h}_{\dot{\boldsymbol{q}}}) &= -\frac{\partial L}{\partial \boldsymbol{q}} \\
\frac{\partial}{\partial \boldsymbol{h}_b}\boldsymbol{H}(\boldsymbol{o}_{ib}, \boldsymbol{R}_{ib}, \boldsymbol{q}, \boldsymbol{h}_b, \boldsymbol{h}_{\dot{\boldsymbol{q}}}) &= \boldsymbol{\nu}_b \\
\frac{\partial}{\partial \boldsymbol{h}_{\dot{\boldsymbol{q}}}}\boldsymbol{H}(\boldsymbol{o}_{ib}, \boldsymbol{R}_{ib}, \boldsymbol{q}, \boldsymbol{h}_b, \boldsymbol{h}_{\dot{\boldsymbol{q}}}) &= \dot{\boldsymbol{q}}
\end{aligned}
\tag{3.36}
$$

The Hamiltonian is the total mechanical energy of the system, which is conserved on the trajectory on the cotangent bundle:

$$
\begin{aligned}
\dot{\boldsymbol{H}} &= \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{o}_{ib}}\dot{\boldsymbol{o}}_{ib} + \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{R}_{ib}}\dot{\boldsymbol{R}}_{ib} + \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} + \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{h}_b}\dot{\boldsymbol{h}}_b + \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{h}_{\boldsymbol{q}}}\dot{\boldsymbol{h}}_{\boldsymbol{q}} \\
&= -\boldsymbol{v}_{ib}^T\boldsymbol{R}_{ib}^T\frac{\partial L}{\partial \boldsymbol{o}_{ib}} + \boldsymbol{\omega}_{ib}^T(\boldsymbol{r}_i \times \frac{\partial L}{\partial \boldsymbol{r}_i}) - \dot{\boldsymbol{q}}^T\frac{\partial L}{\partial \boldsymbol{q}} + \boldsymbol{\nu}_b^T(\frac{d}{dt}\frac{\partial L}{\boldsymbol{\nu}_b}) + \dot{\boldsymbol{q}}^T(\frac{d}{dt}\frac{\partial L}{\dot{\boldsymbol{q}}}) \\
&= \boldsymbol{\nu}_b^T(\frac{d}{dt}\frac{\partial L}{\boldsymbol{\nu}_b} + \begin{bmatrix} -\boldsymbol{R}_{ib}^T\frac{\partial L}{\partial \boldsymbol{o}_{ib}} \\ \sum_{i=1}^3(\boldsymbol{r}_i \times \frac{\partial L}{\partial \boldsymbol{r}_i}) \end{bmatrix}) + \dot{\boldsymbol{q}}^T(\frac{d}{dt}\frac{\partial L}{\dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}}) \\
&= 0
\end{aligned}
$$

Equations on Motion

We can now derive the equations of motion from (3.30a) and (3.30b) using the Lagrangian (2.49):

$$
\begin{aligned}
\frac{\partial L}{\boldsymbol{\nu}_b} &= \boldsymbol{M}_{bb}\boldsymbol{\nu}_b + \boldsymbol{M}_{bq}\dot{\boldsymbol{q}} \\
\frac{d}{dt}\frac{\partial L}{\boldsymbol{\nu}_b} &= \boldsymbol{M}_{bb}\dot{\boldsymbol{\nu}}_{\boldsymbol{b}} + \dot{\boldsymbol{M}}_{bb}\boldsymbol{\nu}_b + \boldsymbol{M}_{bq}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{M}}_{bq}\dot{\boldsymbol{q}} \\
\frac{\partial L}{\partial \dot{\boldsymbol{q}}} &= \boldsymbol{M}_{bq}^T\boldsymbol{\nu}_b + \boldsymbol{M}_{qq}\dot{\boldsymbol{q}} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} &= \boldsymbol{M}_{bq}^T\dot{\boldsymbol{\nu}}_{\boldsymbol{bb}} + \dot{\boldsymbol{M}}_{bq}^T\boldsymbol{\nu}_b + \boldsymbol{M}_{qq}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{M}}_{qq}\dot{\boldsymbol{q}} \\
\boldsymbol{a}_b^T\frac{\partial L}{\partial \boldsymbol{\nu}_b} &= \boldsymbol{a}_b^T\boldsymbol{M}_{bb}\boldsymbol{\nu}_b + \boldsymbol{a}_b^T\boldsymbol{M}_{bq}\dot{\boldsymbol{q}} \\
\frac{\partial L}{\partial \boldsymbol{q}} &= \frac{1}{2}\nabla_{\boldsymbol{q}}\boldsymbol{\nu}_b^T\boldsymbol{M}_{bb}\boldsymbol{\nu}_b + \nabla_{\boldsymbol{q}}\boldsymbol{\nu}_b^T\boldsymbol{M}_{bq}\dot{\boldsymbol{q}} + \frac{1}{2}\nabla_{\boldsymbol{q}}\dot{\boldsymbol{q}}^T\boldsymbol{M}_{qq}\ddot{\boldsymbol{q}}
\end{aligned}
$$

The operator $\nabla_{\boldsymbol{q}}$ is the n dimensional gradient operator. Introducing the following abbreviations:

$$
\boldsymbol{M}_{b,q} = \begin{bmatrix} \frac{\partial}{\partial q_1}\boldsymbol{\nu}_b^T\boldsymbol{M}_{bb} \\ \vdots \\ \frac{\partial}{\partial q_n}\boldsymbol{\nu}_b^T\boldsymbol{M}_{bb} \end{bmatrix} \qquad \boldsymbol{M}_{bq,q} = \begin{bmatrix} \frac{\partial}{\partial q_1}\boldsymbol{\nu}_b^T\boldsymbol{M}_{bq} \\ \vdots \\ \frac{\partial}{\partial q_n}\boldsymbol{\nu}_b^T\boldsymbol{M}_{bq} \end{bmatrix} \qquad M_{q,q} = \begin{bmatrix} \frac{\partial}{\partial q_1}\dot{\boldsymbol{q}}^T\boldsymbol{M}_{qq} \\ \vdots \\ \frac{\partial}{\partial q_n}\dot{\boldsymbol{q}}^T\boldsymbol{M}_{qq} \end{bmatrix} \tag{3.37}
$$

The matrix $\boldsymbol{M}_{b,q}$ has dimensions $n \times 6$, while the matrices $\boldsymbol{M}_{bq,q}$ and $\boldsymbol{M}_{q,q}$ have dimension $n \times n$. The equations of motion (3.30a) and (3.30b) can be written as:

$$
\begin{bmatrix} \boldsymbol{M}_{bb} & \boldsymbol{M}_{bq} \\ \boldsymbol{M}_{bq}^T & \boldsymbol{M}_{qq} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\nu}}_b \\ \ddot{\boldsymbol{q}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_b & \boldsymbol{C}_{bq} \\ \boldsymbol{C}_{qb} & \boldsymbol{C}_q \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_b \\ \boldsymbol{\tau} \end{bmatrix} \tag{3.38}
$$

where the Coriolis-centrifugal terms have been parameterized by using the matrix:

$$\boldsymbol{C} = \begin{bmatrix} \dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T \boldsymbol{M}_{bb} & \dot{\boldsymbol{M}}_{bq} - \boldsymbol{a}_b^T \boldsymbol{M}_{bq} \\ \dot{\boldsymbol{M}}_{bq}^T - \frac{1}{2}\boldsymbol{M}_{b,q} & \dot{\boldsymbol{M}}_{qq} - \boldsymbol{M}_{bq,q} - \frac{1}{2}\boldsymbol{M}_{q,q} \end{bmatrix} \tag{3.39}$$

The mass matrix is the same as in the Euler-Newton approach (3.40).

Euler-Newton Equations

A compact formulation of the Euler-Newton equation is derived in [GOA13] and in [Gio20]. In this work we need the Euler-Newton equations only for comparing them with the results of the Hamel equations (3.38).

The body velocity of the jth frame is given by $\boldsymbol{\nu}_j = \boldsymbol{A}_{jb}\boldsymbol{\nu}_b + \boldsymbol{J}_{jq}\dot{\boldsymbol{q}}$ where $\boldsymbol{J}_{jq}$ is the body Jacobian which transforms the joint velocities to the velocity of the jth joint relative to the base $\boldsymbol{\nu}_{bj} = \boldsymbol{J}_{jq}\dot{\boldsymbol{q}}$. The equation of motion are:

$$\begin{bmatrix} \boldsymbol{M}_{bb} & \boldsymbol{M}_{bq} \\ \boldsymbol{M}_{bq}^T & \boldsymbol{M}_{qq} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\nu}}_b \\ \ddot{q} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_b^E & \boldsymbol{C}_{bq}^E \\ \boldsymbol{C}_{qb}^E & \boldsymbol{C}_q^E \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_{\boldsymbol{b}} \\ \boldsymbol{\tau} \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{q} \end{bmatrix} \tag{3.40}$$

The first matrix is the mass matrix $\boldsymbol{M}_b$ expressed in the base frame. The second matrix is the Coriolis matrix $\boldsymbol{C}_b$ expressed in the base frame. The variable $\boldsymbol{\nu}_b$ is the 6 dimensional body velocity of the base and $\boldsymbol{q}$ are the joint variables. The external wrench $\boldsymbol{F}_{\boldsymbol{b}}$ acts on the body, while τ are the actuator torques acting on the joint variables.

The block matrix elements for the mass matrix have been defined in (2.51a), (2.51b) and (2.51c). The Coriolis terms are given by

$$\boldsymbol{C}_b^E = \sum_{j=0}^{n} \boldsymbol{A}_{jb}^T \boldsymbol{\Psi}_j \boldsymbol{A}_{jb} + \boldsymbol{A}_{jb}^T \boldsymbol{\Lambda}_j \dot{\boldsymbol{A}}_{jb} \tag{3.41a}$$

$$\boldsymbol{C}_{bq}^E = \sum_{j=0}^{n} \boldsymbol{A}_{jb}^T \boldsymbol{\Psi}_j \boldsymbol{J}_{jq} + \boldsymbol{A}_{jb}^T \boldsymbol{\Lambda}_j \dot{\boldsymbol{J}}_{jq} \tag{3.41b}$$

$$\boldsymbol{C}_{qb}^E = \sum_{j=0}^{n} \boldsymbol{J}_{jq}^T \boldsymbol{\Psi}_j \boldsymbol{A}_{jb} + \boldsymbol{J}_{jq}^T \boldsymbol{\Lambda}_j \dot{\boldsymbol{A}}_{jb} \tag{3.41c}$$

$$\boldsymbol{C}_q^E = \sum_{j=0}^{n} \boldsymbol{J}_{jq}^T \boldsymbol{\Psi}_j \boldsymbol{J}_{jq} + \boldsymbol{J}_{jq}^T \boldsymbol{\Lambda}_j \dot{\boldsymbol{J}}_{jq} \tag{3.41d}$$

where $\Psi = \boldsymbol{\Lambda}_j \boldsymbol{a}_j - \boldsymbol{a}_j^T \boldsymbol{\Lambda}_j$.

Equivalence of the Equations of Motion

The equations of motion from the Hamel equations (3.38) and from the Euler-Newton formulation (3.40) must be the same. However, this is not easy spot.

Proposition 3.1.1 (Equivalence Equations of Motion). *The Hamel equations for $SE(3) \times \mathbb{R}^n$equation (3.38) and the Euler Newton equations (3.40) yield the same equations of motion:*

$$\begin{bmatrix} \boldsymbol{M}_{bb} & \boldsymbol{M}_{bq} \\ \boldsymbol{M}_{bq}^T & \boldsymbol{M}_{qq} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\nu}}_b \\ \ddot{\boldsymbol{q}} \end{bmatrix} + \begin{bmatrix} \dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T \boldsymbol{M}_{bb} & \dot{\boldsymbol{M}}_{bq} - \boldsymbol{a}_b^T \boldsymbol{M}_{bq} \\ \dot{\boldsymbol{M}}_{bq}^T - \frac{1}{2} \boldsymbol{M}_{b,q} & \dot{\boldsymbol{M}}_{qq} - \boldsymbol{M}_{bq,q} - \frac{1}{2} \boldsymbol{M}_{q,q} \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix}$$
$$= \begin{bmatrix} \boldsymbol{M}_{bb} & \boldsymbol{M}_{bq} \\ \boldsymbol{M}_{bq}^T & \boldsymbol{M}_{qq} \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_b^E & \boldsymbol{C}_{bq}^E \\ \boldsymbol{C}_{qb}^E & \boldsymbol{C}_q^E \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix}$$

We need to prove that the Coriolis matrices yield the same equations of motion. To this end, we need some intermediate results.

Lemma 3.1. *The following relations hold for the little adjoint $\boldsymbol{a}_{jb}$ and the big adjoint transformation $\boldsymbol{A}_{jb}$, respectively:*

$$\dot{\boldsymbol{A}}_{jb} = \boldsymbol{A}_{jb}\, \boldsymbol{a}_{jb} \qquad \dot{\boldsymbol{A}}_{jb}^T = \boldsymbol{a}_{jb}^T\, \boldsymbol{A}_{jb}^T \tag{3.42a}$$
$$\dot{\boldsymbol{A}}_{jb}^{-1} = -\, \boldsymbol{a}_{jb}\, \boldsymbol{A}_{jb}^{-1} \qquad \dot{\boldsymbol{A}}_{jb}^{-T} = -\, \boldsymbol{A}_{jb}^{-T}\, \boldsymbol{a}_{jb}^T \tag{3.42b}$$
$$\boldsymbol{A}_{jb}\, \boldsymbol{a}_{jb} = -\, \boldsymbol{a}_{bj}\, \boldsymbol{A}_{jb} \qquad \boldsymbol{a}_{jb}^T \boldsymbol{A}_{jb}^T = -\, \boldsymbol{A}_{jb}^T\, \boldsymbol{a}_{bj}^T \tag{3.42c}$$
$$\boldsymbol{a}_{jb}\, \boldsymbol{A}_{jb}^{-1} = -\, \boldsymbol{A}_{jb}^{-1}\, \boldsymbol{a}_{bj} \qquad \boldsymbol{A}_{jb}^{-T} \boldsymbol{a}_{jb}^T = -\, \boldsymbol{a}_{bj}^T\, \boldsymbol{A}_{jb}^{-T} \tag{3.42d}$$
$$\boldsymbol{a}_j = \boldsymbol{A}_{jb} (\, \boldsymbol{a}_b - \boldsymbol{a}_{jb})\, \boldsymbol{A}_{jb}^{-1} \qquad \boldsymbol{a}_j^T = \boldsymbol{A}_{jb}^{-T} (\, \boldsymbol{a}_b^T - \boldsymbol{a}_{jb}^T)\, \boldsymbol{A}_{jb}^T \tag{3.42e}$$

Proof. Equation (3.42a) is taken from [Gio20]. Equation (3.42c) and equation (3.42d) follow directly from (2.37c) by setting $x = z = b$ and $y = j$ then we get $0 = \boldsymbol{a}_{bb} = \boldsymbol{A}_{bj} \boldsymbol{a}_{bj} \boldsymbol{A}_{bj}^{-1} + \boldsymbol{a}_{jb}$. Solving for $\boldsymbol{A}_{jb} \boldsymbol{a}_{jb}$ with $\boldsymbol{A}_{bj} = \boldsymbol{A}_{bj}^{-1}$ and gives the result.
Equation (3.42e) also directly follows from (2.37c) by setting $x = i$ $y = j$ and $z = b$. Then we get $\boldsymbol{a}_b = \boldsymbol{A}_{bj} \boldsymbol{a}_j \boldsymbol{A}_{bj}^{-1} + \boldsymbol{a}_{jb}$. Solving for $\boldsymbol{a}_j$ gives the result
Eq (3.42b): $\dot{\boldsymbol{A}}_{jb}^{-1} = \dot{\boldsymbol{A}}_{bj} = \boldsymbol{A}_{jb}^{-1} \boldsymbol{a}_{bj} = -\boldsymbol{a}_{jb}\, \boldsymbol{A}_{jb}^{-1}$.

Proof of Proposition 3.1.1
For the first row of the equations of motion we have to show:
$\boldsymbol{C}_b^E\, \boldsymbol{\nu}_b + \boldsymbol{C}_{bq}^E \dot{\boldsymbol{q}} = \boldsymbol{C}_b\, \boldsymbol{\nu}_b + \boldsymbol{C}_{bq} \dot{\boldsymbol{q}}$. The superscript E stand for the Euler-Newton equations (3.40). By using (3.41a), (3.41b) and the first row of (3.40), the Coriolis terms for Euler-Newton without the passivity term are:

$$\boldsymbol{C}_b^E\, \boldsymbol{\nu}_b + \boldsymbol{C}_{bq}^E \dot{\boldsymbol{q}} =$$
$$\sum_{j=0}^{n} \boldsymbol{A}_{jb}^T\, \boldsymbol{\Lambda}_j\, \dot{\boldsymbol{A}}_{jb}\, \boldsymbol{\nu}_b + \boldsymbol{A}_{jb}^T\, \boldsymbol{\Lambda}_j\, \dot{\boldsymbol{J}}_{jq} \dot{\boldsymbol{q}} - \boldsymbol{A}_{jb}^T\, \boldsymbol{a}_j^T\, \boldsymbol{\Lambda}_j\, \boldsymbol{\nu}_j \tag{3.43}$$

where (2.39) ha been used. Taking the Coriolis terms for the Hamel equations (3.39) and using identities (3.42a) and (3.42e) from Lemma 3.1 we get

$$\begin{aligned}
\boldsymbol{C}_b\,\boldsymbol{\nu}_b =&(\,\dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T \boldsymbol{M}_{bb})\,\boldsymbol{\nu}_b \\
=&(\,\boldsymbol{a}_{jb}^T\,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb} + \,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb} - \,\boldsymbol{a}_b^T\,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb})\,\boldsymbol{\nu}_b \\
=&((\,\boldsymbol{a}_{jb}^T - \,\boldsymbol{a}_b^T)\,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb} + \,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb})\,\boldsymbol{\nu}_b \\
=&\,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb}\,\boldsymbol{\nu}_b - \,\boldsymbol{A}_{jb}^T\,\boldsymbol{a}_j^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b \qquad (3.44)\\
\boldsymbol{C}_{bq}\dot{\boldsymbol{q}} =&(\,\dot{\boldsymbol{M}}_{bq} - \,\boldsymbol{a}_b^T\,\boldsymbol{M}_{bq})\dot{\boldsymbol{q}} \\
=&\,(\,\boldsymbol{a}_{jb}^T\,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jb} + \,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{J}}_{jb} - \,\boldsymbol{a}_b^T\,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jb})\dot{\boldsymbol{q}} \\
=&\,((\,\boldsymbol{a}_{jb}^T - \,\boldsymbol{a}_b^T)\,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jb} + \,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{J}}_{jb})\dot{\boldsymbol{q}} \\
=&\,\,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{J}}_{jq}\dot{\boldsymbol{q}} - \,\boldsymbol{A}_{jb}^T\,\boldsymbol{a}_j^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \qquad (3.45)\\
\boldsymbol{C}_b\,\boldsymbol{\nu}_b + \boldsymbol{C}_{bq}\dot{\boldsymbol{q}} =&\,\,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb}\,\boldsymbol{\nu}_b + \,\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{J}}_{jq}\dot{\boldsymbol{q}} - \,\boldsymbol{A}_{jb}^T\,\boldsymbol{a}_j^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{\nu}_j \qquad (3.46)
\end{aligned}$$

where relation (2.39), as well as the expressions of the sub-blocks $\boldsymbol{M}_{bb}$ (2.51a), $\boldsymbol{M}_{bq}$ (2.51b) and $\boldsymbol{M}_{qq}$ (2.51c) have been used. The r.h.s. of (3.46) is the same as the r.h.s of (3.43). This proves the equivalence of the first row of the Hamel and Euler-Newton Coriolis-centrifugal terms.

For the second row of the equations of motion we have to show:
$\boldsymbol{C}_{qb}^E\,\boldsymbol{\nu}_b + \boldsymbol{C}_q^E\dot{\boldsymbol{q}} \overset{!}{=} \boldsymbol{C}_{qb}\,\boldsymbol{\nu}_b + \boldsymbol{C}_q\dot{\boldsymbol{q}}$.
The Coriolis terms for Euler-Newton are:

$$\boldsymbol{C}_{qb}^E\,\boldsymbol{\nu}_b + \boldsymbol{C}_q^E\dot{\boldsymbol{q}} = -\,\boldsymbol{J}_{jq}^T\,\boldsymbol{a}_j^T\,\Lambda_j\,\boldsymbol{\nu}_j + \boldsymbol{J}_{jq}^T\boldsymbol{\Lambda}_j\dot{\boldsymbol{A}}_{jb}\,\boldsymbol{\nu}_b + \boldsymbol{J}_{jq}^T\boldsymbol{\Lambda}_j\dot{\boldsymbol{J}}_{jq}\dot{\boldsymbol{q}}$$

We need some additional notation. First we define matrices similar to ones used for the equations of motion (3.37):

$$\boldsymbol{A}_{jb,q} = \begin{bmatrix} \frac{\partial}{\partial q_1}\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b \\ \vdots \\ \frac{\partial}{\partial q_n}\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b \end{bmatrix} \qquad \boldsymbol{J}_{jq,q} = \begin{bmatrix} \frac{\partial}{\partial q_1}\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \\ \vdots \\ \frac{\partial}{\partial q_n}\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \end{bmatrix} \qquad (3.47)$$

The gradient matrices multiplied by the velocities can be expressed in terms of these quantities:

$$\boldsymbol{M}_{b,q}\,\boldsymbol{\nu}_b = 2\boldsymbol{A}_{jb,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b \qquad (3.48)$$
$$\boldsymbol{M}_{bq,q}^T\,\boldsymbol{\nu}_b = \boldsymbol{J}_{jq,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b + \boldsymbol{A}_{jb,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \qquad (3.49)$$
$$\boldsymbol{M}_{q,q}\dot{\boldsymbol{q}} = 2\boldsymbol{J}_{jq,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \qquad (3.50)$$

Relation (3.48) follows from the definitions of $\boldsymbol{M}_{b,q}$ (3.37) and $\boldsymbol{A}_{jb,q}$ (3.47):

$$\begin{aligned}
&\boldsymbol{M}_{b,q}\,\boldsymbol{\nu}_b = \\
&\begin{bmatrix} \frac{\partial}{\partial q_1}(\,\boldsymbol{\nu}_b^T\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b) \\ \vdots \\ \frac{\partial}{\partial q_n}\,\boldsymbol{\nu}_b^T\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b) \end{bmatrix} = 2\begin{bmatrix} (\frac{\partial}{\partial q_1}\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b)^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b \\ \vdots \\ (\frac{\partial}{\partial q_n}\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b)^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b \end{bmatrix} \\
&= 2\boldsymbol{A}_{jb,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b && (3.51) \\
& && (3.52)
\end{aligned}$$

The relation (3.49) follows from the definitions of $\boldsymbol{M}_{bq,q}$ (3.37), $\boldsymbol{A}_{jb,q}$ and $\boldsymbol{J}_{jq,q}$ (3.47):

$$\begin{aligned}
&\boldsymbol{M}_{bq,q}^T\,\boldsymbol{\nu}_b = \\
&\begin{bmatrix} \frac{\partial}{\partial q_1}(\dot{\boldsymbol{q}}^T\boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b) \\ \vdots \\ \frac{\partial}{\partial q_n}\,\boldsymbol{\nu}_b^T\boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b) \end{bmatrix} = \begin{bmatrix} (\frac{\partial}{\partial q_1}\boldsymbol{J}_{jq}\dot{\boldsymbol{q}})^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b \\ \vdots \\ (\frac{\partial}{\partial q_n}\boldsymbol{J}_{jq}\dot{\boldsymbol{q}})^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b \end{bmatrix} + \begin{bmatrix} \dot{\boldsymbol{q}}^T\boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j(\frac{\partial}{\partial q_1}\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b) \\ \vdots \\ \dot{\boldsymbol{q}}^T\boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j(\frac{\partial}{\partial q_1}\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b) \end{bmatrix} \\
&= \boldsymbol{J}_{jq,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b + \boldsymbol{A}_{jb,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}}
\end{aligned}$$

Relation (3.50) follows from the definitions of $\boldsymbol{M}_{q,q}$ (3.37) and $\boldsymbol{J}_{jq,q}$ (3.47):

$$\begin{aligned}
&\boldsymbol{M}_{q,q}\dot{\boldsymbol{q}} = \\
&\begin{bmatrix} \frac{\partial}{\partial q_1}(\dot{\boldsymbol{q}}^T\,\boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}}) \\ \vdots \\ \frac{\partial}{\partial q_1}(\dot{\boldsymbol{q}}^T\,\boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}}) \end{bmatrix} = 2\begin{bmatrix} (\frac{\partial}{\partial q_1}\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}})^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \\ \vdots \\ (\frac{\partial}{\partial q_1}\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}})^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \end{bmatrix} \\
&= 2\boldsymbol{J}_{jq,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} && (3.53) \\
& && (3.54)
\end{aligned}$$

Lemma 3.2. *The following relation holds for $\boldsymbol{J}_{jq,q}$:*

$$\boldsymbol{J}_{jq,q} - \dot{\boldsymbol{J}}_{jq} = \boldsymbol{a}_{\nu_{bj}}\,\boldsymbol{J}_{jq} \tag{3.55}$$

Proof. The ith column of the term $\dot{\boldsymbol{J}}^i_{jq}$ in the Lee algebra is:

$$\begin{aligned}
(\dot{\boldsymbol{J}}^i_{jq})\check{} &= \frac{d}{dt}(\boldsymbol{H}_{bj}^{-1}\frac{\partial \boldsymbol{H}_{bj}}{\partial q_i}) \\
&= (\frac{\partial \boldsymbol{H}_{bj}^{-1}}{\partial q_k}\frac{\partial \boldsymbol{H}_{bj}}{\partial q_i} + \boldsymbol{H}_{bj}^{-1}\frac{\partial \boldsymbol{H}_{bj}}{\partial q_k \partial q_i})\dot{q}_k
\end{aligned}$$

The ith column of the term $\boldsymbol{J}_{jq,q}$ (3.47) can be written as $\boldsymbol{J}^i_{jq,q} = \frac{\partial}{\partial q_i}\boldsymbol{J}^i_{jq}\dot{q}_k$. In the Lee algebra it is:

$$\begin{aligned}(\boldsymbol{J}^i_{jq,q})\hat{} &= \frac{\partial}{\partial q_i}(\boldsymbol{J}^k_{jq})\hat{}\dot{q}_k \\ &= (\frac{\partial \boldsymbol{H}^{-1}_{bj}}{\partial q_i}\frac{\partial \boldsymbol{H}_{bj}}{\partial q_k} + \boldsymbol{H}^{-1}_{bj}\frac{\partial \boldsymbol{H}_{bj}}{\partial q_k \partial q_i})\dot{q}_k\end{aligned}$$

The term with the little adjoint is:

$$\begin{aligned}\boldsymbol{ad}_{\nu_j}\hat{\boldsymbol{J}}^i_j &= \hat{\boldsymbol{\nu}}_j\hat{\boldsymbol{J}}^i_j - \hat{\boldsymbol{J}}^i_j\hat{\boldsymbol{\nu}}_j \\ &= (\boldsymbol{H}^{-1}_{bj}\dot{\boldsymbol{H}}_{bj})\boldsymbol{H}^{-1}_{bj}\frac{\partial \boldsymbol{H}_{bj}}{\partial q_i} - \boldsymbol{H}^{-1}_{bj}\frac{\partial \boldsymbol{H}_{bj}}{\partial q_i}(\boldsymbol{H}^{-1}_{bj}\dot{\boldsymbol{H}}_{bj}) \\ &= -\dot{\boldsymbol{H}}^{-1}_{bj}\frac{\partial \boldsymbol{H}_{bj}}{\partial q_i} + \frac{\partial \boldsymbol{H}^{-1}_{bj}}{\partial q_i}\dot{\boldsymbol{H}}_{bj}\end{aligned}$$

Subtracting the terms concludes the proof:

$$\begin{aligned}(\boldsymbol{J}^i_{jq,q})\hat{} - (\dot{\boldsymbol{J}}^i_{jq})\hat{} &= \frac{\partial \boldsymbol{H}^{-1}_{bj}}{\partial q_i}\dot{\boldsymbol{H}}_{bj} - \dot{\boldsymbol{H}}^{-1}_{bj}\frac{\partial \boldsymbol{H}_{bj}}{\partial q_i} \\ &= \boldsymbol{ad}_{\nu_j}\hat{\boldsymbol{J}}^i_j\end{aligned}$$

Lemma 3.3. *The following relation holds for* $\boldsymbol{A}_{jb,q}$*:*

$$\boldsymbol{A}_{jb,q} = \boldsymbol{a}_{\boldsymbol{A}_{jb}\nu_b}\boldsymbol{J}_{jq} \tag{3.56}$$

Proof. The ith element of $\boldsymbol{A}_{jb,q}$ is $\frac{\partial}{\partial q_i}\boldsymbol{A}_{jb}\boldsymbol{\nu}_b$. Therefore:

$$\begin{aligned}\hat{\boldsymbol{A}}^i_{jb,q} &= \frac{\partial}{\partial q_i}(\boldsymbol{H}_{jb}\boldsymbol{\nu}_b\boldsymbol{H}^{-1}_{jb}) \\ &= (\frac{\partial}{\partial q_i}\boldsymbol{H}_{jb})\boldsymbol{\nu}_b\boldsymbol{H}^{-1}_{jb} + \boldsymbol{H}_{jb}\boldsymbol{\nu}_b(\frac{\partial}{\partial q_i}\boldsymbol{H}^{-1}_{jb}) \\ &= (\frac{\partial}{\partial q_i}\boldsymbol{H}^{-1}_{bj})\boldsymbol{\nu}_b\boldsymbol{H}^{-1}_{jb} + \boldsymbol{H}_{jb}\boldsymbol{\nu}_b\boldsymbol{H}^{-1}_{jb}\boldsymbol{H}^{-1}_{bj}(\frac{\partial}{\partial q_i}\boldsymbol{H}^{-1}_{jb}) \\ &= -\boldsymbol{H}^{-1}_{bj}(\frac{\partial}{\partial q_i}\boldsymbol{H}_{bj})\boldsymbol{H}_{jb}\boldsymbol{\nu}_b\boldsymbol{H}^{-1}_{jb} + \boldsymbol{H}_{jb}\boldsymbol{\nu}_b\boldsymbol{H}^{-1}_{jb}\boldsymbol{H}^{-1}_{bj}(\frac{\partial}{\partial q_i}\boldsymbol{H}_{bj}) \\ &= -\boldsymbol{J}^i_{jq}\boldsymbol{H}_{jb}\boldsymbol{\nu}_b\boldsymbol{H}^{-1}_{jb} + \boldsymbol{H}_{jb}\boldsymbol{\nu}_b\boldsymbol{H}^{-1}_{jb}\boldsymbol{J}^i_{jq} \\ &= [\boldsymbol{H}_{jb}\boldsymbol{\nu}_b\boldsymbol{H}^{-1}_{jb}, \boldsymbol{J}^i_{jq}]\end{aligned}$$

Using these relations, the identities from Lemma 3.1 and (3.48), (3.49), (3.50) the Coriolis terms for second row of the Hamel equations (3.38) are given by:

$$\begin{aligned}
\boldsymbol{C}_{qb}\,\boldsymbol{\nu}_b &= (\,\dot{\boldsymbol{M}}_{bq}^T - \boldsymbol{M}_{bq,q}^T - \frac{1}{2}\boldsymbol{M}_{b,q})\,\boldsymbol{\nu}_b \\
&= \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb}\,\boldsymbol{\nu}_b + \dot{\boldsymbol{J}}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b - \boldsymbol{J}_{jq,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b - \boldsymbol{A}_{jb,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} - \boldsymbol{A}_{jb,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b \\
&= \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb}\,\boldsymbol{\nu}_b - \boldsymbol{J}_{jq}^T\,\boldsymbol{a}_{bj}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b - \boldsymbol{J}_{jq}^T\,\boldsymbol{a}_{\boldsymbol{A}_{jb}\boldsymbol{\nu}_b}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b - \boldsymbol{A}_{jb,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \\
&= \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb}\,\boldsymbol{\nu}_b - \boldsymbol{J}_{jq}^T\,\boldsymbol{a}_{j}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b - \boldsymbol{J}_{jq}^T\,\boldsymbol{a}_{\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \qquad (3.57)\\
\boldsymbol{C}_{q}\dot{\boldsymbol{q}} &= (\,\dot{\boldsymbol{M}}_{qq} - \frac{1}{2}\boldsymbol{M}_{q,q})\dot{\boldsymbol{q}} \\
&= \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{J}}_{jq}\dot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} - \boldsymbol{J}_{jq,q}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \\
&= \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{J}}_{jq}\dot{\boldsymbol{q}} - \boldsymbol{J}_{jq}^T\,\boldsymbol{a}_{bj}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \qquad (3.58)
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{C}_{qb}\,\boldsymbol{\nu}_b + \boldsymbol{C}_{q}\dot{\boldsymbol{q}} \\
&= \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb}\,\boldsymbol{\nu}_b + \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{J}}_{jq}\dot{\boldsymbol{q}} - (\,\boldsymbol{J}_{jq}^T\,\boldsymbol{a}_{j}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b + \boldsymbol{J}_{jq}^T\,\boldsymbol{a}_{\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} + \boldsymbol{J}_{jq}^T\,\boldsymbol{a}_{bj}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}}) \\
&= \boldsymbol{J}_{jq}^T\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb}\,\boldsymbol{\nu}_b + \boldsymbol{J}_{jq}^T\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{J}}_{jq}\dot{\boldsymbol{q}} - \boldsymbol{J}_{jq}^T\,\boldsymbol{a}_{j}^T\,\Lambda_j\,\boldsymbol{\nu}_j
\end{aligned} \qquad (3.59)$$

For the last step, the last 2 terms in the bracket are added to $\boldsymbol{J}_{jq}^T\,\boldsymbol{a}_j^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}}$. The result can then be added to the first term in the bracket.

It is important to note that the sub-blocks $\boldsymbol{C}_{bq}$ (3.57) and $\boldsymbol{C}_q$ (3.58) of Coriolis-centrifugal terms in the Hamel's equations are different from the Newton Euler derivation (3.41c) and (3.41d). The matrix $\boldsymbol{C}_{qb}$ in (3.57) has an additional term $-\boldsymbol{J}_{jq}^T\,\boldsymbol{a}_{\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}}$ compared to the Newton Euler term $\boldsymbol{C}_{qb}^E$ (3.41c), which is another reason for non-passivity of the Coriolis matrix. For the passive formulation of the Coriolis matrix, we will add $\boldsymbol{J}_{jq}^T\,\boldsymbol{a}_{\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}}$ to $\boldsymbol{C}_{qb}$ and subtract the same term from $\boldsymbol{C}_q$.

3.2. Conserved Quantities and the Center of Mass Frame

3.2.1. Constants of Motion

Proposition 3.2.1 (Constants of Motion). *Given a Lagrangian of the form* (2.49)

$$L = \frac{1}{2}\begin{bmatrix}\boldsymbol{\nu}_b\\ \dot{\boldsymbol{q}}\end{bmatrix}^T \begin{bmatrix}\boldsymbol{M}_{bb} & \boldsymbol{M}_{bq}\\ \boldsymbol{M}_{bq}^T & \boldsymbol{M}_{qq}\end{bmatrix} \begin{bmatrix}\boldsymbol{\nu}_b\\ \dot{\boldsymbol{q}}\end{bmatrix}$$

and a quantity $\boldsymbol{h}_c = \boldsymbol{A}_{cb}^{-T}\boldsymbol{h}_b$ with $\boldsymbol{h}_b = \frac{\partial L}{\partial \boldsymbol{v}_b}$. The quantity $\boldsymbol{h}_c$ is a constant of motion, if and only if the relation $\boldsymbol{a}_{cb}^T\boldsymbol{h}_b = \boldsymbol{a}_{ib}^T\boldsymbol{h}_b$ is fulfilled. This implies that the value of $\boldsymbol{h}_c$ can only change, due to external forces:

$$\dot{\boldsymbol{h}}_c = \boldsymbol{A}_{cb}^{-T}\boldsymbol{F_b} \qquad \Longleftrightarrow \qquad \boldsymbol{a}_{cb}^T\boldsymbol{h}_b = \boldsymbol{a}_{ib}^T\boldsymbol{h}_b \qquad (3.60)$$

Proof. First we show that $\boldsymbol{h}_c$ is a constant of motion, if the relation $\boldsymbol{a}_{cb}^T\boldsymbol{h}_b = \boldsymbol{a}_{ib}^T\boldsymbol{h}_b$ is fulfilled. Using (3.42b) the relation implies $\dot{\boldsymbol{A}}_{cb}^{-T} = -\boldsymbol{A}_{cb}^{-T}\,\boldsymbol{a}_{ib}^T$.

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{h}_c &= \frac{d}{dt}(\boldsymbol{A}_{cb}^{-T}\boldsymbol{h}_b) \\
&= \boldsymbol{A}_{cb}^{-T}\dot{\boldsymbol{h}}_b + \dot{\boldsymbol{A}}_{cb}^{-T}\boldsymbol{h}_b \\
&= \boldsymbol{A}_{cb}^{-T}(\dot{\boldsymbol{h}}_b - \boldsymbol{a}_{ib}^T\boldsymbol{h}_b) \\
&= \boldsymbol{A}_{cb}^{-T}(\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\nu}_b} - \boldsymbol{a}_b^T\frac{\partial L}{\partial \boldsymbol{\nu}_b}) \\
&= \boldsymbol{A}_{cb}^{-T}\boldsymbol{F_b}
\end{aligned}$$

Where the Hamel equations of motion (3.30a) were used. For the second part of the proof we show that if $\boldsymbol{h}_c$ is a constant of motion, then relation $\boldsymbol{a}_{cb}^T\boldsymbol{h}_b = \boldsymbol{a}_{ib}^T\boldsymbol{h}_b$ is fulfilled.

$$\begin{aligned}
\frac{d}{dt}(\boldsymbol{h}_c) &= \boldsymbol{A}_{cb}^{-T}\boldsymbol{F_b} \\
&= \boldsymbol{A}_{cb}^{-T}(\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\nu}_b} - \boldsymbol{a}_b^T\frac{\partial L}{\partial \boldsymbol{\nu}_b}) \\
&= \boldsymbol{A}_{cb}^{-T}(\dot{\boldsymbol{h}}_b - \boldsymbol{a}_{ib}^T\boldsymbol{h}_b)
\end{aligned}$$

on the other hand,

$$\begin{aligned}
\frac{d}{dt}(\boldsymbol{h}_c) &= \frac{d}{dt}(\boldsymbol{A}_{cb}^{-T}\boldsymbol{h}_b) \\
&= \boldsymbol{A}_{cb}^{-T}(\dot{\boldsymbol{h}}_b - \boldsymbol{a}_{cb}^T\boldsymbol{h}_b)
\end{aligned}$$

Comparing the 2 expressions for $\dot{\boldsymbol{h}}_c$ implies the relation $\boldsymbol{a}_{cb}^T\boldsymbol{h}_b = \boldsymbol{a}_{ib}^T\boldsymbol{h}_b$ is fulfilled. □

The proposition states that the total momentum expressed in an arbitrary frame c is a constant of motion if and only if the condition is fulfilled. It gives the same constants of motion as Noether's theorem [Noe18]. However, it is more approachable than Noether's theorem, since it requires no knowledge of differential geometry to check the condition and the link between the equations of motion and the conserved quantity is apparent. Also, the proposition allows to easily check the frames, where the total momentum is conserved.

The important point to notice is that the condition $\boldsymbol{a}_{cb}^T\boldsymbol{h}_b = \boldsymbol{a}_{ib}^T\boldsymbol{h}_b$ does not imply that $\boldsymbol{a}_{cb}$ and $\boldsymbol{a}_{ib}$ are the same. To the contrary, the interesting cases are those, where the condition is fulfilled, but $\boldsymbol{a}_{cb}$ and $\boldsymbol{a}_{ib}$ are not the same.

The proposition has two immediate consequences: First, the total momentum expressed in the inertial frame, $\boldsymbol{h}_i = \boldsymbol{A}_{ib}^{-T}\boldsymbol{h}_b$, is a constant of motion, since the condition is trivially fulfilled by setting $c = i$.

$$\dot{\boldsymbol{h}}_i = \boldsymbol{A}_{ib}^{-T}\boldsymbol{F_b} \tag{3.61}$$

Second, the total momentum expressed in frame c is conserved if and only if:

$$\dot{\boldsymbol{h}}_c = \boldsymbol{A}_{cb}^{-T}\boldsymbol{F_b} \qquad \iff \qquad \boldsymbol{a}_{ci}^T\boldsymbol{h}_i = 0 \tag{3.62}$$

Proof. The left hand side of the relation of Proposition 3.2.1 can be rewritten using (2.37c): $\boldsymbol{a}_{cb}^T \boldsymbol{h}_b = \boldsymbol{A}_{ib}^T \boldsymbol{a}_{ci}^T \boldsymbol{A}_{ib}^{-T} \boldsymbol{h}_b + \boldsymbol{a}_{ib}^T \boldsymbol{h}_b = \boldsymbol{A}_{ib}^T \boldsymbol{a}_{ci}^T \boldsymbol{h}_i + \boldsymbol{a}_{ib}^T \boldsymbol{h}_b$ Equating this to right hand side gives the results. □

The condition $\boldsymbol{a}_{ci}^T \boldsymbol{h}_i = 0$ can only be fulfilled, if frame c is aligned with the inertial frame $\boldsymbol{\omega}_{ic} = 0$. To this end, $\boldsymbol{\omega}_{ic} = 0$ is a necessary condition for the total momentum being conserved in frame c.

3.2.2. The Center of Mass

The center of mass $\boldsymbol{o}_{ic}$for n bodies is given by the weighted average of the positions of the masses. Denoting $m_{tot} = \sum_{j=0}^{n} m_j$, the center of mass is:

$$\boldsymbol{o}_{ic} = \frac{1}{m_{tot}} \sum_{j=0}^{n} m_j \boldsymbol{o}_{ij} \tag{3.63}$$

Using the relation $\boldsymbol{o}_{ij} = \boldsymbol{o}_{ib} + \boldsymbol{R}_{ib} \boldsymbol{o}_{bj}$ the center of mass can also be written as

$$\boldsymbol{o}_{ic} = \boldsymbol{o}_{ib} + \boldsymbol{R}_{ib} \boldsymbol{o}_{bc} \qquad \text{with} \qquad \boldsymbol{o}_{bc} = \frac{1}{m_{tot}} \sum_{j=0}^{n} m_j \boldsymbol{o}_{bj} \tag{3.64}$$

3.2.3. Total Momentum and Centroidal Momentum

The total momentum and the locked velocity, both expressed in frame b are defined as [STNN17]:

$$\boldsymbol{h}_b = \boldsymbol{M}_{bb} \boldsymbol{\nu}_b + \boldsymbol{M}_{bq} \dot{\boldsymbol{q}} \tag{3.65}$$

$$\boldsymbol{\nu}_b^{loc} = \boldsymbol{\nu}_b + \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq} \dot{\boldsymbol{q}} \tag{3.66}$$

The total momentum and the locked velocity, both expressed in the inertial frame are [STNN17]:

$$\boldsymbol{h}_i = \boldsymbol{A}_{ib}^{-T} \boldsymbol{h}_b \tag{3.67}$$

$$\boldsymbol{\nu}_i^{loc} = \boldsymbol{A}_{ib} \boldsymbol{\nu}_b^{loc} \tag{3.68}$$

The total momentum expressed in the inertial frame contains the total linear and angular momentum $\boldsymbol{h}_i = [\boldsymbol{p}_i; \boldsymbol{l}_i]$, where the linear momentum is $\boldsymbol{p}_i = m_{tot} \dot{\boldsymbol{o}}_{ic}$ The total momentum $\boldsymbol{h}_i$ is a constant of motion (3.61). The total momentum can also be expressed in a frame c [Gio20]:

$$\boldsymbol{h}_c = \boldsymbol{A}_{cb}^{-T} \boldsymbol{h}_b \tag{3.69}$$

In the robotics literature the center of mass frame c is usually aligned with the inertial frame with its origin at the center of mass $\boldsymbol{o}_{ic}$. If the center of mass frame is aligned with the inertial frame, its pose $\boldsymbol{H}_{ic}$ has no rotation relative to the inertial frame. The transformations $\boldsymbol{H}_{ic}$ and $\boldsymbol{H}_{cb}$ are:

$$\boldsymbol{H}_{ic} = \begin{bmatrix} \mathbb{I} & \boldsymbol{o}_{ic} \\ \boldsymbol{0} & 1 \end{bmatrix} \qquad \boldsymbol{H}_{cb} = \begin{bmatrix} \boldsymbol{R}_{ib} & \boldsymbol{o}_{ib} - \boldsymbol{o}_{ic} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

The total momentum expressed in frame c defined above is called centroidal momentum [OGL13]. The centroidal momentum is also a constant of motion, since the condition $\boldsymbol{a}_{ci}^T \boldsymbol{h}_i = 0$ (3.62) is fulfilled:

$$\boldsymbol{a}_{ci}^T \boldsymbol{h}_i = \begin{bmatrix} \hat{0} & 0 \\ -\hat{\dot{\boldsymbol{o}}}_{ic} & \hat{0} \end{bmatrix} \begin{bmatrix} m_{tot} \dot{\boldsymbol{o}}_{ic} \\ \boldsymbol{l}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

3.2.4. The Orientation of the CoM Frame

As discussed in the previous section, the CoM frame is usually aligned with the inertial frame. To this end, the question arises, if it possible to find a *natural* orientation the CoM frame. An obvious candidate for aligning the CoM frame with is the locked velocity $\boldsymbol{\nu}_b^{loc}$. The pose $\boldsymbol{H}_{ic} = \boldsymbol{H}_{ib}\boldsymbol{H}_{bc}$ must fulfill the alignment condition that the spatial velocity of $\boldsymbol{H}_{ic}$ equals the spatial locked velocity $\boldsymbol{\nu}_i^{loc}$:

$$(\dot{\boldsymbol{H}}_{ic}\boldsymbol{H}_{ic}^{-1})^{\vee} = \boldsymbol{\nu}_i^{loc}$$

While in any case the velocity $\boldsymbol{\nu}_i^{loc}$ can be time integrated to give a pose, it is not guaranteed that the resulting pose $\boldsymbol{H}_{ic}$ is independent of the path taken and only depends on the configuration variables. Saccon et. al. [STNN17] provide an algebraic condition to check, whether the locked velocity defines a frame orientation that only depends on the configuration. This condition is translated to our notation: We want to find the unknown pose $\boldsymbol{H}_{bc}$ and define $\boldsymbol{T}(\boldsymbol{q}) = \boldsymbol{M}_b^{-1}\boldsymbol{M}_{bq}$. The left hand side of the alignment conditions reads as:

$$\begin{aligned} \dot{\boldsymbol{H}}_{ic}\boldsymbol{H}_{ic}^{-1} &= \boldsymbol{H}_{ib}\boldsymbol{H}_{ib}^{-1}\dot{\boldsymbol{H}}_{ib}\boldsymbol{H}_{ib}^{-1} + \boldsymbol{H}_{ib}(\dot{\boldsymbol{H}}_{bc}\boldsymbol{H}_{bc}^{-1})\boldsymbol{H}_{ib}^{-1} \\ (\dot{\boldsymbol{H}}_{ic}\boldsymbol{H}_{ic}^{-1})^{\vee} &= \boldsymbol{A}_{ib}\boldsymbol{\nu}_b + \boldsymbol{A}_{ib}(\dot{\boldsymbol{H}}_{bc}\boldsymbol{H}_{bc}^{-1})^{\vee} \end{aligned}$$

Comparing with the right hand side, $\boldsymbol{\nu}_i^{loc} = \boldsymbol{A}_{ib}\boldsymbol{\nu}_b + \boldsymbol{A}_{ib}\boldsymbol{T}_{bq}(\boldsymbol{q})\dot{\boldsymbol{q}}$, results in an alignment condition for $\boldsymbol{H}_{bc}$[1]:

$$(\dot{\boldsymbol{H}}_{bc}\boldsymbol{H}_{bc}^{-1})^{\vee} = \boldsymbol{T}(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{3.70}$$

[1]The velocity $(\dot{\boldsymbol{H}}_{bc}\boldsymbol{H}_{bc}^{-1})^{\vee}$ can be written as $(\dot{\boldsymbol{H}}_{bc}\boldsymbol{H}_{bc}^{-1})^{\vee} = \begin{bmatrix} \dot{\boldsymbol{o}}_{bc} - \boldsymbol{R}_{bc}\hat{\boldsymbol{\omega}}_{bc}\boldsymbol{R}_{bc}^T\boldsymbol{o}_{bc} \\ \boldsymbol{R}_{bc}\boldsymbol{\omega}_{bc} \end{bmatrix}$. Due to the frame alignment condition, $(\dot{\boldsymbol{H}}_{bc}\boldsymbol{H}_{bc}^{-1})^{\vee} = \boldsymbol{M}_b^{-1}\boldsymbol{M}_{bq}\dot{\boldsymbol{q}}$, the term $\boldsymbol{M}_b^{-1}\boldsymbol{M}_{bq}\dot{\boldsymbol{q}}$ also must be of the form $\boldsymbol{M}_b^{-1}\boldsymbol{M}_{bq}\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{\boldsymbol{o}}_{bc} - \boldsymbol{R}_{bc}\hat{\boldsymbol{\omega}}_{bc}\boldsymbol{R}_{bc}^T\boldsymbol{o}_{bc} \\ \boldsymbol{R}_{bc}\boldsymbol{\omega}_{bc} \end{bmatrix}$

which results in the partial derivative expressions:

$$
\begin{aligned}
(\frac{\partial \boldsymbol{H}_{bc}}{\partial \boldsymbol{q}_i}\boldsymbol{H}_{bc}^{-1})^{\vee} &= \boldsymbol{T}^i(\boldsymbol{q}) \\
\frac{\partial \boldsymbol{H}_{bc}}{\partial \boldsymbol{q}_i} &= \hat{\boldsymbol{T}}^i(\boldsymbol{q})\boldsymbol{H}_{bc}
\end{aligned}
\tag{3.71}
$$

Since $\frac{\partial \boldsymbol{H}_{bc}}{\partial \boldsymbol{q}_i}\boldsymbol{H}_{bc}^{-1} \in se(n)$ and $\frac{\partial \boldsymbol{H}_{bc}}{\partial \boldsymbol{q}_i}\boldsymbol{H}_{bc}^{-1} = \hat{\boldsymbol{T}}_i(\boldsymbol{q})$, the hat operator means to put the elements of $\boldsymbol{T}_i(\boldsymbol{q})$ into a $se(n)$ matrix. Integrability requires that the second derivatives of $\boldsymbol{H}_{bc}$ commute:

$$
\frac{\partial}{\partial \boldsymbol{q}_j}\frac{\partial \boldsymbol{H}_{bc}}{\partial \boldsymbol{q}_i} = \frac{\partial}{\partial \boldsymbol{q}_i}\frac{\partial \boldsymbol{H}_{bc}}{\partial \boldsymbol{q}_j}
$$

Since the left hand side is:

$$
\frac{\partial}{\partial \boldsymbol{q}_j}\frac{\partial \boldsymbol{H}_{bc}}{\partial \boldsymbol{q}_i} = \frac{\partial}{\partial \boldsymbol{q}_j}(\hat{\boldsymbol{T}}^i\boldsymbol{H}_{bc}) = (\frac{\partial \hat{\boldsymbol{T}}^i}{\partial \boldsymbol{q}_j} + \hat{\boldsymbol{T}}^i\hat{\boldsymbol{T}}^j)\boldsymbol{H}_{bc}
$$

The right hand side is obtained by swapping i and j. The integrability condition for $\boldsymbol{H}_{bc}$ [STNN17, eq. 20] is:

$$
\frac{\partial \hat{\boldsymbol{T}}^i}{\partial \boldsymbol{q}_j} - \frac{\partial \hat{\boldsymbol{T}}^j}{\partial \boldsymbol{q}_i} + \hat{\boldsymbol{T}}^i\hat{\boldsymbol{T}}^j - \hat{\boldsymbol{T}}^j\hat{\boldsymbol{T}}^i = \hat{\boldsymbol{0}}
\tag{3.72}
$$

It is shown in [STNN17] that the condition for the integrability of the CoM frame is necessary and sufficient.

Integrability of the CoM Frame in SE(2)

Lemma 3.4. *The integrability condition* (3.72) *for* $SE(2)$ *is given by:*

$$
\begin{aligned}
&\frac{\partial \boldsymbol{T}_{\omega}^i}{\partial \boldsymbol{q}_j} - \frac{\partial \boldsymbol{T}_{\omega}^j}{\partial \boldsymbol{q}_i} = 0 \\
&\frac{\partial \boldsymbol{T}_{v}^i}{\partial \boldsymbol{q}_j} - \frac{\partial \boldsymbol{T}_{v}^j}{\partial \boldsymbol{q}_i} + \boldsymbol{T}_{\omega}^i\boldsymbol{S}\boldsymbol{T}_{v}^j - \boldsymbol{T}_{\omega}^j\boldsymbol{S}\boldsymbol{T}_{v}^i = 0
\end{aligned}
\tag{3.73}
$$

with $\boldsymbol{T}^i = [\boldsymbol{T}_v^i; \boldsymbol{T}_{\omega}^i]$.

Proof. The matrix $\hat{\boldsymbol{T}}^i$ for $se(2)$ is: $\hat{\boldsymbol{T}}^i = \begin{bmatrix} \boldsymbol{T}_{\omega}^i\boldsymbol{S} & \boldsymbol{T}_v^i \\ \boldsymbol{0} & 0 \end{bmatrix}$. The basis for $se(2)$ is $\boldsymbol{S} = [0 - 1; 10]$.

Plugging $\hat{\boldsymbol{T}}^i$ into (3.72) gives:

$$\begin{aligned}\hat{\mathbf{0}} &\stackrel{!}{=} \frac{\partial \hat{\boldsymbol{T}}^i}{\partial \boldsymbol{q}_j} - \frac{\partial \hat{\boldsymbol{T}}^j}{\partial \boldsymbol{q}_i} + \hat{\boldsymbol{T}}^i \hat{\boldsymbol{T}}^j - \hat{\boldsymbol{T}}^j \hat{\boldsymbol{T}}^i \\ &= \begin{bmatrix} \frac{\partial \boldsymbol{T}^i_\omega}{\partial \boldsymbol{q}_j}\boldsymbol{S} - \frac{\partial \boldsymbol{T}^j_\omega}{\partial \boldsymbol{q}_i}\boldsymbol{S} - (\boldsymbol{T}^i_\omega \boldsymbol{T}^j_\omega - \boldsymbol{T}^j_\omega \boldsymbol{T}^i_\omega)\mathbb{I}_{2\times 2} & \frac{\partial \boldsymbol{T}^i_v}{\partial \boldsymbol{q}_j} - \frac{\partial \boldsymbol{T}^j_v}{\partial \boldsymbol{q}_i} + \boldsymbol{T}^i_\omega \boldsymbol{S} \boldsymbol{T}^j_v - \boldsymbol{T}^j_\omega \boldsymbol{S} \boldsymbol{T}^i_v \\ \mathbf{0} & 0 \end{bmatrix}\end{aligned}$$

Since $\boldsymbol{T}^i_\omega$ is a scalar function, the term $\boldsymbol{T}^i_\omega \boldsymbol{T}^j_\omega - \boldsymbol{T}^j_\omega \boldsymbol{T}^i_\omega$ vanishes. □

The condition is always fulfilled, if there is only one joint variable.

The angular velocity $\omega_{bc}(\boldsymbol{q}(t))$ in terms of $\boldsymbol{T}_\omega$ follows directly from the alignment condition (3.70): $\omega_{bc} = \boldsymbol{T}_\omega \dot{\boldsymbol{q}}$. The angle $\theta_{bc}(t)$ can be obtained from $\omega_{bc}(\boldsymbol{q}(t))$ by time integration, regardless whether the integrability condition (3.73) holds:

$$\theta_{bc}(t) - \theta_{bc}(t_0) = \int_{t_0}^{t} \boldsymbol{T}^i_\omega(\boldsymbol{q}(t'))\, \dot{\boldsymbol{q}}_i(t')\, dt' \tag{3.74}$$

Since $d\boldsymbol{q}_i(t) = \dot{\boldsymbol{q}}_i(t)\, dt$, the integral can also be written as:

$$\theta_{bc}(t) - \theta_{bc}(t_0) = \int_{t_0}^{t} \boldsymbol{T}^i_\omega(\boldsymbol{q}(t'))\, d\boldsymbol{q}_i(t') \tag{3.75}$$

If the integrability condition (3.73) does not hold, the value of the angle $\theta_{bc}(t)$ depends on the whole trajectory of $\boldsymbol{q}$ and is therefore not a function of the endpoints $\boldsymbol{q}(t_0)$ and $\boldsymbol{q}(t)$. If the integrability condition (3.73) does holds, the value of $\theta_{bc}(t)$ is independent of the path taken and only a function of the endpoints i.e $\theta_{bc}(t) = \theta_{bc}(\boldsymbol{q}(t))$. Therefore, one can choose any path between $\boldsymbol{q}(t_0)$ and $\boldsymbol{q}(t)$ to arrive at the same result for $\theta_{bc}(\boldsymbol{q}(t))$. In particular, a linear path can be taken to solve the integral. In this case the trajectories $\boldsymbol{q}(t)$ have a constant velocity $\boldsymbol{v}$, i.e. $q_i(t) = \boldsymbol{v}_i t$, $d\boldsymbol{q}_i(t) = \boldsymbol{v}_i dt$) and the integral becomes:

$$\theta_{bc}(\boldsymbol{v}_i, t) - \theta_{bc}(t_0) = \int_{t_0}^{\boldsymbol{v}_i, t_0} \boldsymbol{T}^i_\omega(\boldsymbol{v}_i t)\, \boldsymbol{v}_i dt \tag{3.76}$$

The solution of the integral is a function of $\boldsymbol{v}_i$ and t. The result can be turned into the function $\theta_{bc}(\boldsymbol{q}(t))$, by substituting $\boldsymbol{v}_i$ with $\boldsymbol{q}_i/t$ [2].

[2] The integral of a total derivative is the integral over a path. It is not the integrals over the variables separately, since different paths would be taken for the different integrals. For example, the function $f(x, y) = x\, y$ has the total derivative $df(x, y) = ydx + xdy$. The integral of df must again give $f(x, y)$. However, taking the integral over the variables, $\int ydx + \int xdy = 2x\, y$, does not yield the correct result. In contrast, choosing a linear path, $x = v_x\, t$, $y = v_y\, t$, $dx = v_x dt$ and $dy = v_y dt$ the integral becomes: $\int df = v_x\, v_y \int 2\, t' dt' = v_x\, v_y\, t^2 = x\, y$.

3.3. Decoupling and Invariance

3.3.1. Transformation of Equations of Motion

The equations of motion in variables $\boldsymbol{v}_x$ given the mass matrix $\boldsymbol{M}_x$ and the Coriolis matrix $\boldsymbol{C}_x$ are:

$$\boldsymbol{M}_x\dot{\boldsymbol{v}}_x + \boldsymbol{C}_x\boldsymbol{v}_x = \boldsymbol{F}_x$$

The index x highlights that the equations of motion do no have to be expressed in the base frame. The transformation relations hold for any system matrices $\boldsymbol{M}$ and $\boldsymbol{C}$. Transforming the variables to $\boldsymbol{v}_y = \boldsymbol{T}\boldsymbol{v}_x$ using a transformation $\boldsymbol{T}$, yields the equations of motion in variables $\boldsymbol{v}_y$:

$$\boldsymbol{M}_y\dot{\boldsymbol{v}}_y + \boldsymbol{C}_y\boldsymbol{v}_y = \boldsymbol{T}^{-T}\boldsymbol{F}_x$$

where the system matrices transform as:

$$\boldsymbol{M}_y = \boldsymbol{T}^{-T}\boldsymbol{M}_x\boldsymbol{T}^{-1} \qquad \boldsymbol{C}_y = \boldsymbol{T}^{-T}\boldsymbol{C}_x\boldsymbol{T}^{-1} - \boldsymbol{M}_y\dot{\boldsymbol{T}}\boldsymbol{T}^{-1} \tag{3.77}$$

Proof: Multiply the original equation with $\boldsymbol{T}^{-T}$ from the left and replace $\boldsymbol{v}_x$ with $\boldsymbol{T}^{-1}\boldsymbol{v}_y$ and $\dot{\boldsymbol{v}}_x$ with $\boldsymbol{T}^{-1}\dot{\boldsymbol{v}}_y + \dot{\boldsymbol{T}}^{-1}\boldsymbol{v}_y$.

3.3.2. Decoupling Transform

The goal of this section is to find all transformed representations of the equations of motion (3.38) where the transformed base is either decoupled from the transformed joints or there is an invariance structure 1.3.1. To this end, we take an unknown transformation $\boldsymbol{T}$, transform the equations of motion with $\boldsymbol{T}$ and solve for $\boldsymbol{T}$, where the transformed equations of motion decouple or have an invariance structure.

The transformations $\boldsymbol{T}$ must have certain properties that restrict the general form of the transformation. First, the transformation $\boldsymbol{T}$ should define an allocation mapping, such that the base wrench is fully decoupled i.e. a virtual force $\boldsymbol{F}_y$ should not influence the base wrench $\boldsymbol{F_b}$:

$$\begin{bmatrix}\boldsymbol{F_b}\\ \boldsymbol{\tau}\end{bmatrix} = \begin{bmatrix}\boldsymbol{T}_x^T & \boldsymbol{T}_{yx}^T\\ \boldsymbol{T}_{xy}^T & \boldsymbol{T}_y^T\end{bmatrix}\begin{bmatrix}\boldsymbol{F}_x\\ \boldsymbol{F}_y\end{bmatrix} \tag{3.78}$$

Therefore, the block matrix element $\boldsymbol{T}_{yx}$ must be zero. Second, any transformation $\boldsymbol{T}$ must diagonalize the mass matrix to ensure that the equations of motion are not coupled by the transformed mass matrix:

$$\boldsymbol{T}^{-T}\boldsymbol{M}\boldsymbol{T}^{-1} \stackrel{!}{=} \begin{bmatrix}\boldsymbol{M}_x & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{M}_y\end{bmatrix} \tag{3.79}$$

From this condition it follows that the off diagonal element must be of the form $\boldsymbol{T}_{xy} = \boldsymbol{T}_x\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq}$ which can be shown by direct calculation. To this end the most general form of the transformation $\boldsymbol{T}$ fulfilling the 2 requirements are:

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_x & \boldsymbol{T}_x \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq} \\ \boldsymbol{0} & \boldsymbol{T}_y \end{bmatrix} \qquad \boldsymbol{T}^T = \begin{bmatrix} \boldsymbol{T}_x^T & \boldsymbol{0} \\ \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} \boldsymbol{T}_x^T & \boldsymbol{T}_y^T \end{bmatrix}$$

$$\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{T}_x^{-1} & -\boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq} \boldsymbol{T}_y^{-1} \\ \boldsymbol{0} & \boldsymbol{T}_y^{-1} \end{bmatrix} \qquad \boldsymbol{T}^{-T} = \begin{bmatrix} \boldsymbol{T}_x^{-T} & \boldsymbol{0} \\ -\boldsymbol{T}_y^{-T} \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} & \boldsymbol{T}_y^{-T} \end{bmatrix} \tag{3.80}$$

Now we have all ingredients in place. Taking the dynamic matrices $\boldsymbol{M}$ (2.50) and $\boldsymbol{C}$ (3.39) and transforming them with the transformation (3.80), by applying the transformation rules for dynamic matrices (3.77), yields the following result:

Proposition 3.3.1 (Decoupling). *Given a free floating robotic system defined by the Lagrangian in $SE(3) \times \mathbb{R}^n$ base frame coordinates* (2.49)*. Transforming the equations of motion with a transformation of the form* (3.80)*, decouples the equations of motion:*

$$\begin{bmatrix} \boldsymbol{M}_x & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_y \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_x \\ \boldsymbol{v}_y \end{bmatrix} \begin{bmatrix} \boldsymbol{C}_x & \boldsymbol{0} \\ \boldsymbol{C}_{yx} & \boldsymbol{C}_y \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_x \\ \boldsymbol{v}_y \end{bmatrix} = \boldsymbol{T}^{-T} \begin{bmatrix} \boldsymbol{F}_b \\ \boldsymbol{\tau} \end{bmatrix} \tag{3.81}$$

with

$$\boldsymbol{M}_x = \boldsymbol{T}_x^{-T} \boldsymbol{M}_{bb} \boldsymbol{T}_x^{-1}$$
$$\boldsymbol{M}_y = \boldsymbol{T}_y^{-T} (\boldsymbol{M}_{qq} - \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq}) \boldsymbol{T}_y^{-1}$$
$$\boldsymbol{C}_x = \boldsymbol{T}_x^{-T} (\dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T \boldsymbol{M}_{bb} - \boldsymbol{M}_{bb} \boldsymbol{T}_x^{-1} \dot{\boldsymbol{T}}_x) \boldsymbol{T}_x^{-1}$$
$$\boldsymbol{C}_{yx} = \boldsymbol{T}_y^{-T} \boldsymbol{C}_{qb} - \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} \boldsymbol{C}_b) \boldsymbol{T}_x^{-1}$$
$$\boldsymbol{C}_y = \boldsymbol{T}_y^{-T} (\boldsymbol{C}_q - \boldsymbol{C}_{qb} \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq} - \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} \boldsymbol{C}_{bq} + \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} \boldsymbol{C}_b \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq} - \boldsymbol{M}_y \dot{\boldsymbol{T}}_y) \boldsymbol{T}_y^{-1}$$

It $\boldsymbol{v}_x$ is a momentum map, also the term C_x vanishes and we have an invariance structure ($\boldsymbol{v}_x$ is a momentum map, if $\boldsymbol{T}_x = \boldsymbol{A}_{cb}^{-T} \boldsymbol{M}_{bb}$ and $\dot{\boldsymbol{A}}_{cb} = \boldsymbol{A}_{cb} \boldsymbol{a}_{ib}$ (3.60)*).*

Proof. : The first part is proven by taking the block matrices of the Coriolis matrix (3.38) and plugging them in transformation matrices from the Appendix A.1 using matrix identities (A.6g). The second part is proven by plugging $\boldsymbol{T}_x$ into $\boldsymbol{C}_x$, noting that $\dot{\boldsymbol{T}}_x = \boldsymbol{A}_{cb}^{-T} (\dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T \boldsymbol{M}_{bb})$ using (3.42b).

$$\begin{aligned} \boldsymbol{C}_x &= \boldsymbol{T}_x^{-T} (\dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T \boldsymbol{M}_{bb} - \boldsymbol{M}_{bb} \boldsymbol{T}_x^{-1} \dot{\boldsymbol{T}}_x) \boldsymbol{T}_x^{-1} \\ &= \boldsymbol{T}_x^{-T} (\dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T \boldsymbol{M}_{bb} - \boldsymbol{M}_{bb} (\boldsymbol{M}_{bb}^{-1} \boldsymbol{A}_{cb}^T)(\boldsymbol{A}_{cb}^{-T} (\dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T \boldsymbol{M}_{bb})) \boldsymbol{T}_x^{-1} \\ & \boldsymbol{T}_x^{-T} (\dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T \boldsymbol{M}_{bb} - (\dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T \boldsymbol{M}_{bb})) \boldsymbol{T}_x^{-1} = 0 \end{aligned}$$

3.3.3. Passive Formulation and Decoupling

The Coriolis Matrix (3.39) is not passive, since the relation for passivity e.q. (1.6) does not hold. A Coriolis matrix is not unique. There are infinitely many Coriolis matrices.

The only condition a Coriolis matrix must fulfill is the equations of motion being valid. To this end, one can add terms to the Coriolis matrix, which vanish when the Coriolis matrix is multiplied by the system velocities, or shift elements of the same row from one Coriolis matrix block element to another. The passive formulation of the Coriolis matrix takes the expressions of the Coriolis matrix, derived for the proof of the equivalence of the equations of motion, as a starting point (3.44), (3.45), (3.57) and (3.58). For the Coriolis matrix entries $\boldsymbol{C}_{qb}$ and $\boldsymbol{C}_q$, we subtract the third $-\boldsymbol{J}_{jq}^T\,\boldsymbol{a}^T_{\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b}\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}}$ term in (3.57), and add it to (3.58). This operation does makes the Hamel Coriolis matrix identical to the Euler-Newton Coriolis matrix without the passivity term. The equations of motion remain unchanged. The passive Coriolis matrix $\boldsymbol{C}^p$ is:

$$\boldsymbol{C}^p = \begin{bmatrix} \boldsymbol{C}^p_b & \boldsymbol{C}^p_{bq} \\ \boldsymbol{C}^p_{qb} & \boldsymbol{C}^p_q \end{bmatrix} \tag{3.82}$$

with block matrix entires:

$$\begin{aligned}
\boldsymbol{C}^p_b &= \dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T\,\boldsymbol{M}_{bb} + \boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{A}_{jb} \\
&= \boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb} - \boldsymbol{A}_{jb}^T\,\boldsymbol{a}_j^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb} + \boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{A}_{jb} && \text{(3.83a)} \\
\boldsymbol{C}^p_{bq} &= \dot{\boldsymbol{M}}_{bq} - \boldsymbol{a}_b^T\,\boldsymbol{M}_{bq} + \boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{J}_{jq} \\
&= \boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb} - \boldsymbol{A}_{jb}^T\,\boldsymbol{a}_j^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb} + \boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{A}_{jb} && \text{(3.83b)} \\
\boldsymbol{C}^p_{qb} &= \dot{\boldsymbol{M}}_{bq}^T - \frac{1}{2}\boldsymbol{M}_{b,q} + \boldsymbol{J}_{jq}^T\,\boldsymbol{a}^T_{\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b}\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} + \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{A}_{jb} \\
&= \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{A}}_{jb} - \boldsymbol{J}_{jq}^T\,\boldsymbol{a}_j^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{A}_{jb} + \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{A}_{jb} && \text{(3.83c)} \\
\boldsymbol{C}^p_q &= \dot{\boldsymbol{M}}_{qq} - \boldsymbol{M}_{bq,q} - \frac{1}{2}\boldsymbol{M}_{q,q} - \boldsymbol{J}_{jq}^T\,\boldsymbol{a}^T_{\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b}\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} + \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{J}_{jq} \\
&= \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\dot{\boldsymbol{J}}_{jq} - \boldsymbol{J}_{jq}^T\,\boldsymbol{a}_j^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{J}_{jq} + \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{J}_{jq} && \text{(3.83d)}
\end{aligned}$$

First we need to prove, that the passive formulation of the Coriolis matrix fulfills the equations of motion. Since we have already proven that the equations of motion are valid without the added terms, we need to show that the added terms vanish, when multiplied by the velocities:

$$\begin{aligned}
0 &\overset{!}{=} \boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b + \boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} + \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{A}_{jb}\,\boldsymbol{\nu}_b + \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\,\boldsymbol{J}_{jq}\dot{\boldsymbol{q}} \\
&= \boldsymbol{A}_{jb}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\boldsymbol{\nu}_j + \boldsymbol{J}_{jq}^T\,\boldsymbol{\Lambda}_j\,\boldsymbol{a}_j\boldsymbol{\nu}_j = 0
\end{aligned}$$

where relation $\boldsymbol{a}_{xy}\boldsymbol{\nu}_{xy} = 0$ from equation (2.17) was used.

The passive formulation of Coriolis matrix has the same form as in the Euler Newton formulation (3.41a), (3.41b), (3.41c) and (3.41d). To this end, we do not need to prove passivity, since this was done in [Gio20, p.37]. We can now use the passive formulation of the Coriolis matrix $\boldsymbol{C}^p$ and apply it to the decoupling transform (3.80) in the same way as we did in the previous chapter.

Proposition 3.3.2 (Passive Decoupling)**.** *Given a free floating robotic system defined by the mass matrix* (2.50) *and the passive Coriolis matrix* (3.82)*. Transforming the equations of motion with a transformation of the form* (3.80)*, yield the equations of motion:*

$$\begin{bmatrix} \boldsymbol{M}_x & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_y \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{v}}_x \\ \dot{\boldsymbol{v}}_y \end{bmatrix} \begin{bmatrix} \boldsymbol{C}_x^p & \boldsymbol{C}_{xy}^p \\ \boldsymbol{C}_{xy}^{p,T} & \boldsymbol{C}_y^p \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_x \\ \boldsymbol{v}_y \end{bmatrix} = \boldsymbol{T}^{-T} \begin{bmatrix} \boldsymbol{F}_b \\ \boldsymbol{\tau} \end{bmatrix} \tag{3.84}$$

with

$$\begin{aligned}
&\boldsymbol{M}_x = \boldsymbol{T}_x^{-T} \boldsymbol{M}_{bb} \boldsymbol{T}_x^{-1} \\
&\boldsymbol{M}_y = \boldsymbol{T}_y^{-T} (\boldsymbol{M}_{qq} - \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq}) \boldsymbol{T}_y^{-1} \\
&\boldsymbol{C}_x^p = \boldsymbol{T}_x^{-T} (\dot{\boldsymbol{M}}_{bb} - \boldsymbol{a}_b^T \boldsymbol{M}_{bb} - \boldsymbol{M}_{bb} \boldsymbol{T}_x^{-1} \dot{\boldsymbol{T}}_x + \boldsymbol{A}_{jb}^T \boldsymbol{\Lambda}_j \boldsymbol{a}_j \boldsymbol{A}_{jb}) \boldsymbol{T}_x^{-1} \\
&\boldsymbol{C}_{xy}^p = \boldsymbol{T}_x^{-T} (\boldsymbol{A}_{jb}^T \boldsymbol{\Lambda}_j \boldsymbol{a}_j \boldsymbol{J}_{jq} - \boldsymbol{A}_{jb}^T \boldsymbol{\Lambda}_j \boldsymbol{a}_j \boldsymbol{A}_{jb} \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq}) \boldsymbol{T}_y^{-1} \\
&\boldsymbol{C}_y^p = \boldsymbol{T}_y^{-T} (\boldsymbol{C}_q^p - \boldsymbol{C}_{qb}^p \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq} - \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} \boldsymbol{C}_{bq}^p + \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} \boldsymbol{C}_b^p \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq}) \boldsymbol{T}_y^{-1} - \boldsymbol{M}_y \dot{\boldsymbol{T}}_y \boldsymbol{T}_y^{-1}
\end{aligned}$$

If $\boldsymbol{v}_x$ is a centroidal momentum map, the invariance structure holds:

$$\boldsymbol{C}_x \boldsymbol{v}_x + \boldsymbol{C}_x \boldsymbol{v}_y = 0 \tag{3.85}$$

($\boldsymbol{v}_x$ is a momentum map, if $\boldsymbol{T}_x = \boldsymbol{A}_{cb}^{-T} \boldsymbol{M}_{bb}$ and $\boldsymbol{v}_x$ is a constant of motion (3.60)*)*

Proof. : For the proof of the first part, the block matrices of the passive Coriolis matrix (3.82) are plugging into the transformation relations from the Appendix A.1 by using matrix identities (A.6g). This directly gives the results for $\boldsymbol{C}_x^p$ and for $\boldsymbol{C}_y^p$. For $\boldsymbol{C}_{xy}^p$ the result can be simplified by plugging the passive Coriolis terms in and removing canceling terms.

$$\boldsymbol{C}_{xy}^p = \boldsymbol{T}_x^{-T} (\boldsymbol{A}_{jb}^T \boldsymbol{\Lambda}_j \boldsymbol{a}_j \boldsymbol{J}_{jq} - \boldsymbol{A}_{jb}^T \boldsymbol{\Lambda}_j \boldsymbol{a}_j \boldsymbol{A}_{jb} \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq}) \boldsymbol{T}_y^{-1} \tag{3.86}$$

For proving the second part of the Proposition, we have to show, that $\boldsymbol{C}_x^p \boldsymbol{v}_x + \boldsymbol{C}_{xy}^p \boldsymbol{v}_y = 0$. Plugging the passive coriolis terms into the definition of $\boldsymbol{C}_x$ and $\boldsymbol{C}_y$ gives after removing canceling terms:

$$\begin{aligned}
&\boldsymbol{C}_x \boldsymbol{\nu}_b = \boldsymbol{A}_{cb} \boldsymbol{M}_b^{-1} \boldsymbol{A}_{jb}^T \boldsymbol{\Lambda}_j \boldsymbol{a}_j \boldsymbol{A}_{jb} (\boldsymbol{\nu}_j + \boldsymbol{M}_b^{-1} \boldsymbol{M}_{bq} \dot{\boldsymbol{q}}) \\
&\boldsymbol{C}_y \dot{\boldsymbol{q}} = \boldsymbol{A}_{cb} \boldsymbol{M}_b^{-1} \boldsymbol{A}_{jb}^T \boldsymbol{\Lambda}_j \boldsymbol{a}_j \boldsymbol{A}_{jb} (\boldsymbol{J}_{jq} \dot{\boldsymbol{q}} - \boldsymbol{M}_b^{-1} \boldsymbol{M}_{bq} \dot{\boldsymbol{q}}) \\
&\boldsymbol{C}_x^p \boldsymbol{v}_x + \boldsymbol{C}_{xy}^p \boldsymbol{v}_y = 0
\end{aligned}$$

3.3.4. An Axiomatic Approach to Internal-External Velocity Decomposition

For control it is desirable to be able to separately control the motion arising from external forces and the internal motion. The decomposition of the dynamics into external and internal motion is extensively studied in [Gio20]. Here we want to look at the decomposition from the perspective of the transform, which generates the decomposition. We start with an unknown transformation $\boldsymbol{I}$ and state necessary conditions for the existence of an external and an internal velocity. Each condition imposes a constraint on the form of the transformation. The goal is to find out, how the shape of the transformation changes with the conditions imposed and to study the consequences.

Given an unknown transform $\boldsymbol{I}$, we want to decompose the generalized velocities into an external and an internal component:

$$\begin{bmatrix} \boldsymbol{\nu}^{ext} \\ \boldsymbol{\nu}^{int} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix} \tag{3.87}$$

The transform is a $(6+6) \times (6+n)$ matrix. It is assumed that the square matrix $\boldsymbol{A}$ and $(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{\#}\boldsymbol{C})$ are invertible and that the pseudo inverse $\boldsymbol{D}^{\#}$[3] and $(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{\#}$ exists. The inverse of I is:

$$\boldsymbol{I}^{-1} = \begin{bmatrix} (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{\#}\boldsymbol{C})^{-1} & -\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{\#} \\ -\boldsymbol{D}^{\#}\boldsymbol{C}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{\#}\boldsymbol{C})^{-1} & (\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{\#} \end{bmatrix} \tag{3.88}$$

The transformed equations of motion (3.38) are given by:

$$\begin{bmatrix} \boldsymbol{M}_e & \boldsymbol{M}_{ei} \\ \boldsymbol{M}_{ie} & \boldsymbol{M}_i \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\nu}}^{ext} \\ \dot{\boldsymbol{\nu}}^{int} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_e & \boldsymbol{C}_{ei} \\ \boldsymbol{C}_{ie} & \boldsymbol{C}_i \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}^{ext} \\ \boldsymbol{\nu}^{int} \end{bmatrix} = \boldsymbol{T}^{-T} \begin{bmatrix} \boldsymbol{F_b} \\ \boldsymbol{\tau} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}^{ext} \\ \boldsymbol{F}^{int} \end{bmatrix} \tag{3.89}$$

The external velocity and the internal velocity must have the following properties:

1. The internal velocity must depend on the joint velocities $\dot{\boldsymbol{q}}$.
2. The external force $\boldsymbol{F}^{ext}$ must not depend on the actuation of the joints $\boldsymbol{\tau}$.
3. The dynamics of the external velocity must be decoupled from the internal velocity with respect to the mass matrix. This implies: $\boldsymbol{M}_{ei}\dot{\boldsymbol{\nu}}^{int} = 0$
4. The dynamics of the external velocity must be decoupled from the internal velocity with respect to the Coriolis matrix. This implies: $\boldsymbol{C}_{ei}\boldsymbol{\nu}^{int} = 0$ or $\boldsymbol{C}_{ii}\boldsymbol{\nu}^{ext} + C_{ei}\boldsymbol{\nu}^{int} = 0$

The first condition implies that the block matrix $\boldsymbol{D}$ must not be zero. The second necessary condition implies for the transformation:

$$\boldsymbol{F}^{ext} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{\#}\boldsymbol{C})^{-T}(\boldsymbol{F}_b - \boldsymbol{C}^T\boldsymbol{D}^{\#T}\boldsymbol{\tau})$$

The condition must hold for any $\boldsymbol{\tau}$, which implies that $\boldsymbol{C}^T\boldsymbol{D}^{\#T}$ must vanish. It follows that $\boldsymbol{C} = 0$, since $\boldsymbol{D}$ must not be zero. The shape of the transform I becomes:

$$\boldsymbol{I} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{D} \end{bmatrix} \qquad \boldsymbol{I}^{-1} = \begin{bmatrix} \boldsymbol{A}^{-1} & -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{D}^{\#} \\ \boldsymbol{0} & \boldsymbol{D}^{\#} \end{bmatrix} \tag{3.90}$$

[3] The pseudo-inverse $\boldsymbol{D}^{\#}$ can either be the Moore-Pensorse pseudo inverse , i.e. $\boldsymbol{D}^{\#} = \boldsymbol{D}^T(\boldsymbol{D}\boldsymbol{D}^T)^{-1}$, or a dynamically consistent pseudo inverse, i.e. $\boldsymbol{D}^{\#} = \boldsymbol{M}\boldsymbol{D}^T(\boldsymbol{D}\boldsymbol{M}\boldsymbol{D}^T)^{-1}$ for some some invertible matrix $\boldsymbol{M}$, as long as $\boldsymbol{D}\boldsymbol{D}^{\#} = \mathbb{I}_{6\times 6}$ holds.

The immediate consequence of the first two conditions imposed is that any internal velocity only depends on the joint velocities. The third condition requires $\boldsymbol{M}_{ei}\dot{\boldsymbol{\nu}}^{int} = 0$. The mass matrix is transformed by $\boldsymbol{M}_{\boldsymbol{I}} = \boldsymbol{I}^{-T}\boldsymbol{M}\boldsymbol{I}^{-1}$. The term $\boldsymbol{M}_{ei}\dot{\boldsymbol{\nu}}^{int}$ is:

$$0 \overset{!}{=} \boldsymbol{M}_{ei}\dot{\boldsymbol{\nu}}^{int} = \boldsymbol{A}^{-T}(\boldsymbol{M}_{bq} - \boldsymbol{M}_{bb}\boldsymbol{A}^{-1}\boldsymbol{B})(\ddot{\boldsymbol{q}} + \boldsymbol{D}^{\#}\dot{\boldsymbol{D}}\dot{\boldsymbol{q}}) \tag{3.91}$$

The condition must hold for any $\ddot{\boldsymbol{q}}$ and any $\dot{\boldsymbol{q}}$. Therefore: $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq}$. The shape of the transform I after the second condition imposed becomes:

$$\boldsymbol{I} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{A}\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq} \\ \boldsymbol{0} & \boldsymbol{D} \end{bmatrix} \qquad \boldsymbol{I}^{-1} = \begin{bmatrix} \boldsymbol{A}^{-1} & -\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq}\boldsymbol{D}^{\#} \\ \boldsymbol{0} & \boldsymbol{D}^{\#} \end{bmatrix} \tag{3.92}$$

and the transformed mass matrix is:

$$\boldsymbol{M}_{\boldsymbol{I}} = \begin{bmatrix} \boldsymbol{A}^{-T}\boldsymbol{M}_{bb}\boldsymbol{A}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}^{\#T}(\boldsymbol{M}_{qq} - \boldsymbol{M}_{bq}^{T}\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq})\boldsymbol{D}^{\#} \end{bmatrix} \tag{3.93}$$

The fourth condition requires that either $\boldsymbol{C}_{ei}\boldsymbol{\nu}^{int} = 0$ or $\boldsymbol{C}_{ii}\boldsymbol{\nu}^{ext} + C_{ei}\boldsymbol{\nu}^{int} = 0$. Using (A.5a),(A.5b):

$$\begin{aligned} 0 &\overset{!}{=} \boldsymbol{C}_{ei}\boldsymbol{\nu}^{int} \\ &= \boldsymbol{A}^{-T}[\boldsymbol{C}_{bq} + (\dot{\boldsymbol{M}}_{bb} - \boldsymbol{C}_{b})\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq} - \dot{\boldsymbol{M}}_{bq}]\dot{\boldsymbol{q}} \\ &\quad \text{or} \\ 0 &\overset{!}{=} \boldsymbol{C}_{i}\boldsymbol{\nu}^{ext} + C_{ei}\boldsymbol{\nu}^{int} \\ &= \boldsymbol{A}^{-T}[\boldsymbol{C}_{b} - \boldsymbol{M}_{bb}\boldsymbol{A}^{-1}\dot{\boldsymbol{A}}]\boldsymbol{\nu}_{b} \\ &+ \boldsymbol{A}^{-T}[\boldsymbol{C}_{bq} + (\dot{\boldsymbol{M}}_{bb} - \boldsymbol{C}_{b})\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq} - \dot{\boldsymbol{M}}_{bq}]\dot{\boldsymbol{q}} \end{aligned} \tag{3.94}$$

There are multiple possibilities for the conditions to be fulfilled. If $\boldsymbol{M}_{e}\boldsymbol{\nu}^{ext}$ is a constant of motion, the condition $\boldsymbol{C}_{i}\boldsymbol{\nu}^{ext} + C_{ei}\boldsymbol{\nu}^{int}$ is fulfilled (3.3.2). In this case, the transformation matrix $\boldsymbol{A}$ must the adjoint matrix: $\boldsymbol{A} = \boldsymbol{A}_{cb}$ and the center of mass frame must be aligned with the inertial frame. In this case, the transformation $\boldsymbol{I}$ becomes:

$$\boldsymbol{I} = \begin{bmatrix} \boldsymbol{A}_{cb} & \boldsymbol{A}_{cb}\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq} \\ \boldsymbol{0} & \boldsymbol{D} \end{bmatrix} \qquad \boldsymbol{I}^{-1} = \begin{bmatrix} \boldsymbol{A}_{cb}^{-1} & -\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq}\boldsymbol{D}^{\#} \\ \boldsymbol{0} & \boldsymbol{D}^{\#} \end{bmatrix} \tag{3.95}$$

subject to the condition $\boldsymbol{a}_{cb}^{T}(\boldsymbol{M}_{bb}\boldsymbol{\nu}_{b} + \boldsymbol{M}_{bq}\dot{\boldsymbol{q}}) = \boldsymbol{a}_{ib}^{T}(\boldsymbol{M}_{bb}\boldsymbol{\nu}_{b} + \boldsymbol{M}_{bq}\dot{\boldsymbol{q}})$.

The term $\boldsymbol{C}_{ei} = \boldsymbol{0}$ vanishes, if the Coriolis matrix is non-passive (3.3.1). If the Coriolis matrix is passive, its off-diagonal term $\boldsymbol{C}_{xy}^{p}$ is (3.3.2).

$$\boldsymbol{C}_{xy}^{p}\boldsymbol{\nu}^{int} = \boldsymbol{A}^{-T}(\boldsymbol{A}_{jb}^{T}\boldsymbol{\Lambda}_{j}\boldsymbol{a}_{j}\boldsymbol{J}_{jq} - \boldsymbol{A}_{jb}^{T}\boldsymbol{\Lambda}_{j}\boldsymbol{a}_{j}\boldsymbol{A}_{jb}\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq})\dot{\boldsymbol{q}}$$

It might be possible that $\boldsymbol{C}_{xy}^{p} = \boldsymbol{0}$, if the mass matrix has a special structure.

Up to now, both, the external and internal velocity are abstract concepts. They do not relate to the body velocities and no frames are assigned to them. Therefore we state additional requirements:

5. The external force is expressed in a center of mass frame of some orientation[4].

6. The velocity of a body is composed of the sum of an external velocity and an internal velocity of the body.

From requirement 5 it follows that the transformation block matrix $\boldsymbol{A}$ is the adjoint matrix $\boldsymbol{A}_{cb}$, which transforms the total momentum expressed in frame b to frame c.
Requirement 6 states for the body velocity j: $\boldsymbol{\nu}_j = \boldsymbol{\nu}_j^{ext} + \boldsymbol{\nu}_j^{int}$. Since the body velocity j is: $\boldsymbol{\nu}_j = \boldsymbol{A}_{jb}\boldsymbol{\nu}_b + \boldsymbol{J}_{jq}\dot{\boldsymbol{q}}$ and the external velocity expressed in frame j is: $\boldsymbol{\nu}_j^{ext} = \boldsymbol{A}_{jb}\boldsymbol{\nu}_b + \boldsymbol{A}_{jb}\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq}\dot{\boldsymbol{q}}$, the internal velocity must be $\boldsymbol{\nu}_j^{int} = \boldsymbol{J}_{jq}^{int}\dot{\boldsymbol{q}} = (\boldsymbol{J}_{jq} - \boldsymbol{A}_{jb}\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq})\dot{\boldsymbol{q}}$.

Finally, the external-internal velocity decomposition is given by the transformation $\boldsymbol{I}$:

$$\boldsymbol{I} = \begin{bmatrix} \boldsymbol{A}_{cb} & \boldsymbol{A}_{cb}\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq} \\ \boldsymbol{0} & \boldsymbol{J}_j^{int} \end{bmatrix} \qquad \boldsymbol{I}^{-1} = \begin{bmatrix} \boldsymbol{A}_{cb}^{-1} & -\boldsymbol{M}_b^{-1}\boldsymbol{M}_{bq}\boldsymbol{J}_j^{int,\#} \\ \boldsymbol{0} & \boldsymbol{J}_j^{int,\#} \end{bmatrix} \tag{3.96}$$

with $\boldsymbol{J}_j^{int} = \boldsymbol{J}_{jq} - \boldsymbol{A}_{jb}\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq}$

For the case that CoM frame is aligned to the inertial frame, the resulting external and internal velocities are the same as in [Gio20, eq. (A.4b)]. The axiomatic approach allows in addition to make statements for the case that the CoM frame is not aligned to the inertial frame, by using the results of decoupling (see Prop. 3.3.1 and Prop. 3.3.2). If the CoM frame is not aligned to the inertial frame, the total momentum expressed in the CoM frame, $\boldsymbol{M}_e\boldsymbol{\nu}^{ext}$, is not a constant of motion (3.60). Therefore, requirement 4 can only be fulfilled by decoupling: $\boldsymbol{C}_{ei}\boldsymbol{\nu}^{int} = 0$. This is the case, if the non-passive Coriolis matrix derived from the Hamel equations (3.39) is used. Decoupling is also achieved, if a passive Coriolis matrix diagonalizes under the Internal/external transform.

[4]This requirement defines the meaning of external velocity. The dynamics of an external velocity must not depend on internal motion. This can only be achieved, if the external velocity is expressed in a center-of-mass frame.

4. Application to Hopping Robots

4.1. Theory and Modeling

4.1.1. SE(2) Algebra

In this section the relations derived for $SE(n)$ in Chapter 2.1 are translated to $SE(2)$. The Lee group $SE(2)$ is three dimensional, constituting of a 2 dimensional frame origin, $\boldsymbol{o}_{xy}$ and a rotation angle θ_{xy}. The corresponding Lee algebra $se(2)$ has also 3 dimensions, given by the linear body velocity $\boldsymbol{v}_{xy}$ and the angular velocity ω_{xy}.

We start with the rotation matrix $\boldsymbol{R}_{ib}(\theta)$, which is the rotational part of the rigid body transformation from the base frame b to the inertial frame i. The angle θ is measured from the x-axis of the inertial frame to the x-axis of the body frame. The 2×2 rotation matrix is given by:

$$
\begin{aligned}
\boldsymbol{R}_{ib}(\theta) &= \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} = \theta \boldsymbol{S} \\
\boldsymbol{S} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\end{aligned}
$$

The $se(2)$ element of the angular body velocity, $\hat{\boldsymbol{\omega}}_b = \boldsymbol{R}_{ob}^T \dot{\boldsymbol{R}}_{ob}$, is:

$$
\hat{\boldsymbol{\omega}}_b = \omega_b \boldsymbol{S} \tag{4.1}
$$

The skew matrix $\boldsymbol{S}$ has some useful properties, which can easily be verified:

$$
\begin{aligned}
\boldsymbol{S}^T &= -\boldsymbol{S} \\
\boldsymbol{S}\boldsymbol{S}^T &= \boldsymbol{S}^T \boldsymbol{S} = -\boldsymbol{S}\boldsymbol{S} = \mathbb{I}_{2x2} \\
\boldsymbol{R}_{xy}\boldsymbol{S} &= \boldsymbol{S}\boldsymbol{R}_{xy}
\end{aligned}
$$

The $SE(2)$ group element $\boldsymbol{H}_{xy} \in SO(2)$ (2.3) is:

$$
\boldsymbol{H}_{xy} = \begin{bmatrix} \boldsymbol{R}_{xy} & \boldsymbol{o}_{xy} \\ \boldsymbol{0} & 1 \end{bmatrix} \qquad \dot{\boldsymbol{H}}_{xy} = \begin{bmatrix} \omega_{xy}\boldsymbol{R}_{xy}\boldsymbol{S} & \dot{\boldsymbol{o}}_{xy} \\ \boldsymbol{0} & 0 \end{bmatrix} \tag{4.2}
$$

The $se(2)$ element of body velocity $\hat{\boldsymbol{\nu}}_{xy} = \boldsymbol{H}_{xy}^{-1} \dot{\boldsymbol{H}}_{xy}$ (2.7) is:

$$
\begin{bmatrix} \hat{\boldsymbol{\omega}}_{xy} & \boldsymbol{v}_{xy} \\ \boldsymbol{0} & 0 \end{bmatrix} = \begin{bmatrix} \omega_{xy}\boldsymbol{S} & \boldsymbol{R}_{xy}^T \dot{\boldsymbol{o}}_{xy} \\ \boldsymbol{0} & 0 \end{bmatrix}
$$

From this follows that the linear body velocity is $\boldsymbol{v}_{xy} = \boldsymbol{R}_{xy}^T \dot{\boldsymbol{o}}_{xy}$. The big adjoint transformation $\boldsymbol{Ad}_{xy}$ is for any $\hat{\boldsymbol{\nu}}_1 \in se(2)$ (2.10):

$$\begin{aligned}
\boldsymbol{Ad}_{xy}\hat{\boldsymbol{\nu}}_1 &= \begin{bmatrix} \boldsymbol{R}_{xy}\hat{\boldsymbol{\omega}}_1\boldsymbol{R}_{xy}^T & \boldsymbol{R}_{xy}\boldsymbol{v}_1 - \boldsymbol{R}_{xy}\hat{\boldsymbol{\omega}}_1\boldsymbol{R}_{xy}^T\boldsymbol{o}_{xy} \\ \boldsymbol{0} & 0 \end{bmatrix} \\
&= \begin{bmatrix} \omega_1\boldsymbol{S} & \boldsymbol{R}_{xy}\boldsymbol{v}_1 - \omega_1\boldsymbol{S}\boldsymbol{o}_{xy} \\ \boldsymbol{0} & 0 \end{bmatrix} \\
\boldsymbol{A}_{xy} &= \begin{bmatrix} \boldsymbol{R}_{xy} & -\boldsymbol{S}\boldsymbol{o}_{xy} \\ \boldsymbol{0} & 1 \end{bmatrix}
\end{aligned}$$

The little adjoint transformation $\boldsymbol{ad}_{\nu_{xy}}$ is for any $\hat{\boldsymbol{\nu}}_1 \in se(2)$ (2.11):

$$\begin{aligned}
\boldsymbol{ad}_{\nu_{xy}}\hat{\boldsymbol{\nu}}_1 &= \begin{bmatrix} \hat{\boldsymbol{\omega}}_{xy}\hat{\boldsymbol{\omega}}_1 - \hat{\boldsymbol{\omega}}_1\hat{\boldsymbol{\omega}}_{xy} & \hat{\boldsymbol{\omega}}_{xy}\boldsymbol{v}_1 - \hat{\boldsymbol{\omega}}_1\boldsymbol{v}_{xy} \\ \boldsymbol{0} & 0 \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{0}_{2\times 2} & \omega_{xy}\boldsymbol{S}\boldsymbol{v}_1 - \omega_1\boldsymbol{S}\boldsymbol{v}_{xy} \\ \boldsymbol{0} & 0 \end{bmatrix} \\
\boldsymbol{a}_{xy} &= \begin{bmatrix} \omega_{xy}\boldsymbol{S} & -\boldsymbol{S}\boldsymbol{v}_{xy} \\ \boldsymbol{0} & 0 \end{bmatrix}
\end{aligned}$$

4.1.2. Transformations for Free Floating Robots on SE(2)

In this section we calculate the link twist Jacobian $\boldsymbol{J}_{jq}$. The i-th column of $\boldsymbol{J}_{jq}$ for $SE(n)$ is (2.23)

$$\boldsymbol{J}_{jq}^i = \begin{bmatrix} \boldsymbol{R}_{bj}^T \frac{\partial \boldsymbol{o}_{bj}}{\partial q_i} \\ (\boldsymbol{R}_{bj}^T \frac{\partial \boldsymbol{R}_{bj}}{\partial q_i})^\vee \end{bmatrix} \tag{4.3}$$

using the relations

$$\boldsymbol{R}_{xy}^T \frac{\partial \boldsymbol{R}_{xy}(\theta_{xy})}{\partial \theta_{xy}} = \boldsymbol{S} \tag{4.4}$$

$$\boldsymbol{R}_{xy}^T \frac{\partial \boldsymbol{R}_{xy}(\boldsymbol{q})}{\partial \boldsymbol{q}_i} = \boldsymbol{S}\frac{\partial \theta_{xy}(\boldsymbol{q}_i)}{\partial \boldsymbol{q}_i} \tag{4.5}$$

the ith column of the $SE(2)$ link twist Jacobian for $j > 0$ becomes:

$$\boldsymbol{J}_{jq}^i = \begin{bmatrix} \boldsymbol{R}_{bj}^T \frac{\partial \boldsymbol{o}_{bj}}{\partial q_i} \\ \frac{\partial \theta_{bj}}{\partial q_i} \end{bmatrix} \tag{4.6}$$

The Jacobian $\boldsymbol{J}_{jb}$ for $SE(2)$ corresponding to (2.47) is

$$\begin{aligned}
\boldsymbol{J}_{0b} &= \begin{bmatrix} \mathbb{I}_{3\times 3} & \boldsymbol{0}_{3\times n} \end{bmatrix} \\
\boldsymbol{J}_{jb} &= \begin{bmatrix} \boldsymbol{A}_{jb}(\boldsymbol{q}) & \boldsymbol{J}_{jq} \end{bmatrix}
\end{aligned} \tag{4.7}$$

4.1.3. Lagrange Function on SE(2)

The adjoint $\boldsymbol{A}_{j_{ci},j}$ for $SE(2)$ corresponding to (2.44) is

$$\boldsymbol{A}_{j_{ci},j} = \begin{bmatrix} \mathbb{I}_{2\times 2} & \boldsymbol{S}\boldsymbol{o}_{j,j_{ci}} \\ \boldsymbol{0} & 1 \end{bmatrix} \tag{4.8}$$

The mass matrix for joint j for $SE(2)$ corresponding to (2.45) is

$$\boldsymbol{\Lambda}_j = \sum_{i=1}^{n_j} \begin{bmatrix} m_{j_{ci}}\mathbb{I}_{2\times 2} & m_{j_{ci}}\boldsymbol{S}\boldsymbol{o}_{j,j_{ci}} \\ -m_{j_{ci}}\boldsymbol{o}_{j,j_{ci}}^T\boldsymbol{S} & I_{j_{ci}} + m_{j_{ci}}\boldsymbol{o}_{j,j_{ci}}^T\boldsymbol{o}_{j,j_{ci}} \end{bmatrix} \tag{4.9}$$

The mass matrix for $SE(2)$ (2.50) in terms of $\boldsymbol{J}_{jb}$ is

$$\boldsymbol{M}_b = \sum_{i=0}^{n} \boldsymbol{J}_{jb}^T \boldsymbol{\Lambda}_j \boldsymbol{J}_{jb} \tag{4.10}$$

The Coriolis matrix can be calculated through the Hamel equations (3.39) or defined passively with the Euler-Lagrange method (3.41a),(3.41b),(3.41c) and (3.41d).

The gravitational potential for a system with n_j masses per joint is given by (2.52). Assuming that gravity points downwards on the y-axis, the gravity potential becomes:

$$U_g(\boldsymbol{o}_{ib}, \boldsymbol{\theta}_{ib}, \boldsymbol{q}) = m\, g\boldsymbol{o}_{ib} + g\boldsymbol{r}_2^T \sum_{j=0}^{n}\sum_{i=1}^{n_j} m_{j_{ci}}\boldsymbol{o}_{b,j_{ci}}(\boldsymbol{q}) \tag{4.11}$$

where m is the total mass of the system and $\boldsymbol{r}_2$ is the second row of $\boldsymbol{R}_{ib}$. The gravity system matrix $\boldsymbol{G}$ (2.53) for $SE(2)$ is

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{R}_{ib}^T \frac{\partial U_g}{\partial \boldsymbol{o}_{ib}} \\ \frac{\partial U_g}{\partial \theta_{ib}} \\ \frac{\partial U_g}{\partial \boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} m\, g\sin(\theta_{ib}) \\ m\, g\cos(\theta_{ib}) \\ \frac{\partial U_g}{\partial \theta_{ib}} \\ \frac{\partial U_g}{\partial \boldsymbol{q}} \end{bmatrix} \tag{4.12}$$

4.1.4. Modeling the Floor

A viscous-elastic floor is modeled, which acts on the foot of the robot. The inertial floor forces have a horizontal and a vertical component, denoted by ${}_i\boldsymbol{F}_{floor}^x$ and ${}_i\boldsymbol{F}_{floor}^y$. They are modeled as spring with damping. The vertical floor force is constrained to be positive, in order to make sure that the floor does not pull the robot. The inertial floor forces are given by:

$${}_i\boldsymbol{F}_{floor}^x = K_{floor}^x \boldsymbol{o}_{if}^x - D_{floor}^x \dot{\boldsymbol{o}}_{if}^x \tag{4.13}$$

$${}_i\boldsymbol{F}_{floor}^y = \max(K_{floor}^y \boldsymbol{o}_{if}^y - D_{floor}^y \dot{\boldsymbol{o}}_{if}^y, 0) \tag{4.14}$$

The floor forces are active only when the foot touches the ground. The inertial foot forces have to be transformed to be base frame. This is achieved with the transformation dual to the transformation of the base velocity to the inertial foot velocity: $\dot{\boldsymbol{o}}_{if} = \boldsymbol{J}_{\dot{o}_{if}} \begin{bmatrix} \boldsymbol{\nu}_b \\ \dot{\boldsymbol{q}} \end{bmatrix}$:

$$\boldsymbol{\tau}_b^{floor} = \boldsymbol{J}_{\dot{o}_{if}}^T \, {}_i\boldsymbol{F}_{floor} \tag{4.15}$$

4.1.5. Control Strategy

The robot is controlled by an an impedance controller [ASOH07] of virtual springs. A radial spring acts on the line between the CoM and the foot, r_{cf}, and a polar spring acts on the angle $\alpha_{i,cf}$ between the vertical line and r_{cf}. The radial spring is active during both, flight and stance. The polar spring is active during stance only. In addition to the virtual springs an energy controller is used to maintain a desired level of energy. The radial controller containing the energy controller is always the same. There are 2 types of polar controllers, which differ in the definition of the angle α to be controlled.

Control Structure

The structure of the full controller looks like:

Stance

$$\begin{aligned} \tau_E &= -K_E(E - E^{des})\dot{r}_{cf} \\ \tau_r &= -K_r(r_{cf} - r_{cf}^{des}) - D_r\,\dot{r}_{cf} + \tau_E && (4.16) \\ \tau_p &= -K_p(\alpha - \,\alpha^{des}) - D_p\,\dot{\alpha} && (4.17) \end{aligned}$$

Flight

$$\begin{aligned} \tau_E &= -K_E(E - E^{des})\dot{r}_{cf} \\ \tau_r &= -K_r(r_{cf} - r_{cf}^{des}) - D_r\dot{r}_{cf} + \tau_E && (4.18) \\ \tau_p &= 0 && (4.19) \end{aligned}$$

The energy controller tries to maintain a desired level of energy E^{des}. The value of E^{des} is usually chosen to be the initial energy E_0. The control output τ_E is proportional to the velocity of length between the center of mass and the foot [GG12]. It is added to the radial spring controller. The radial controller tries to maintain a desired length between the center of mass and the foot. It operates during stance and flight with the same control parameters. The value of r_{cf}^{des} is chosen as the initial CoM-foot length. The polar controller is a virtual polar spring, which tries to maintain the angle α^{des}. The polar controller operates only during stance. The definition of α depends on the type of polar controller used.

Controller I and II

Controller I is under-actuated and uses only radial control. Controller II is fully actuated using both, radial and polar control. The angle $\alpha_{i,cf}$ from the negative inertial y axis to the line from the center of mass to the foot, is used as α for polar control (see Fig. (4.1) for the definition of the reference axes). The vectors from the center of mass to the foot, expressed in the inertial frame and the CoM frame, respectively are:

$$\begin{aligned} {}_i\boldsymbol{o}_{cf} &= \boldsymbol{R}_{ib}^T(\boldsymbol{o}_{bf} - \boldsymbol{o}_{bc}) \\ \boldsymbol{o}_{cf} &= \boldsymbol{R}_{bc}^T(\boldsymbol{o}_{bf} - \boldsymbol{o}_{bc}) \end{aligned} \tag{4.20}$$

Their components are denoted ${}_i\boldsymbol{o}_{cf} = [{}_i\boldsymbol{o}_{cf}^x; {}_i\boldsymbol{o}_{cf}^y]$ and $\boldsymbol{o}_{cf} = [\boldsymbol{o}_{cf}^x; \boldsymbol{o}_{cf}^y]$. The corresponding velocities are given by the Jacobians:

$$\begin{aligned} {}_i\dot{\boldsymbol{o}}_{cf} &= \boldsymbol{J}_{{}_i\dot{\boldsymbol{o}}_{cf}}\boldsymbol{v}_b \\ \dot{\boldsymbol{o}}_{cf} &= \boldsymbol{R}_{cf}\boldsymbol{J}_{\boldsymbol{v}_f^{int}}\boldsymbol{v}_b \end{aligned} \tag{4.21}$$

The distance r_{cf} and the angle ${}_i\alpha_{i,cf}$ are obtained from ${}_i\boldsymbol{o}_{cf}$:

$$\begin{aligned} {}_i\boldsymbol{o}_{cf} &= \begin{bmatrix} r_{cf}\ \sin({}_i\alpha_{cf}) \\ -r_{cf}\ \cos({}_i\alpha_{cf}) \end{bmatrix} \\ r_{cf} &= |\boldsymbol{o}_{cf}| \\ \alpha_{i,cf} &= \text{atan2}({}_i\boldsymbol{o}_{cf}^y, {}_i\boldsymbol{o}_{cf}^x) + \frac{\pi}{2} \end{aligned}$$

The transformation from Cartesian to polar velocities is given by:

$$\begin{aligned} \dot{r}_{cf} &= \boldsymbol{J}_r({}_i\boldsymbol{o}_{cf}){}_i\dot{\boldsymbol{o}}_{cf} \\ \dot{\alpha}_{i,cf} &= \boldsymbol{J}_\alpha({}_i\boldsymbol{o}_{cf}){}_i\dot{\boldsymbol{o}}_{cf} \end{aligned} \tag{4.22}$$

with the polar and radial Jacobians:

$$\boldsymbol{J}_r({}_i\boldsymbol{o}_{cf}) = \begin{bmatrix} \frac{{}_i\boldsymbol{o}_{cf}^x}{r_{cf}} & \frac{{}_i\boldsymbol{o}_{cf}^y}{r_{cf}} \end{bmatrix} \qquad \boldsymbol{J}_\alpha({}_i\boldsymbol{o}_{cf}) = \begin{bmatrix} \frac{-{}_i\boldsymbol{o}_{cf}^y}{r_{cf}^2} & \frac{{}_i\boldsymbol{o}_{cf}^x}{r_{cf}^2} \end{bmatrix} \tag{4.23}$$

Finally the transformations from the generalized base velocities $\boldsymbol{v}_b$ to $\dot{r}_{cf}$ and $\dot{\alpha}_{i,cf}$ are:

$$\begin{aligned} \dot{r}_{cf} &= \boldsymbol{J}_r({}_i\boldsymbol{o}_{cf})\boldsymbol{R}_{cf}\boldsymbol{J}_{\boldsymbol{v}_f^{int}}\boldsymbol{v}_b \\ \dot{\alpha}_{i,cf} &= \boldsymbol{J}_\alpha({}_i\boldsymbol{o}_{cf})\boldsymbol{J}_{{}_i\dot{\boldsymbol{o}}_{cf}}\boldsymbol{v}_b \end{aligned} \tag{4.24}$$

The control outputs τ_r and τ_p transformed to the base frame, give the external forces:

$$\begin{bmatrix} \boldsymbol{F}_b \\ \tau \end{bmatrix}^{control} = \boldsymbol{J}_{\boldsymbol{v}_f^{int}}^T \boldsymbol{R}_{cf}^T \boldsymbol{J}_r^T({}_i\boldsymbol{o}_{cf})\tau_r + \boldsymbol{J}_{{}_i\dot{\boldsymbol{o}}_{cf}}^T \boldsymbol{J}_\alpha^T({}_i\boldsymbol{o}_{cf})\tau_p \tag{4.25}$$

Where $\boldsymbol{F}_b$ is the three dimensional wrench acting on the base. Due to use of the internal velocity, the radial component in (4.25) only excites the joint torques τ, whereas the polar term also excites the angular component of $\boldsymbol{F}_b$. The linear force component in $\boldsymbol{F}_b$ is always zero.

Controller III

The radial control part for controller III is the same as for controller II. Controller III controls the orientation of the CoM frame, which is aligned to the locked velocity (Sect. (3.2.4)), instead of the orientation of the leg. The angle to be controlled, α_c , is the angle from the negative inertial y axis to the x axis of the center of mass frame.

$$\alpha_c = \theta_b + \theta_{bc}$$
$$\dot{\alpha}_c = \boldsymbol{J}_{\dot{\alpha}_c} \boldsymbol{v}_b$$

The control outputs τ_r and τ_p transformed to the base frame, give the external forces:

$$\begin{bmatrix} \boldsymbol{F}_b \\ \tau \end{bmatrix}^{control} = \boldsymbol{J}^T_{\boldsymbol{v}_f^{int}} \boldsymbol{R}^T_{cf} \boldsymbol{J}^T_r ({}_i\boldsymbol{o}_{cf}) \tau_r + \boldsymbol{J}^T_{\dot{\alpha}_c} \tau_p \tag{4.26}$$

4.1.6. Frame Conventions

The transformations between joints depend on the frame convention used. We assume that the origin of frame j is along the x axis of frame $j-1$. This means that the rotation matrix only depends on q_j in the case of a rotational joint and that a prismatic joint moves along the x axis of frame $j-1$:

$$\begin{aligned} &j > 0\text{: Rotational Joint} \qquad && \boldsymbol{o}_{j-1,j} = \begin{bmatrix} l_j \\ 0 \end{bmatrix} \qquad && \theta_{j-1,j} = q_j \\ &j > 0\text{: Prismatic Joint} \qquad && \boldsymbol{o}_{j-1,j} = \begin{bmatrix} q_j \\ 0 \end{bmatrix} \qquad && \theta_{j-1,j} = 0 \end{aligned} \tag{4.27}$$

The quantity l_j is a constant. The body frame 0 corresponds to the base frame b.

Given the frame convention defined, the kinematics of a floating base robot in $SE(2)$ with n joints is fully specified, if all inputs in the Table are provided.

Joint	$\boldsymbol{o}^x_{j-1,j}(q_j)$	$\boldsymbol{o}^y_{j-1,j}(q_j)$	$\theta_{j-1,j}(q_j)$
1	$\boldsymbol{o}^x_{b1}(q_1)$	0	$\theta_{b1}(q_i)$
2	$\boldsymbol{o}^x_{12}(q_2)$	0	$\theta_{12}(q_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\boldsymbol{o}^x_{n-1,n}(q_n)$	0	$\theta_{n-1,n}(q_n)$

Table 4.1.: Kinematic Inputs for a Free Floating Robot with n Joints

The inputs in the Table depend on the joint type:

Joint Type	$\boldsymbol{o}^x_{j-1,j}(q_j)$	$\boldsymbol{o}^y_{j-1,j}(q_j)$	$\theta_{j-1,j}(q_j)$
Revolute	$\boldsymbol{o}^x_{j-1,j}$	0	q_j
Prismatic	q_j	0	0

Table 4.2.: Kinematic Inputs per Joint Type

The dynamics of a free floating robot with n joints and j_n masses of joint j is fully specified by the providing the masses for each joint and their centers of mass relative to frame j:

Joint	n_j	$m_{j_{ci}}$	$\boldsymbol{o}^x_{j,j_{ci}}$	$\boldsymbol{o}^y_{j,j_{ci}}$	$I_{j_{ci}}$
0	1	$m_{0,c1}$	$\boldsymbol{o}^x_{0,0_{c1}}$	$\boldsymbol{o}^y_{0,0_{c1}}$	$I_{0_{c1}}$
0	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	n_0	$m_{0,cn}$	$\boldsymbol{o}^x_{0,0_{cn}}$	$\boldsymbol{o}^y_{0,0_{cn}}$	$I_{0_{cn}}$
1	1	$m_{1,c1}$	$\boldsymbol{o}^x_{1,1_{c1}}$	$\boldsymbol{o}^y_{1,1_{c1}}$	$I_{1_{c1}}$
1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	n_1	$m_{1,cn}$	$\boldsymbol{o}^x_{1,1_{cn}}$	$\boldsymbol{o}^y_{1,1_{cn}}$	$I_{1_{c1}}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1	$m_{n,c1}$	$\boldsymbol{o}^x_{n,n_{c1}}$	$\boldsymbol{o}^y_{n,n_{c1}}$	$I_{n_{cn}}$
n	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	n_n	$m_{n,cn}$	$\boldsymbol{o}^x_{n,n_{cn}}$	$\boldsymbol{o}^y_{n,n_{cn}}$	$I_{n_{cn}}$

Table 4.3.: Dynamic Inputs for a Free Floating Robot with n Joints

4.2. Simulation of a Prismatic Hopper in the Plane

The prismatic hopper consists of a base and a foot and a prismatic joint between them, representing the leg (see Fig. 4.1). The foot and the base consist of point masses (see Tab. 4.5). The rest length of the leg is $12cm$. The radial controller acts between the CoM and the foot and is active during both, stance and flight (see Fig. 4.2). The center of mass is always on the leg. The rest length of the radial spring is $0.114cm$. The polar string is only active during stance (see Fig. 4.2).

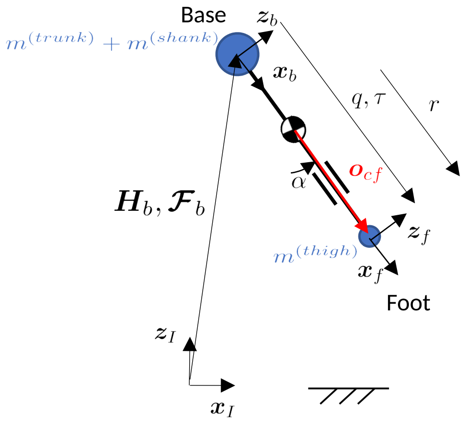


Figure 4.1.: Drawing of the prismatic hopper

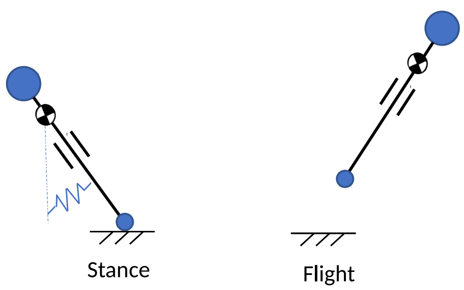


Figure 4.2.: The prismatic leg during stance and during flight. The polar spring is only active during stance.

4.2.1. Setup and Simulation Parameters

The kinematic inputs (see Tab 4.4) and the dynamics parameters (see Tab. 4.5) are fed into the Symbolic Lee Algebra Toolbox, described in Appendix A.2, which calculates the dynamic matrices and the Jacobians for control.

Joint	$o^x_{j-1,j}(q_j)$	$o^y_{j-1,j}(q_j)$	$\theta_{j-1,j}(q_j)$
1	q_1	0	0

Table 4.4.: Kinematic parameters for the prismatic leg

$\alpha_{i,cf}(0)[deg]$	$\boldsymbol{o}_b^x$	$\boldsymbol{o}_b^y$	θ_b	q_1	$\boldsymbol{v}_b^x$	$\boldsymbol{v}_b^y$	ω_b	$\dot{q}_1$
0	0	0.3	$-\pi/2$	0.114	0	0	0	0

Table 4.7.: Initial conditions used all simulations of the prismatic leg

Joint	n_j	$m_{j_{ci}}$	$\boldsymbol{o}_{j,j_{ci}}^x$	$\boldsymbol{o}_{j,j_{ci}}^y$	$I_{j_{ci}}$
0	1	0.957	0	0	0
1	1	0.050	0	0	0

Table 4.5.: Dynamics parameters for the prismatic leg

4.2.2. Simulation Results for the Prismatic Hopper

The prismatic leg has 4 configuration variables $(\boldsymbol{o}_b^x, \boldsymbol{o}_b^y, \theta_b, q_1)$ and 4 velocities $(\boldsymbol{v}_b^x, \boldsymbol{v}_b^y, \omega_b, \dot{q}_1)$. It has only one prismatic joint q_1 and the foot is a point. The parameters for the viscous-elastic floor are displayed in Table (4.6).

Direction	K_{floor}	D_{floor}
x	10000	1000
y	10000	1000

Table 4.6.: Floor parameters used for all simulations

During hopping forward, the leg picks up an angular momentum at lift-off. Since there is no polar control during flight, the leg rotates during flight. To this end, it is hard to find stable forward gaits by trial and error. Therefore, we restrict ourselves to hopping in place for the prismatic leg.

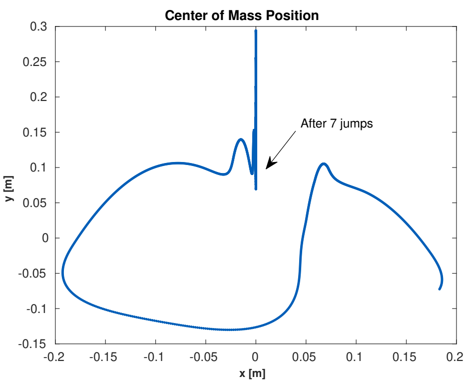
For hopping in place, the initial angle of the base is chosen, such that the foot points vertically downwards and the fall-off height is $30cm$. The resulting initial conditions are displayed in Table (4.7). These initial conditions are used for all experiments for the prismatic leg. In the subsequent experiments the influence of different control parameters are examined. Experiment I uses under actuated control to examine, how long the leg remains upright without polar stabilization. Experiment II uses fully actuated control without the energy controller, while experiment III uses both, fully actuated control and energy control. In all experiments no damping is used. The control parameters and commanding values for all experiments of the prismatic leg are summarized in Table (4.8).

Experiment	K_r/D_r	K_p/D_p	K_E	r_{cf}^{des}	$\alpha_{i,cf}^{des}$	E^{des}
EXP I	2200/0	0/0	0	0.114	n.a	n.a
EXP II	2200/0	6/0	0	0.114	0	n.a
EXP III	2200/0	6/0	20	0.114	0	E_0 =2.9047

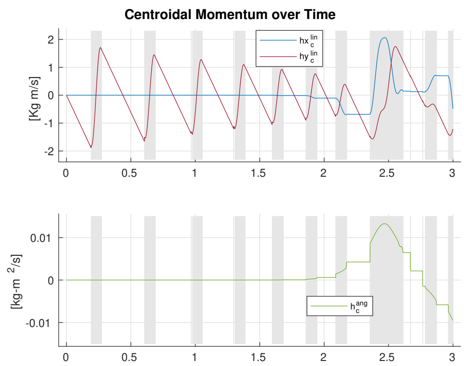
Table 4.8.: Control parameters and commanding values used for the different experiments of the prismatic leg. The polar gains are active only during stance, whereas the radial and energy gains are active during both, stance and flight. No damping is used in any experiment.

Experiment I - Under Actuated Hopping in Place

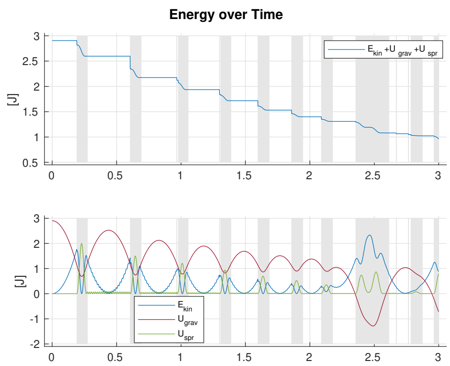
The first experiment examines, how long the foot remains upright without polar control and without energy control (see Tab. 4.8 for control parameters). The results are depicted in the Figures (4.3). Figure (4.3a) displays the hopping pattern of the center of mass. One can clearly see that the leg starts to tilt after some time due to simulation noise. This happens after 7 jumps (see Fig. 4.3d). The result is to be expected, since the upright leg without polar control is in an unstable equilibrium.



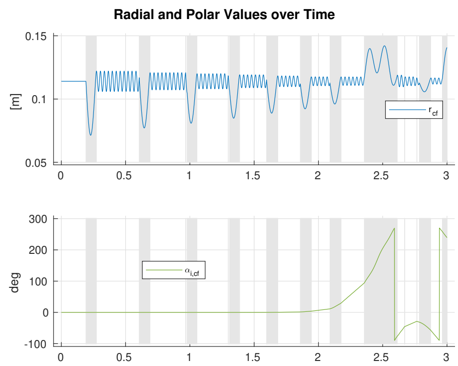
(a) Hopping pattern of the Center of Mass



(b) Linear and angular momentum



(c) Kinetic, gravity and spring energy



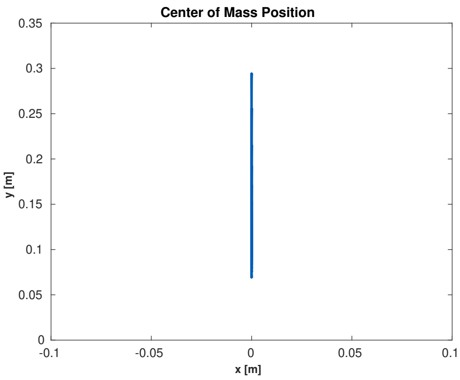
(d) Radial and polar quantities

Figure 4.3.: Results from experiment I of the prismatic leg for under-actuated control without energy controller. The the leg starts to tilt after 7 hops due to disturbances arising from simulation noise, since it is in an unstable equilibrium.

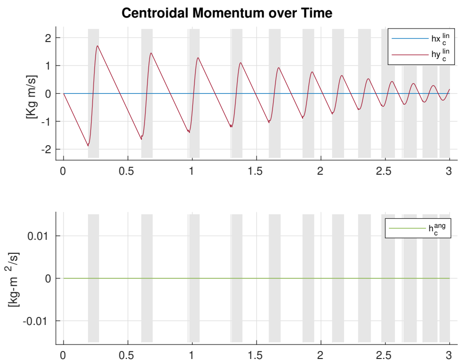
Experiment II - Fully Actuated Hopping in Place

In experiment II, we switch on polar control during stance, in order to stabilize the otherwise unstable polar dynamics around the vertical. A stiffness gain of 6 N/m and no damping was used (see Tab. 4.8 for control parameters).

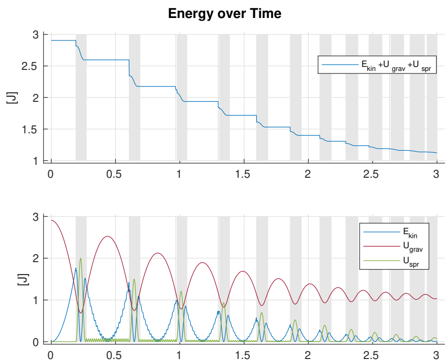
The results for this experiment are displayed in the Figures (4.4). The polar controller is able to stabilize the leg, such that it remains upright (see Fig. 4.4a). The energy (see Fig.4.5c) and the vertical momentum (see Fig. 4.4d) decrease due to dissipation in the ground. This leads to a reduction of the hopping height after each jump and eventually to a final rest position after 12 jumps. This behavior can be mitigated by reintroducing the energy lost at each jump by using an energy regulator and is done in the Experiment III.



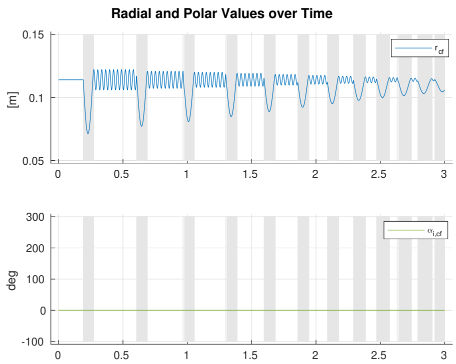
(a) Hopping pattern of the Center of Mass



(b) Linear and angular momentum



(c) Kinetic, gravity and spring energy



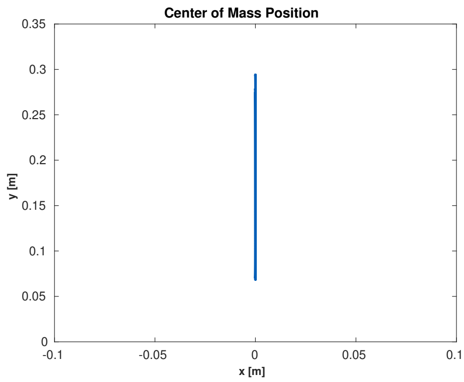
(d) Radial and polar quantities

Figure 4.4.: Results from experiment II of the prismatic leg: Fully actuated hopping in place without energy control. The leg successfully hops in place, but looses energy.

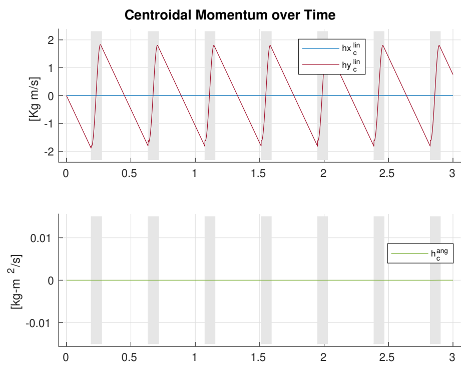
Experiment III - Fully Actuated Hopping in Place with Energy Recovery

In experiment III the energy controller is switched on with a gain $K_E = 20$ (see Tab. 4.8 for control parameters). The commanding value for the energy is the initial energy. The energy controller regulates the total mechanical energy E. The control effort is proportional to the velocity of the joint. The energy controller operates during both, stance and flight.

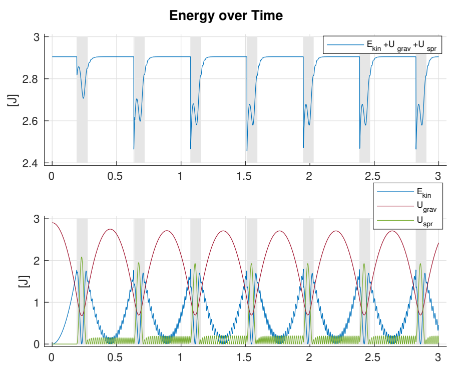
It is to be expected that during flight the controller only increases the average energy of the spring, since it has nothing to push against. The results with a proportional term of $K_E = 20$ for the energy controller are depicted in the figures (4.5). The energy controller is able to recover the desired level of energy after each jump (see Fig. 4.5c). However, it is not able to recover the energy fully during stance. A bit of energy recovered during flight, is stored in the spring. The average spring energy during flight increases slightly from hop to hop (see Fig. 4.5c).



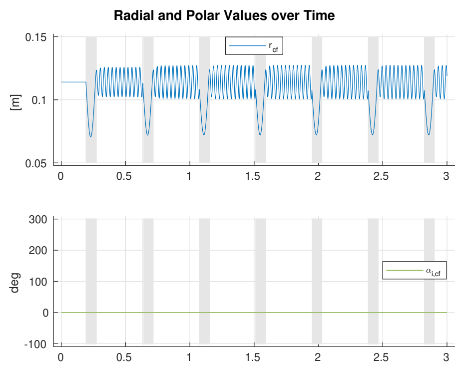
(a) Hopping pattern of the Center of Mass



(b) Linear and angular momentum



(c) Kinetic, gravity and spring energy



(d) Radial and polar quantities

Figure 4.5.: Results from experiment III of the prismatic leg: Fully actuated hopping in place with energy control. The energy regulator is able to maintain the desired level of energy. The energy is not fully recovered during stance, leading to a slight upward trend of the average spring energy during flight.

4.3. Simulation of a Revolute Hopper in the Plane

The rotational leg consists of a point trunk, a tight, a shank and a mass less point foot. Between the shank and the thigh a knee is modeled as a revolute joint (see Fig. 4.6). The length of both, thigh and shank are $12cm$ (see Tab. 4.9). In the middle between the trunk and the knee is a thigh point-mass and in the middle between the knee and the foot is a shank point-mass (see Tab. 4.10). The angle of the base is measured from the inertial vertical to the thigh link. The joint angle is measured from the thigh to the shank.

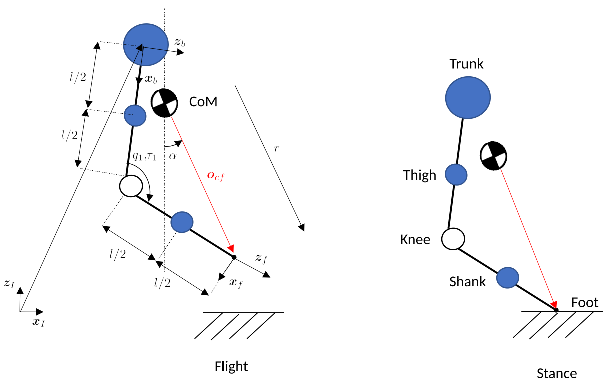


Figure 4.6.: Drawing of the rotational leg

4.3.1. Setup and Simulation Parameters

The kinematic parameters (see Tab. 4.9) and dynamic inputs (see Tab. 4.10) are fed into the Symbolic Lee Algebra Toolbox described in Appendix A.2, which calculates the dynamic matrices and the Jacobians for control. The length of both, the thigh and the shank is $l = 12cm$.

Joint	$\boldsymbol{o}^x_{i-1,i}(q_i)$	$\boldsymbol{o}^y_{i-1,i}(q_i)$	$\theta_{i-1,i}(q_i)$
1	$l = 0.12m$	0	q_1
foot	$l = 0.12m$	0	0

Table 4.9.: Kinematic parameters for the rotational leg

Joint	n_j	m_{jci}	$\boldsymbol{o}^x_{j,jci}$	$\boldsymbol{o}^y_{j,jci}$	I_{jci}
0	1	$m_{00} = 0.835kg$	0	0	0
0	1	$m_{01} = 0.122kg$	$l/2 = 0.06m$	0	0
1	1	$m_{11} = 0.05kg$	$l/2 = 0.06m$	0	0

Table 4.10.: Dynamics parameters for the rotational leg

4.3.2. Simulation Results for Revolute Hopper

The rotational leg with only one joint is a minimalistic model of a leg. During flight it has no means to influence the angular momentum and therefore cannot position the leg to a desired angle of attack before touch down . In the rigid-body model used herein, the foot is a point and therefore cannot exert a torque during stance, since there is no ankle joint. There is no means to control the orientation and the angular momentum of the leg.

The revolute hopper has 4 configuration variables $(\boldsymbol{o}^x_b, \boldsymbol{o}^y_b, \theta_b, q_1)$ and 4 velocities $(\boldsymbol{v}^x_b, \boldsymbol{v}^y_b, \omega_b, \dot{q}_1)$. The parameters for the viscous-elastic floor are kept constant during all simulations (see Tab. 4.11).

Direction	K_{floor}	D_{floor}
x	10000	1000
y	10000	1000

Table 4.11.: Floor parameters used for all experiments of the rotational leg

The control parameters and the commanding values used in the different experiments are summarized in Table (4.12).

Experiment	K_r/D_r	K_p/D_p	K_E	r^{des}_{cf}	$\alpha^{des}_{i,cf}$	E^{des}
EXP I	2200/0	0/0	0	$r_{cf}(0) = 0.1411$	n.a	n.a
EXP II	2200/0	6/0	0	$r_{cf}(0) = 0.1411$	0	n.a
EXP III	2200/0	6/0	20	$r_{cf}(0) = 0.1411$	0	$E(0) = 2.8382$
EXP IV	2200/20	6/0	20	$r_{cf}(0) = 0.1411$	0	$E(0) = 2.8382$
EXP V	2200/20	6/0	20	$r_{cf}(0) = 0.1411$	0	$E(0) = 2.8382$
EXP VI	2200/20	6/0	20	$r_{cf}(0) = 0.1411$	0	$E(0) = 2.8382$

Table 4.12.: Summary of the control parameters and of the commanding values for the simulations of the revolute hopper. The polar gains are active during stance only, whereas the radial and energy gains are active during both, stance and flight. Experiments IV-VI use radial damping during flight only.

The angular momentum after lift-off is a crucial parameter for the behavior of the leg. Since the leg has no means to influence the angular momentum during flight, the angular momentum after lift-off has to be very small, in order not to rotate the leg too much during flight. There are several parameters that influence the angular momentum after lift-off:

- The trajectory of $\alpha_{i,cf}$ during stance
- The trajectory of r_{cf} during stance
- The viscous-elasticity of the floor

The tuning strategy employed was to fix $\alpha_{i,cf} = 0$, such that the line from the CoM to the foot is vertical and to fix the initial height of the base at $\boldsymbol{o}_{ib}^{y} = 0.3$. Then the initial joint angle q_1 was varied and the angular momentum after the first take-off was measured. The result is depicted in Figure (4.7). The minimum angular momentum is at the initial joint angle $q_1 = 102deg$. This value is selected as an initial condition for all simulations (see Tab. 4.13).

Experiment	$\alpha_{i,cf}(0)[deg]$	$\boldsymbol{o}_b^x$	$\mathbf{0}_b^y$	θ_{ib}	q_1	vx_b	vy_b	ω_{ib}	$\dot{q}_1$
EXP I	0	0	0.3	-2.5174	1.7802	0	0	0	0
EXP II	0	0	0.3	-2.5174	1.7802	0	0	0	0
EXP III	0	0	0.3	-2.5174	1.7802	0	0	0	0
EXP VI	0	0	0.3	-2.5174	1.7802	0	0	0	0
EXP V	30	0	0.3	-1.9938	1.7802	0	0	0	0
EXP VI	-0.4	2.7676	0.2574	-2.5243	1.7802	-0.1919	0.1263	0.3301	0

Table 4.13.: Initial conditions used for the simulations of the rotational leg

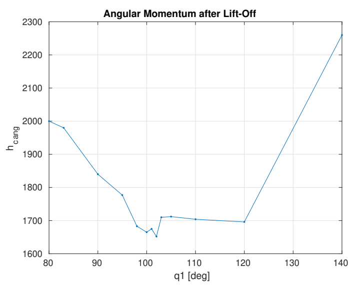
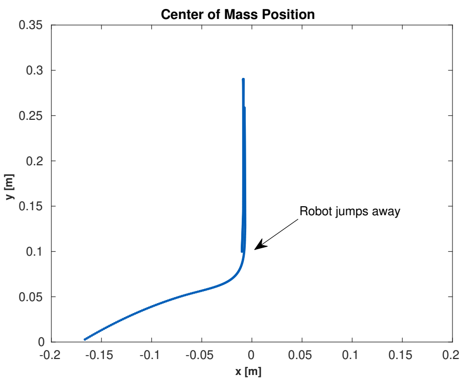


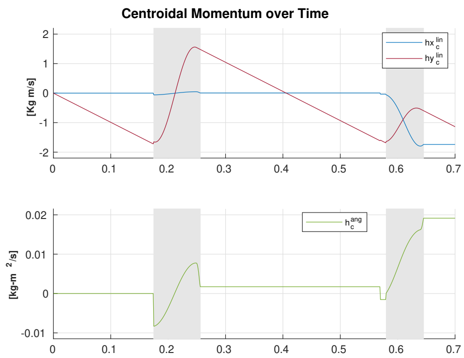
Figure 4.7.: Parameter tuning for the rotational leg based on simulations. The response of the angular momentum after the first lift-off as a function of the initial joint angle q_1 is analyzed and shown in the Figure. The angle $\alpha_{i,cf} = 0$ and the fall-off height $\boldsymbol{o}_{ib}^{y} = 0.3$ are fixed. The minimum value of the angular momentum at $q_1 = 102deg$ is taken as the initial condition for all experiments.

Experiment I - Under Actuated Hopping in Place

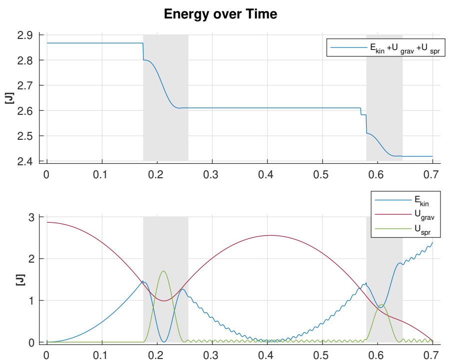
The first experiment examines how long the leg remains upright without polar control. In this experiment energy control is switched off (i.e. $K_E = 0$). The only active controller is the radial spring. The results are shown in Figure (4.8). After the first lift-off the leg obtains a positive angular momentum (see Fig. 4.8b), which is high enough to rotate the com-foot line r_{cf} by about 30 degrees until the next touch down (see Fig. 4.8d). Therefore it cannot make another hop, but jumps away.



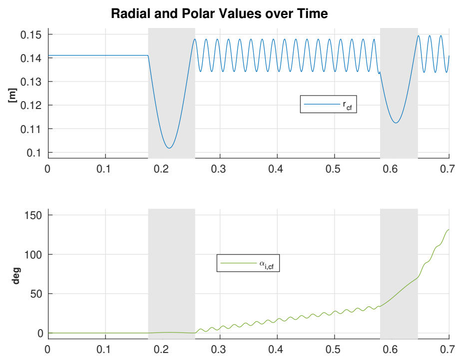
(a) Hopping pattern of the Center of Mass



(b) Linear and angular momentum



(c) Kinetic, gravity and spring energy

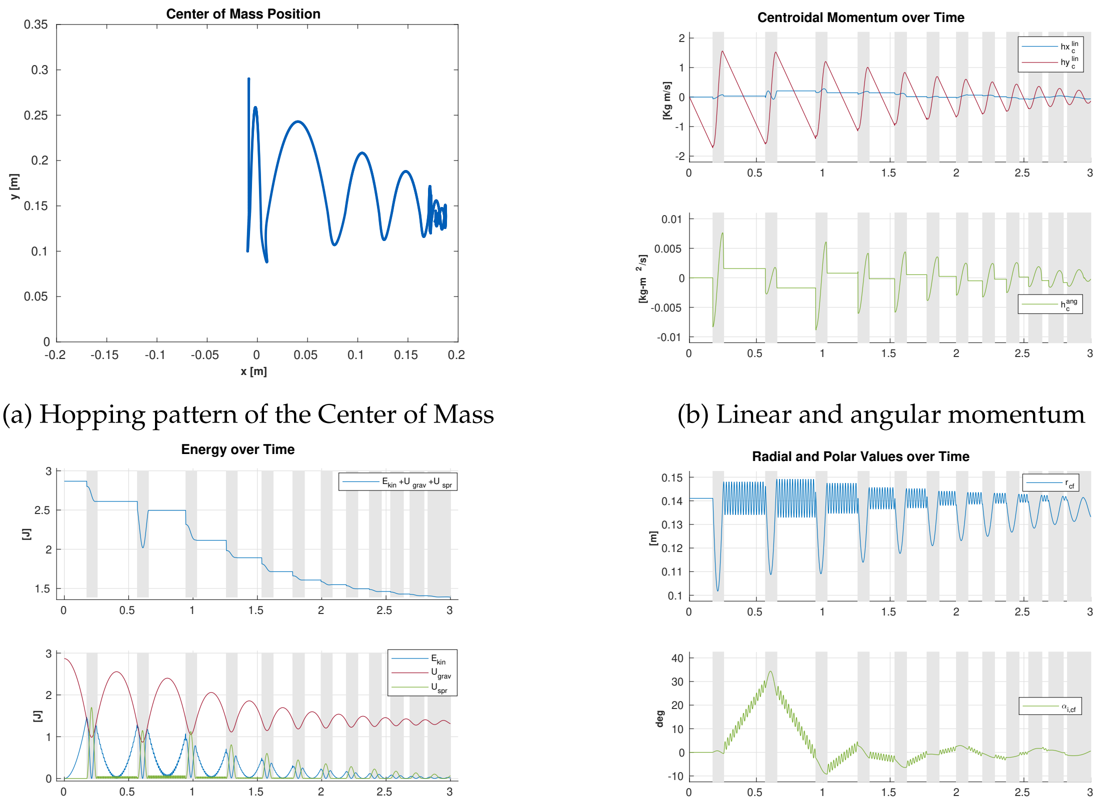


(d) Radial and polar quantities

Figure 4.8.: Results from Experiment I of the rotational leg: Under-actuated control without energy control. At the fist lift-off, the leg gets a positive angular momentum, which rotates the leg during flight by $30\,deg$. As a result, at the second touch down it cannot make another hop, but drifts away.

Experiment II - Fully Actuated Hopping in Place without Energy Control

In experiment II, polar controller II is used to bring the orientation of the foot-CoM axis to its initial value of 0. The polar controller uses a gain of 6 Nm/rad and no damping. No energy control is used (i.e. $K_E = 0$).



(a) Hopping pattern of the Center of Mass

(b) Linear and angular momentum

(c) Kinetic, gravity and spring energy

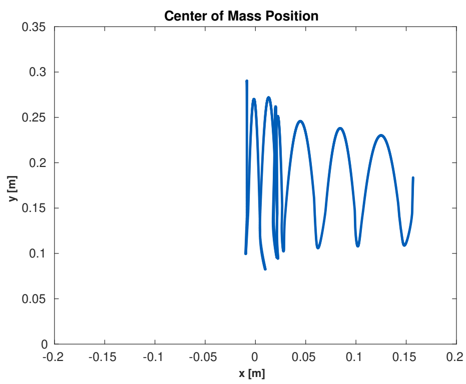
(d) Radial and polar quantities

Figure 4.9.: Results from Experiment II of the rotational leg: Fully actuated control without energy control. The leg makes a relatively stable forward movement. The reason for the stability, is the alternating sign of the angular momentum from hop to hop. While such a drifting hopping might be acceptable sometimes, it has the drawback that the drift is not predictable and the motion does not look natural.

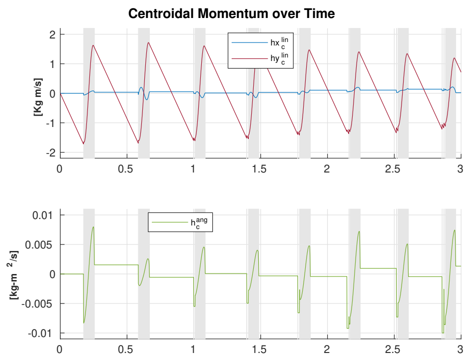
The results are depicted in Figure (4.9). Figure (4.9a) shows a somewhat stable hopping gait with forward drift. This indicates that the use of the polar springs helps stabilizing the hopping, but still alone is not sufficient to achieve hopping in place for a revolute robot. The polar control is not able to drive the angle to the desired value of 0 (see Fig. 4.8d). The reason for the relative the stability is the change of sign of the angular momentum from hop to hop (see Fig. 4.9b). While such a gait might be a viable strategy for hopping in place, it is not a natural movement for hopping forward.

Experiment III and IV - Fully-Actuated Hopping in Place with Energy Control

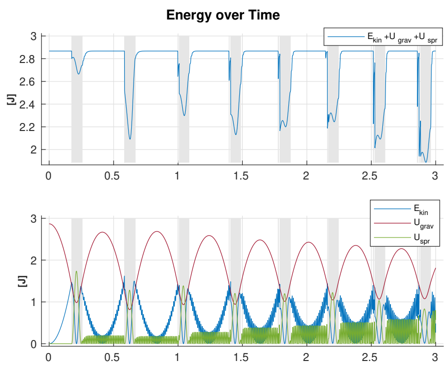
For experiment III energy control is activated with a gain of $K_E = 10$. The results are displayed in Figure (4.10).



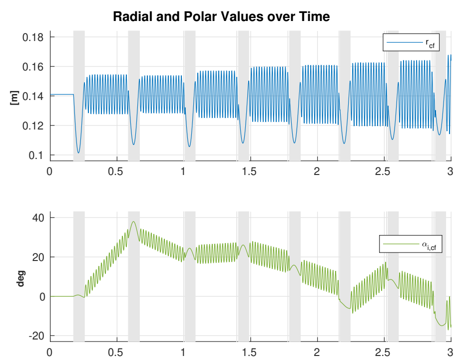
(a) Hopping pattern of the Center of Mass



(b) Linear and angular momentum



(c) Kinetic, gravity and spring energy



(d) Radial and polar quantities

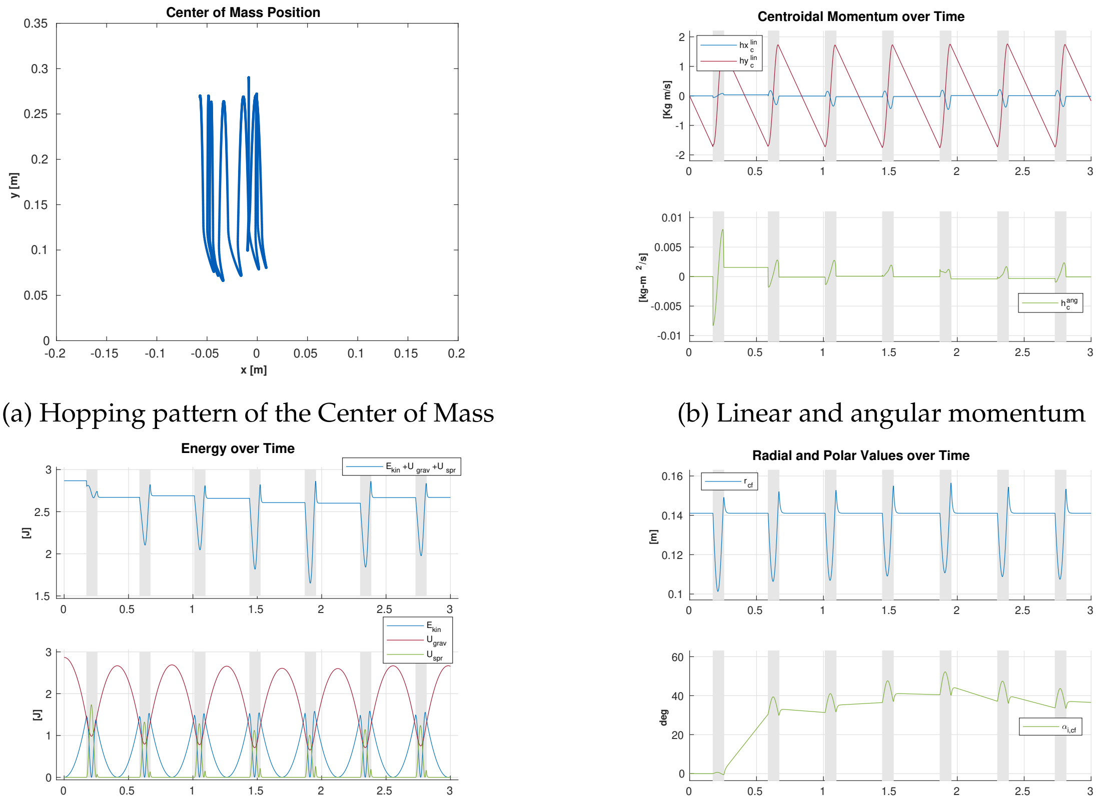
Figure 4.10.: Results from Experiment III of the rotational leg: Fully actuated control with energy control. The energy controller is able to maintain the level of energy. The leg makes a relatively stable forward movement. The reason for the stability is the alternating sign of the angular momentum from hop to hop. While such a gait might be a viable strategy for hopping in place, it is not a natural movement for hopping forward.

The energy controller is capable of maintaining the initial level of energy (4.10c) and the relatively stable gait remains (4.10a). As in experiment II, the reason for relative the stability arises from a gait, which changes sign of the angular momentum during flight from hop to hop (4.10b). While such a gait might be a viable strategy for hopping place, it is certainly not natural for hopping forward.

In the previous experiments there were strong vibrations in the virtual radial spring during flight, leading to oscillations in the knee. This is not desirable, because it is not a behavior observed in nature and because systematically finding gaits which are stable in the presence of such oscillations is difficult.

In experiment IV the radial damping is activated during flight ($D_r = 20$) to remove radial oscillations and to stabilize the touch down state. All other control parameters including initial conditions are the same in experiment III. The results are displayed in Figure (4.11). The controller successfully removes the vibrations in the spring, as well as in the spring energy (see Fig. 4.11d and Fig. 4.11c). Different to experiment III, the total mechanical energy during flight changes between jumps, but keeps well near the initial mechanical energy (see Fig. 4.11c). The pattern of changing sign of the angular momentum from experiment III is removed. (4.11b). Instead, the angular momentum during flight becomes so small that the leg rotates only around $3deg$ during flight (4.11d). This leads to a much stabler gait. The leg almost hops in place with a small negative horizontal velocity [1].

[1] Simulating for a longer time period shows that the movement converges to a stable gait with a small negative horizontal velocity.



(a) Hopping pattern of the Center of Mass

(b) Linear and angular momentum

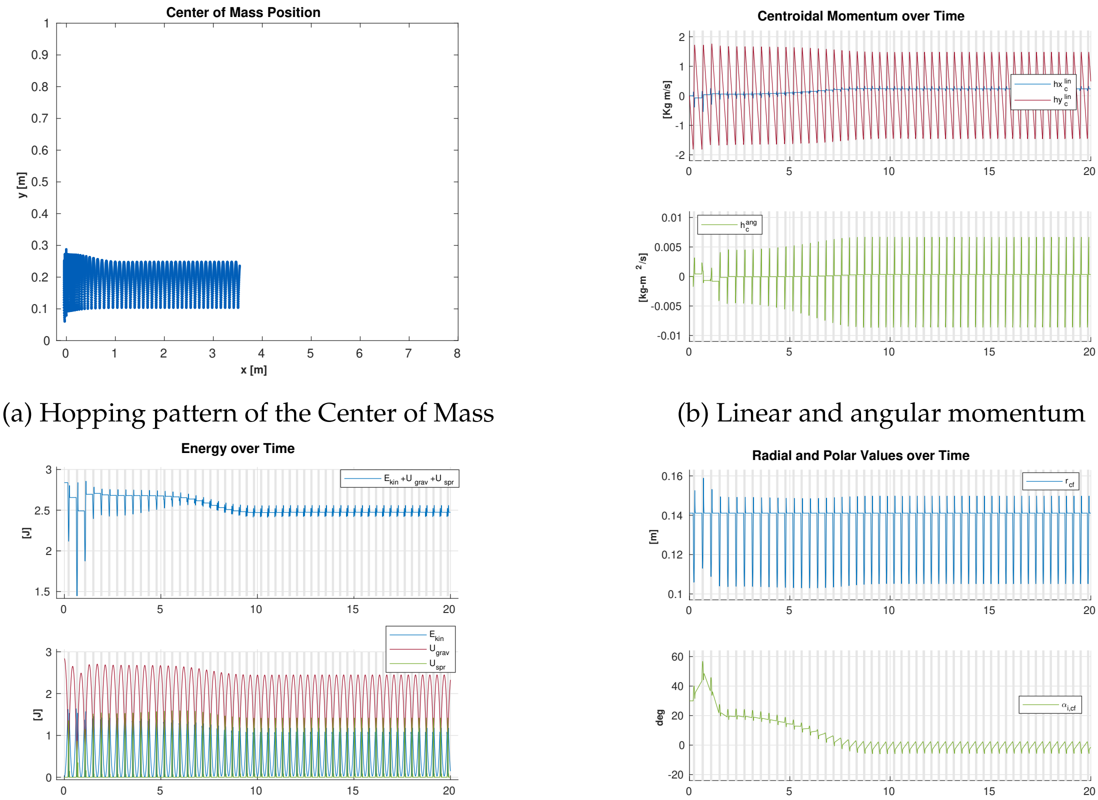
(c) Kinetic, gravity and spring energy

(d) Radial and polar quantities

Figure 4.11.: Results from Experiment IV of the rotational leg: Fully actuated control with energy control and radial damping. Full actuation together with radial damping minimizes the angular momentum during flight, such that the rotation in the flight phase is only about $3deg$. This is the basis for a stable gait. The leg almost hops in place with a small negative horizontal drift.

Experiment V and VI - Fully-Actuated Hopping in Forward with Energy Control

Motivated by the results of experiment IV, the goal was to empirically find a stable hopping forward gait. To this end, the same control parameters and commanding values were used as in experiment IV (see Tab. 4.12) and only the initial conditions were changed. The initial conditions from experiment IV were used as a starting point. Subsequently, the leg was tilted from the vertical and the resulting gaits were analyzed. Note that the commanding value for polar control was kept at $\alpha_{i,cf} = 0$. It turned out that an angle of attack of $\alpha_{i,cf} = 30deg$, measured from the vertical, resulted in interesting gaits (see Tab. 4.13 for initial conditions). The results are displayed in Figure (4.12).



(a) Hopping pattern of the Center of Mass

(b) Linear and angular momentum

(c) Kinetic, gravity and spring energy

(d) Radial and polar quantities

Figure 4.12.: Experiment V: Forward hopping of the rotational leg: Fully actuated control with energy control and radial damping. The initial chaotic movement converges to a stable gait. The level of energy is not maintained, but stabilizes as a lower level. After the transient phase, all states become stationary.

After the free-fall phase, the leg makes chaotic movements but self-stabilizes after a transient phase (see Fig. 4.12a). Such a behavior can also be observed in the SLIP model, where for a small range of initial angles of attack, the movement converges to a stationary gait with a smaller apex that the initial fall-off height [GBS02]. The level of energy is not maintained, but stabilizes at a lower level than the initial energy (see Fig. 4.12c). After the transient phase, all states become stationary (Fig. 4.12).

For a stable gait the leg must not accumulate to much rotation angle during flight as in experiment IV (see Fig. 4.11d). After each stance phase, the angular momentum has to be reset to a low level. Since the leg has no means to control the angular momentum during flight, the reset of the angular momentum has to happen during stance. In this case, this is achieved by a synchronized angular momentum during stance, which has a zero crossing in the middle of the stance phase (see Fig. 4.12b).

The transition from chaotic movement to the stable gait was not systematically analyzed. However, such a behavior is only possible, if the stable attractor has a considerably large basis of attraction. Therefore it can be assumed, that the occurrence of a cyclic gait is insensitive to initial conditions contained in the basin of attraction.

Experiment VI analyzes the stable gait quantitatively. The initial chaotic and transient phase was removed by using a point of the converged trajectory of experiment V as an initial condition for the new simulation. The initial conditions of the simulation of Experiment VI are used at time $t = 16.7010$, which corresponded to an apex of the center of mass.

The results are depicted in figures (4.13) and (4.14) for the whole simulation run of 20 seconds. The figures (4.15) and (4.16) show a zoom in at the end of the simulation. Apart from periodic bursts in the forces of the floor (see Fig. 4.14a), all figures show a stationary behavior. The horizontal linear momentum has a constant pattern, which is only slightly interrupted in the stance phase (see Fig. 4.15b). The interrupts are so small that the the horizontal center of mass position is almost linear (see Fig. 4.15a and Fig. 4.15a). This is similar to the SLIP model, expect that the pattern during stance is different. While the SLIP model has a U-shaped pattern, the horizontal linear momentum of the revolute hopper first undershoots and then overshoots. The pattern of overshooting and subsequently undershooting can also be observed in other quantities, like the energy (see Fig. 4.16e) or the spring length (see Fig. 4.15e). It could be a sign that the control action is not fully in sync with the movement of the leg.

The revolute hopper has important sources of physical energy losses. At touch down, the kinetic energy is dissipated by the floor. The loss at touch down cannot be avoided in any case. The touch-down loss can be approximated by the loss of kinetic energy at the time of touch-down. This measure is a lower bound for the touch down loss, since not all kinetic energy is necessarily dissipated by the floor in one integration step. The touch down loss of experiment VI amounts to $45mJ$ at each touch down.

The second type of loss is the dissipation of energy due to damping of the radial spring during flight. The amount of energy lost depends on the radial spring loading and on the velocity of the spring at lift-off. The radial dissipation loss cannot be avoided for a system with a shank mass, since the spring has to be loaded at lift-off, in order to be able to drag the shank upwards.

It is hard specify an optimum for the radial dissipation loss. However, assuming that all the energy of the radial spring at lift-off is converted to kinetic energy of the shank, the spring energy at lift-off must be at least as high as the gravity potential of the shank mass at its apex, which would correspond to $52mJ$. Since parts of the spring energy is dissipated, this value is a lower bound. In our case, the energy of the spring at lift-off is $80mJ$, corresponding to a an absolute spring extension of $9mm$ or 6% of the rest length (see Fig. 4.15e). Therefore, it seems that the lift-off loss reached, is close to the optimum.

Another source of energy loss is the dissipation of energy by the floor after touch down. It is not possible to calculate this loss directly, since the floor dissipation cannot reliably be integrated using the bursting time series of floor forces (see Fig. 4.14a).

Instead, the unknown floor dissipation after touch down is estimated as the residual between the known energy inflows and the known energy outflows, by means of a cumulative energy flow balance sheet. This analysis also allows to study the energy efficiency of the revolute hopper.

Due to energy conservation, the cumulative energy inflows have to be equal to the cumulative energy outflows at any point in time. The cumulative inflows of energy are the work performed by the external (angular) forces and the joint torques. They are calculated by integrating the power at the power ports of the external angular torque and the joint torque, respectively.

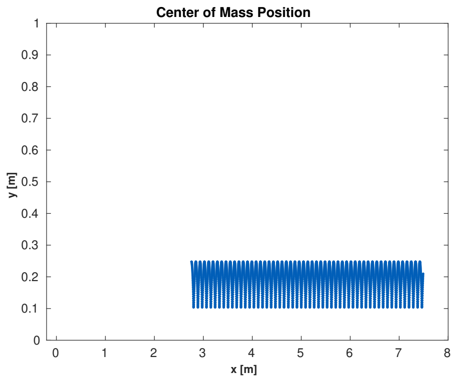
The known cumulative outflows of energy are given by the temporary storage of energy in the virtual springs, the touch down loss estimated above and energy dissipation in the radial controller.

Figure (4.14f) shows the cumulative energy inflows, the external work and the work of joint torques, together with the cumulative outflows of energy. For the whole 20 seconds run of the simulation, $7.15J$ and $4.65J$ of external and joint actuation work, respectively, were provided to the system, totaling $11.8J$. On the outflow side, the touch down losses amounted to $2.9J$, the radial dissipation losses amounted to $5.9J$, the storage of energy in the springs accumulates to zero and the continuous dissipation of energy by the floor after touch down amounted to $3.0J$, totaling $11.8J$ by definition. This analysis shows that the radial dissipation consumes roughly 50% of the energy provided and the touch down loss and the floor dissipation after touch down consumes another 25% each. The judgment whether the controller is energy efficient is a subjective one. However, given that we have a very small shank mass of only 5% of the total mass of the leg, the amounts of energy lost per hop are small relative to the kinetic and gravity potential of the system (see Tab. 4.13). The loss per hop is about 7% of the initial energy of the system. To this end, the control mechanism can be regarded as efficient.

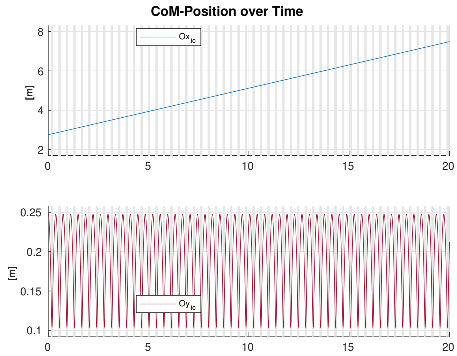
In addition to the results shown here, the effects of using commanding angles for the polar controller different from zero were analyzed. It turned out that commanding values only slightly different from zero ($2 - 3deg$), resulted in large angular momenta during flight, leading to unstable gaits. Therefore, the polar controller should not be looked at as a tracking controller steering to a desired angle at lift-off. Instead, it should be regarded as a stabilizing element, which minimizes the angular momentum and allows the radial controller to perform a stable gait.

In summary, the fully actuated revolute hopper performs stable gaits, which look natural and have an almost linear horizontal angular momentum like the SLIP model. It produces near optimal lift-off losses. It has desirable confined control actions and it is rather insensitive to initial conditions. The price paid herein was the required radial damping during flight, which led to additional energy losses and the requirement of a foot or multiple legs for employing full actuation.

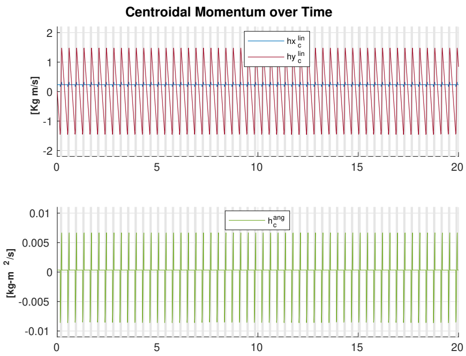
The important question for under-actuated control is: What is the stabilizing element replacing full actuation and what does this imply for the design of the robot. Put differently: What is the minimum viable leg design, which can be expected to be able hop naturally with under-actuated control and how to find under-actuated gaits?



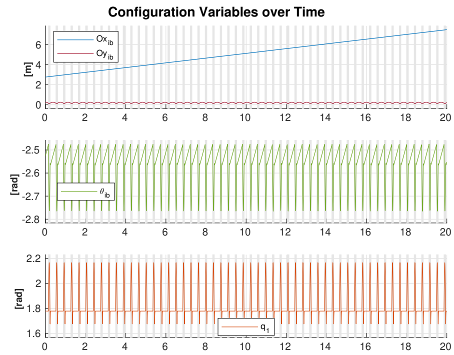
(a) Hopping pattern of the Center of Mass



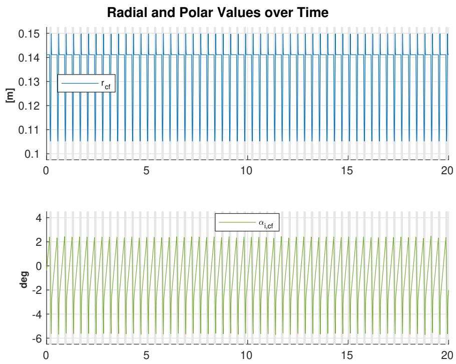
(b) Center of Mass over time



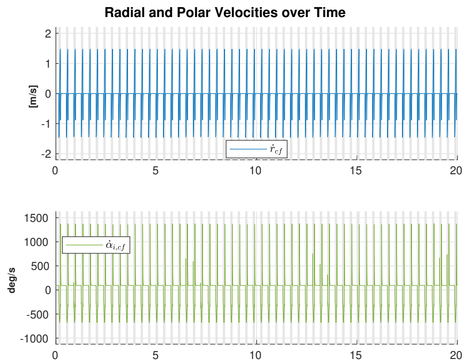
(c) Linear and angular momentum



(d) Configuration Variables

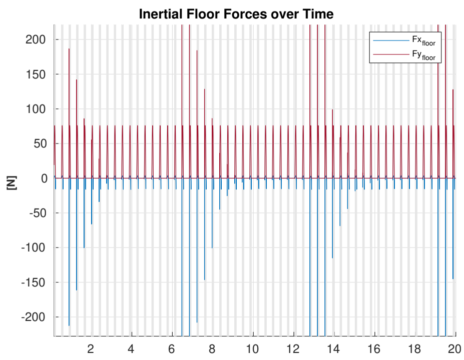


(e) Radial and Polar Values

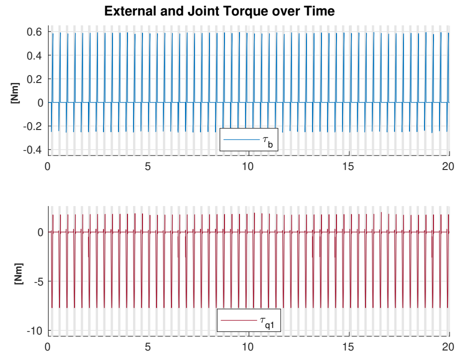


(f) Radial and Polar Velocities

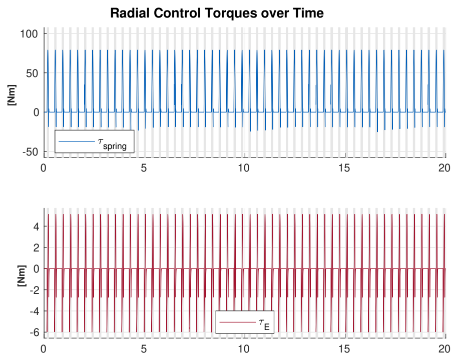
Figure 4.13.: Experiment VI: Forward hopping of the rotational leg: Fully actuated control with energy control and radial damping.



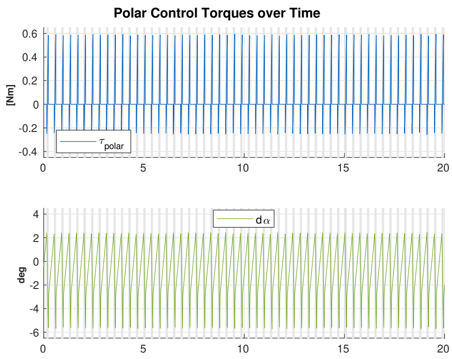
(a) Inertial floor forces



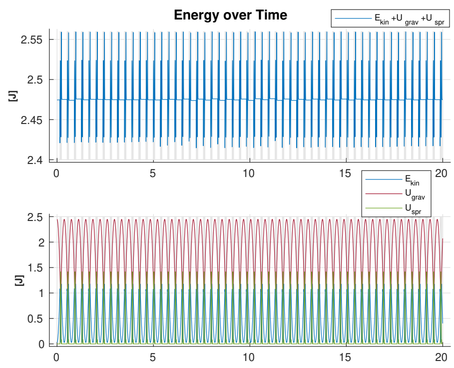
(b) External and Joint Torques



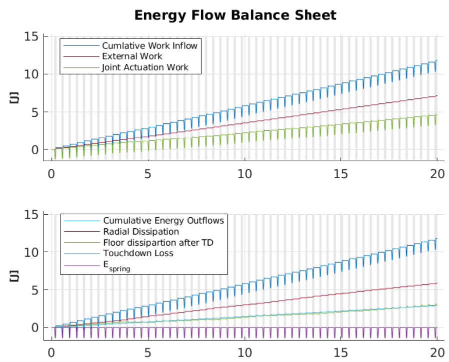
(c) Radial Control



(d) Polar Control

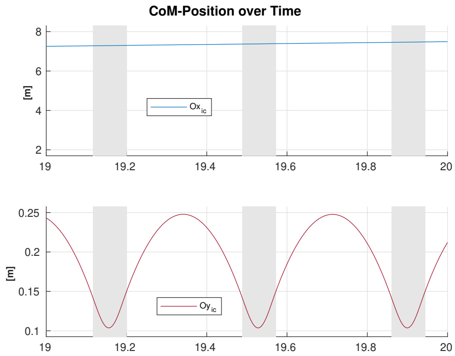


(e) Total mechanical energy and its components

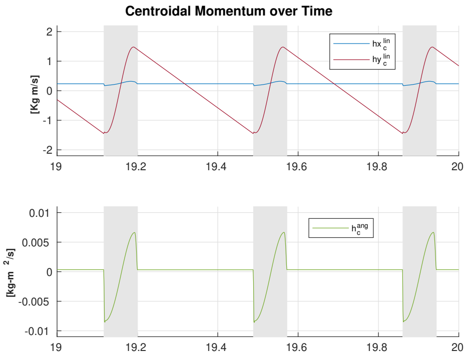


(f) Cumulative energy inflows compared to losses

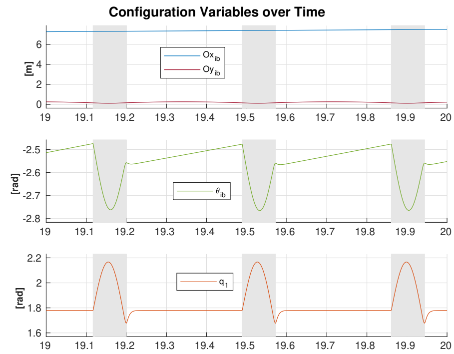
Figure 4.14.: Experiment VI: Forward hopping of the rotational leg: Fully actuated control with energy control and radial damping



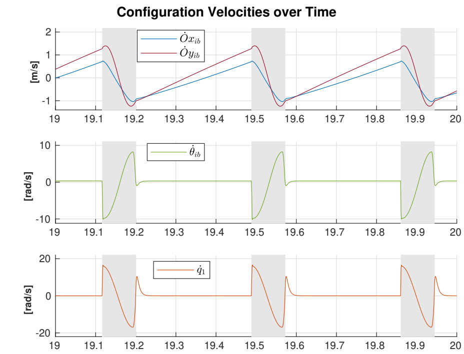
(a) Center of Mass over time



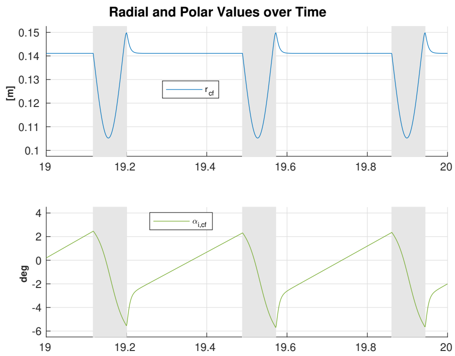
(b) Linear and angular momentum



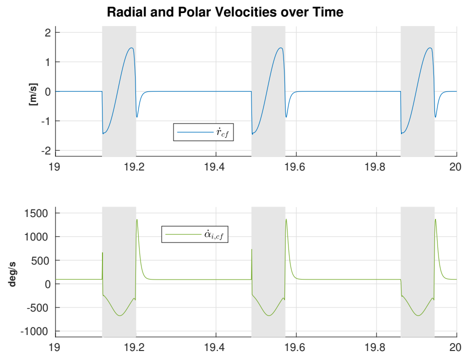
(c) Configuration Variables



(d) Configuration Velocities

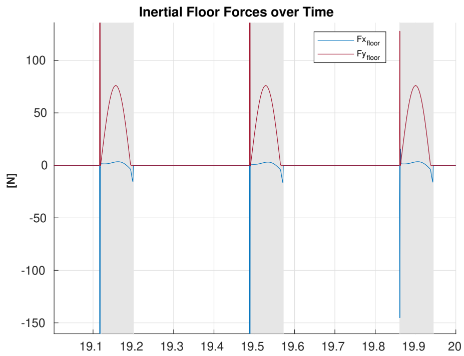


(e) Radial and Polar Values

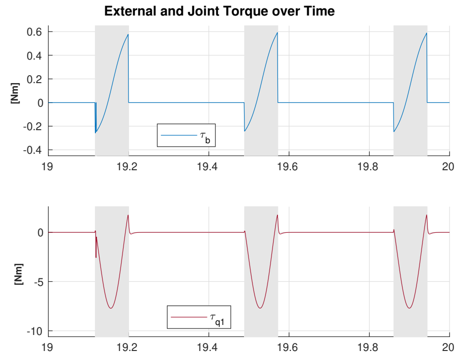


(f) Radial and Polar Velocities

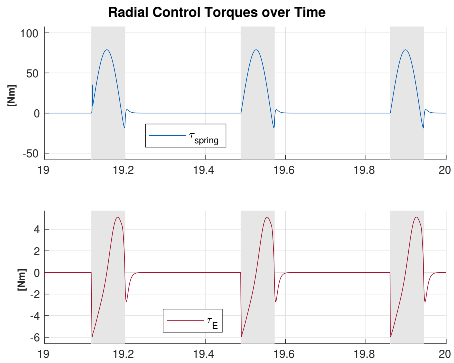
Figure 4.15.: Zoom Experiment VI: Forward hopping of the rotational leg: Fully actuated control with energy control and radial damping



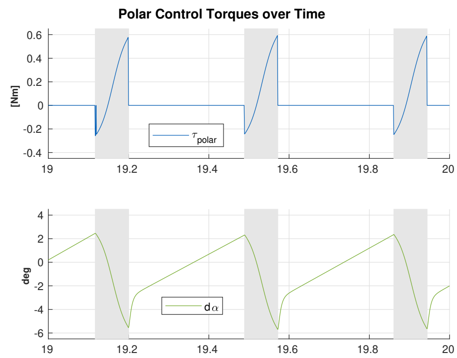
(a) Inertial floor forces



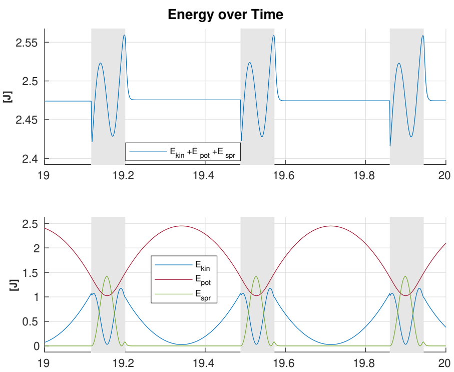
(b) External and Joint Torques



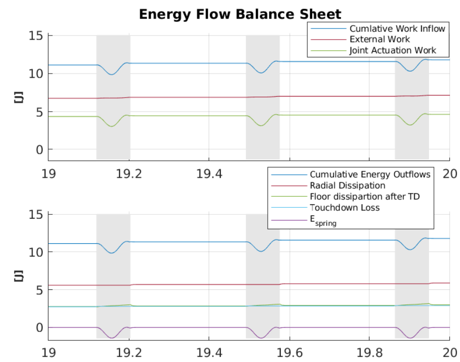
(c) Radial Control



(d) Polar Control



(e) Total mechanical energy and its components



(f) Cumulative energy inflows compared to losses

Figure 4.16.: Zoom Experiment VI: Forward hopping of the rotational leg: Fully actuated control with energy control and radial damping

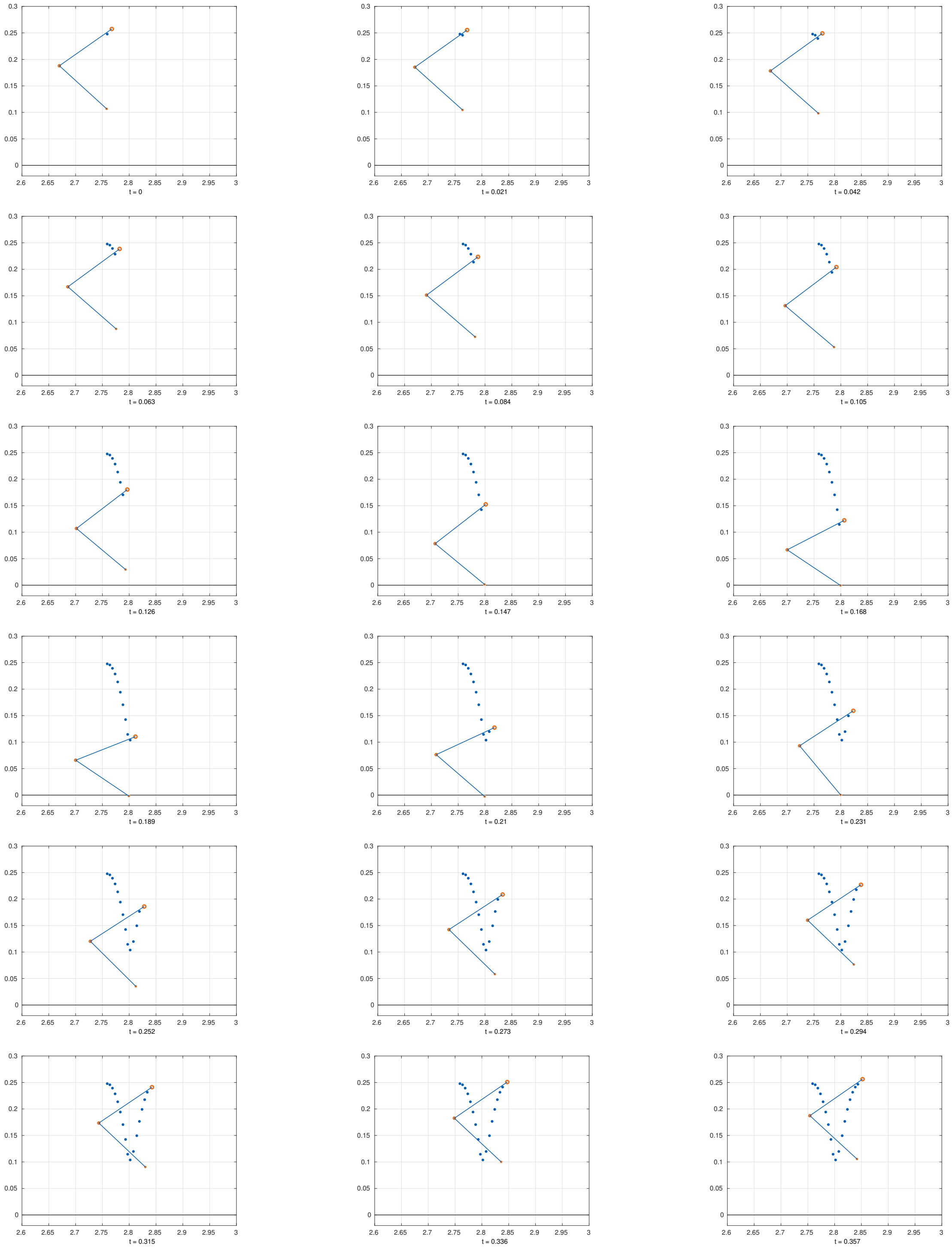


Figure 4.17.: Experiment VI: Screenshots of a hopping cycle

5. Conclusion

The goal of this work was to systematically find decompositions of the system dynamics into external and internal components and to use those decompositions for efficient control strategies.

Using Hamel equations derived for the manifold $SE(3) \times \mathbb{R}^n$, we systematically derived all transformations, which decouple the motion of the base and the motion of the joints. It turns out that any transformation, which decouples the base wrench from the joints torques and simultaneously diagonalizes the mass matrix, decouples the dynamics. Employing a transformation, which results in a constant of motion of the transformed system, leads to an invariance structure. These results hold only for the Bolzmann-Hamel equations, which have a non-passive Coriolis matrix. Therefore, a passive formulation was derived and the conditions for decoupling were analyzed. The passive formulation found is the same as the one derived using the Euler-Newton method. In the passive case, the system does not decouple under any transform and shows invariance, when the transformation of the base velocity results in a constant of motion.

It might be worth investigating this further trying to find a different passive formulation. However, passive decoupling, i.e vanishing of the upper off-diagonal Coriolis block element, required complete decoupling of the base and of the joints, since the off-diagonal blocks of a passive Coriolis matrix are the negative transpose of each other.

The results of decoupling have been applied to hopping robots. Mono-legs robots with one prismatic or one revolute joint were considered.

Impedance control was used to control the system. It consisted of virtual radial springs between the foot and the center of mass and polar springs at the center of mass. In addition an energy controller was used to recover energy losses. For polar control two different controllers are considered. The fist controller assumes a center of mass frame aligned to the inertial frame and controls the inertial orientation of the radial spring. The latter controls the orientation of the center of mass frame aligned to the locked velocity. So far, results were provided for the first type of polar control only.

For the prismatic leg, stable fully actuated hopping in place was demonstrated. Under-actuated hopping in place is unstable by design. Under-actuated and fully-actuated hopping forward with a prismatic leg is devoted to future work.

For the revolute leg, stable fully actuated hopping forward was demonstrated by using radial damping during flight. The occurrence of stable gaits is rather insensitive to initial conditions, all measures are fully stationary and the control actions are confined to desirable values. The gaits look natural with an almost linear horizontal momentum. It turns out that the radial controller acts as the stabilizing element allowing the polar controller to converge to stable gaits. For under-actuated hopping forward, a leg design with elements, that can play the role of the stabilizing element, has to be considered.

To this end, important questions are: How to find stable hopping-forward gaits considering under-actuation during stance? What is the minimum viable leg design, which can be expected to be able hop naturally with under-actuated control. These questions will be addressed in future work.

A. Appendix

A.1. Transformation Relations Equations of Motion

Here we derive expressions for the transformed system matrices $\boldsymbol{M}_T$ $\boldsymbol{C}_T$ for a given Mass matrix and Coriolis matrix the using the general decoupling transform T from (3.80). The mass matrix and the Coriolis matrix have form:

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{bb} & \boldsymbol{M}_{bq} \\ \boldsymbol{M}_{bq}^T & \boldsymbol{M}_{qq} \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_b & \boldsymbol{C}_{bq} \\ \boldsymbol{C}_{qb} & \boldsymbol{C}_q \end{bmatrix} \tag{A.1}$$

The transformation is (3.80):

$$\begin{aligned} \boldsymbol{T} &= \begin{bmatrix} \boldsymbol{T}_x & \boldsymbol{T}_x \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq} \\ \boldsymbol{0} & \boldsymbol{T}_y \end{bmatrix} & \boldsymbol{T}^T &= \begin{bmatrix} \boldsymbol{T}_x^T & \boldsymbol{0} \\ \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} \boldsymbol{T}_x^T & \boldsymbol{T}_y^T \end{bmatrix} \\ \boldsymbol{T}^{-1} &= \begin{bmatrix} \boldsymbol{T}_x^{-1} & -\boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq} \boldsymbol{T}_y^{-1} \\ \boldsymbol{0} & \boldsymbol{T}_y^{-1} \end{bmatrix} & \boldsymbol{T}^{-T} &= \begin{bmatrix} \boldsymbol{T}_x^{-T} & \boldsymbol{0} \\ -\boldsymbol{T}_y^{-T} \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} & \boldsymbol{T}_y^{-T} \end{bmatrix} \end{aligned} \tag{A.2}$$

The transformation rules 3.77 are:

$$\begin{aligned} \boldsymbol{M}_T &= \boldsymbol{T}^{-T} \boldsymbol{M} \boldsymbol{T}^{-1} \\ \boldsymbol{C}_T &= \boldsymbol{T}^{-T} \boldsymbol{C} \boldsymbol{T}^{-1} - \boldsymbol{M}_T \dot{\boldsymbol{T}} \boldsymbol{T}^{-1} \end{aligned}$$

The derivative $\dot{\boldsymbol{T}}$ is given by:

$$\dot{\boldsymbol{T}} = \begin{bmatrix} \dot{\boldsymbol{T}}_x & \dot{\boldsymbol{T}}_{xy} \\ 0 & \dot{\boldsymbol{T}}_y \end{bmatrix}$$

$$\dot{\boldsymbol{T}}_{xy} = \dot{\boldsymbol{T}}_x \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq} + \boldsymbol{T}_x(\dot{\boldsymbol{M}}_{bb}^{-1} \boldsymbol{M}_{bq} + \boldsymbol{M}_{bb}^{-1} \dot{\boldsymbol{M}}_{bq}) \tag{A.3}$$

By plugging in the mass matrix, Coriolis matrix and the transformations, a simple but tedious calculation yields:

$$\boldsymbol{M}_T = \begin{bmatrix} \boldsymbol{M}_x & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_y \end{bmatrix} \qquad \boldsymbol{C}_T = \begin{bmatrix} \boldsymbol{C}_x & \boldsymbol{C}_{xy} \\ \boldsymbol{C}_{yx} & \boldsymbol{C}_y \end{bmatrix}$$

with the block matrices given by:

$$\boldsymbol{M}_x = \boldsymbol{T}_x^{-1} \boldsymbol{M}_{bb} \boldsymbol{T}_x^{-T} \tag{A.4a}$$
$$\boldsymbol{M}_y = \boldsymbol{T}_y^{-T} (\boldsymbol{M}_{qq} - \boldsymbol{M}_{bq}^T \boldsymbol{M}_{bb}^{-1} \boldsymbol{M}_{bq}) \boldsymbol{T}_y^{-1} \tag{A.4b}$$

and

$$\boldsymbol{C}_x = \boldsymbol{T}_x^{-T}(\boldsymbol{C}_b - \boldsymbol{M}_{bb}\boldsymbol{T}_x^{-1}\dot{\boldsymbol{T}}_x)\boldsymbol{T}_x^{-1} \tag{A.5a}$$

$$\boldsymbol{C}_{xy} = \boldsymbol{T}_x^{-T}(\boldsymbol{C}_{bq} - \boldsymbol{C}_b\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq} - \boldsymbol{M}_{bb}\dot{\boldsymbol{M}}_{bb}^{-1}\boldsymbol{M}_{bq} - \dot{\boldsymbol{M}}_{bq})\boldsymbol{T}_y^{-1} \tag{A.5b}$$

$$\boldsymbol{C}_{yx} = \boldsymbol{T}_y^{-T}(\boldsymbol{C}_{qb} - \boldsymbol{M}_{bq}^T\boldsymbol{M}_{bb}^{-1}\boldsymbol{C}_b)\boldsymbol{T}_x^{-1} \tag{A.5c}$$

$$\boldsymbol{C}_y = \boldsymbol{T}_y^{-T}(\boldsymbol{C}_q - \boldsymbol{C}_{qb}\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq} + \boldsymbol{M}_{bq}^T\boldsymbol{M}_{bb}^{-1}(\boldsymbol{C}_b\boldsymbol{M}_{bb}^{-1}\boldsymbol{M}_{bq} - \boldsymbol{C}_{bq})\boldsymbol{T}_y^{-1} - \boldsymbol{M}_y\dot{\boldsymbol{T}}_y\boldsymbol{T}_y^{-1} \tag{A.5d}$$

A.2. Symbolic Lee Algebra Toolbox SE(2) x R^n

The toolbox implements many of the relations derived in Section 4.1. It is implemented in Matlab using the Matlab symbolic toolbox. The configuration variables and generalized velocities are implemented as functions of time. Therefore it is possible to take the symbolic time derivative of any matrix expression. The Lee Algebra toolbox allows to access elements of a time function matrix using indices. This is not possible in the Matlab symbolic toolbox. The function symmat converts a matrix of Matlab time functions to a matrix, which allows access to the elements. The notation of the Lee Algebra Toolbox is very much aligned with the notation used in this work. Therefore is is easy to understand and write code, if one is used to this notation.

One function worth mentioning is printsimmat, which converts symbolic matrices to matrices used in the numeric simulation. The converted matrices contain the state vector used in the simulation and are printed on screen. To this end, the simulation program does not contain complex code, but only function bodies containing the converted matrices. This makes the simulation program easily adaptable to different robots.

Function	Description
R_xy = rotationmatrix_R_xy(theta_xy)	generates 2×2 rotation matrix
H_xy = framepose_H_xy(theta_xy,O_xy)	calculates the rigid body transform $\boldsymbol{H}_{xy}$
O_xy = frameorigin_o_xy(H_xy);	gets the origin of the frame from H_{xy}
R_xy = frameorientaton_R_o_xy(H_xy)	gets the Rotation matrix out of H_{xy}
theta_xy = frameangle_theta_o_xy(H_xy)	gets the angle from H_{xy}
A_xy = Adjoint_A_xy(H_xy)	big adjoint matrix
a_xy = adjoint_a_xy(H_xy)	little adjoint matrix
a_xy = bodyvelocity2adjoint_a_xy(nu_xy)	little adjoint matrix from body velocity
Jodot_xy = Jacobian_Jodot_xy(O_xy,xq_ib)	Jacobian: $\dot{\boldsymbol{o}}_{xy} = \boldsymbol{J}_{\dot{\boldsymbol{o}}_{xy}} \boldsymbol{v}_b$
Jnu_xy = Jacobian_Jnu_xy(H_xy,xq_ib)	Jacobian: $\boldsymbol{\nu}_{xy} = \boldsymbol{J}_{\boldsymbol{\nu}_{xy}} \boldsymbol{v}_b$
J_ydot = Jacobian_Jydot(H_xy,xq_ib);	Jacobian: $\dot{\boldsymbol{y}}(\boldsymbol{o}_{ib}, \theta_{ib}, \boldsymbol{q}) = \boldsymbol{J}_{\dot{\boldsymbol{y}}} \boldsymbol{v}_b$
[Lam_j, mc_j, oc_j] = massmatrix(_Lamda_j(o_jci_mat,mj,Ij)	calculates the mass matrix for a link
[M, Res] = vec2mat(symVec,Vars)	calculates matrix M from column vector symVec: symVec = M * Vars + Res. The result is not unique
M12 = row12(M)	gets the first 2 rows from matrix M
printsimmat(M,symvarStr,simvarStr)	converts symbolic matrix into a matrix for numeric simulation.
M = symmat(X)	converts a symbolic time function to a symbolic matrix. The elements of M can be accessed by indices

Table A.1.: List of Functions in the Symbolic Lee Algebra Toolbox

A.2.1. Symbolic Sample Program

As an example for an application of the symbolic Lee Algebra Toolbox, the program that symbolically calculated all relation required for the revolute hopper is printed.

```
% symbolic_rotatinalTorsoLeg2D.m
%
% calculates the all relations required for simulation
% of a 2D rotational foot with 3 masses
```

```
% The com frame is aligned with locked velocity


clear all
close all

%% User input setup %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
nq = 1;              % number of link variables q
sim_statevar = 'y'; % name of state vector used in numeric simulation
%% run setup: do not touch %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% this setup is the same for all 2D floating robots with nq serial joints
% and nj masses and nj moments of Inertial per joint
%
nv = 3 + nq; % number of configuration variables
ns = 2* nv;  % number of states

% define symbolic variables  and time functions
syms  t g            % symbolic constants time, gravity constant

syms o_ib(t)     [2 1], syms  theta_ib(t), syms     q(t) [nq 1]
syms v_ib(t)     [2 1], syms      w_ib(t), syms  qdot(t) [nq 1]
syms vdot_ib(t)  [2 1], syms   wdot_ib(t), syms qddot(t) [nq 1]

assume(t,'real')
assume(g,'positive')

% collect variables
v0        = symmat([o_ib;    theta_ib;     q]);
v1        = symmat([v_ib;        w_ib;  qdot]);
v2        = symmat([vdot_ib; wdot_ib; qddot]);
nvar    = length(v0);    % number of variables

% collect variable names for substitution
% replace varsdiff     with varsdot
%         varsdotdiff  with varsddot
%         odot_ib      with R_ib * V_ib

```

```
varsdot     = symmat([rotationmatrix_R_xy(theta_ib)*v1(1:2);v1(3:end)]); % time deriv.
varsddot    = v2;  % twice time deriv.

varsdiff    = symmat(diff(v0,t));
varsdotdiff = symmat(diff(v1,t));

% variable conversion from sym to simulation
%
% the function printsimmat(Msym) replaces
% the symbolic time variables in Msym with
% the state vector entry used in the simulation.
% e.g
% nq = 1;
% sim_statevar = 'y';
% simvarStr = string(['y(1)';'y(2)';'y(3)';'y(4)';'y(5)';'y(6)';'y(7)';'y(8)'])
%
%  theta_ib(t) is replaced with y(3) and
%      w_ib(t) is replaced with y(5)

symvarStr  = string([v0;v1]);
simvarStr = strcat(sim_statevar,"(",string((1:ns)'),")");

% define  matrices
S = symmat([0,-1; 1,0]); % cross product matrix
R_ib = rotationmatrix_R_xy(theta_ib); % rotation from b to i

% Base frame vectors and tranformations
xq_ib  =  v0;              % vector of all configuration variables
nuq_ib =  v1;              % generalized velocities
nu_ib  =  nuq_ib(1:3);     % base twist vector

H_ib   = framepose_H_xy(theta_ib,o_ib);

%%% End setup %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% User Inputs %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% additional variables
syms l % length on links
assume(l,'positive')

% kinematic specification: link to link Transform variables
theta_b1 = q1; % link 1
    o_b1 = symmat([l;0]); % link 1

theta_1f = 0;  % link f
    o_1f = symmat([l;0]);  % link f

% dynamic specification: centers of masses in frame j
o_bc1  = [0;  0]; % 2x1 center of mass 1 in link b
o_bc2  = [l/2;0]; % 2x1 center of mass 2 in link b
o_1c1  = [l/2;0]; % 2x1 center of mass 1 in link 1

%% Mass matrices for Links
% link b
o_bci_mat =[o_bc1,o_bc2];  % 2 x nmb centers of mass i in link b

syms m0 I0 [size(o_bci_mat,2) 1]
assume(m0,'positive')
assume(I0,'positive')
I0 = I0 *0; % in our case  momentum of inertia of link b is 0

[Lam_b,mc_b,oc_b] = massmatrix_Lamda_j(o_bci_mat,m0,I0);

% link 1
o_1ci_mat = [o_1c1]; % 2 x nm1 centers of mass i in link b

syms m1 I1 [size(o_1ci_mat,2) 1]
assume(m1,'positive')
assume(I1,'positive')
I1 = I1 *0; % in our case  momentum of inertia of link 1 is 0

[Lam_1,mc_1,oc_1] = massmatrix_Lamda_j(o_1ci_mat,m1,I1);

%% H_xy Transformations
H_b1     = framepose_H_xy(theta_b1,o_b1);
H_1f     = framepose_H_xy(theta_1f,o_1f);

H_i1 = simplify(H_ib * H_b1);
```

```
H_if = simplify(H_i1 * H_1f);
H_bf = simplify(H_b1 * H_1f);

%% Center of mass in frame b
% centers of mass of links b and 1 expressed in frame b
boc_b = oc_b;
boc_1 = row12(H_b1 * [oc_1;1]);

% put results in matrices
boc_mat = [boc_b,boc_1];
 mc_vec = [mc_b;mc_1];

%  center of mass in frame b: o_bc
 o_bc    = simplify(expand(boc_mat*mc_vec/sum(mc_vec)));
odot_bc = simplify(timederivative(o_bc,varsdiff,varsdot));

%% Center of mass in frame i

% inertial CoM of link j
ioc_b = row12(H_ib * [oc_b;1]);
ioc_1 = row12(H_i1 * [oc_1;1]);

% put results in matrices
ioc_mat = [ioc_b,ioc_1];
 mc_vec = [mc_b;mc_1];

% inertial center of mass: o_ic
 o_ic    = simplify(expand(ioc_mat*mc_vec/sum(mc_vec)));
odot_ic = simplify(timederivative((o_ic),varsdiff,varsdot));

%%  Jodot_xy Jacobians:  i_odot_xy = Jodot_xy *nuq_ib
Jodot_i1 = Jacobian_Jodot_xy(frameorigin_o_xy(H_i1),xq_ib);
Jodot_if = Jacobian_Jodot_xy(frameorigin_o_xy(H_if),xq_ib);
Jodot_bf = Jacobian_Jodot_xy(frameorigin_o_xy(H_bf),xq_ib);

%%  Jnu_xy Jacobians:  nu_xy = Jnu_xy *nuq_ib
Jnu_ib   = Jacobian_Jnu_xy(H_ib,xq_ib);
Jnu_i1   = Jacobian_Jnu_xy(H_i1,xq_ib);
Jnu_if   = Jacobian_Jnu_xy(H_if,xq_ib);
Jnu_bf   = Jacobian_Jnu_xy(H_bf,xq_ib);

%% dotJnu_xy: time derivative of Jnu_xy
```

```
dotJnu_ib   = simplify(timederivative(Jnu_ib,varsdiff,varsdot));
dotJnu_i1   = simplify(timederivative(Jnu_i1,varsdiff,varsdot));

%% nu_xy twists:  nu_xy = Jnu_xy *nuq_ib
nu_i1     = simplify(Jnu_i1 * nuq_ib);
nu_if     = simplify(Jnu_if * nuq_ib);

%% big adjoints A_xy:  A_xy = Adjoint_A_xy(H_xy);
A_ib = Adjoint_A_xy(H_ib);

%% little adjoints a_xy: a_xy  = adjoint_a_xy(nu_xy);
a_ib     = adjoint_a_xy(H_ib,varsdiff,varsdot);
a_i1     = adjoint_a_xy(H_i1,varsdiff,varsdot);

%% Mass matrix Mb
Mb     = simplify(expand( Jnu_ib.'* Lam_b *Jnu_ib + Jnu_i1.'*
 ↪  Lam_1 * Jnu_i1 ));
Mbdot = simplify(timederivative(Mb,varsdiff,varsdot));
MbI    = simplify(expand(inv(Mb)));

%% passive coriolis matrix Cbp Euler-Newton
Psi_b      = Lam_b * a_ib - a_ib.'* Lam_b;
Psi_1      = Lam_1 * a_i1 - a_i1.'* Lam_1;

CbPsi      = Jnu_ib.'* Psi_b *   Jnu_ib + Jnu_i1.'* Psi_1 *
 ↪  Jnu_i1;
CbLam      = Jnu_ib.'* Lam_b *dotJnu_ib + Jnu_i1.'* Lam_1
 ↪  *dotJnu_i1;

Cbp        = simplify(expand(CbPsi + CbLam));
Cbpvec     = simplify(expand(Cbp*nuq_ib));

%% Lagrage Method: Coriolis matrix CbL
E_kin       = 1/2 * simplify(expand(nuq_ib.'* Mb *nuq_ib));

%  Coriolis matrix CbL: first 3 rows
CbL(1:3,:) = simplify(expand(Mbdot(1:3,:) -a_ib.'* Mb(1:3,:)));

%  take functional derivative of E_kin with respect to q
%  and convert the results to a matrix
gradL = functionalDerivative(E_kin,q);
gradLmat = simplify(expand(vec2mat(gradL,nuq_ib)));
```

```

%  Coriolis matrix CbL: last n rows
CbL(4,:)   = Mbdot(4,:) - gradLmat;
CbLvec = simplify(expand(CbL*nuq_ib));

%% Gravity vector G
% Potential Energy
E_pot = simplify(expand(g*ioc_mat(2,:) * mc_vec));

% G vector:
Gb      =    functionalDerivative(E_pot,xq_ib);
Gb(1:2) = R_ib.' * Gb(1:2);

%% Center of mass frame H_bc

% w_bc from locked velocity
MbbI = inv(Mb(1:3,1:3));
w_bc = MbbI(3,:)*Mb(1:3,4:end)*qdot;
w_bc = collect(w_bc,cos(q1));

%  xtheta_cb is obtained by integrating w_cb: this was done in
%  integrate_w_bc_rotationalLeg2D.m
A = m01*m11 + m02*m11;
B = 2*m01*m11 + m02*m11;
a = m01*m02 + 5*m01*m11 + 2*m02*m11;
b = 2*B;

theta_bc = (B*q1)/b +  2*atan( (tan(q1/2)*(a^2 - b^2)^(1/2))/(a + b) ) *  (A*b - B*a)/(b*(a^2 - b^2)^(1/2));
% theta_bc = (B*qv)/b +  2*atan2( tan(qv/2)*(a^2 - b^2)^(1/2), a + b ) *  (A*b - B*a)/(b*(a^2 - b^2)^(1/2));

% Transformations H_bc, H_ic
H_bc = framepose_H_xy(theta_bc,o_bc);
H_ic = H_ib * H_bc;
H_cf = inv(H_bc)*H_bf;

%%  Com-Foot vectors: b_o_cf, i_o_cf and o_cf
% b_o_cf = o_bf - o_bc;  i_o_cf   = R_ib * b_o_cf

b_o_cf = frameorigin_o_xy(H_bf) - o_bc;
i_o_cf   = R_ib * b_o_cf;
```

```
R_bc = H_bc(1:2,1:2);
R_bf = H_bf(1:2,1:2);
R_cf = R_bc.' * R_bf;

o_cf  = R_bc.' * b_o_cf;

%%  Jodot_cy Jacobians:  i_odot_cy = Jacobian_Jodot_xy *nuq_ib
Jiodot_cf = Jacobian_Jodot_xy(i_o_cf,xq_ib);
 Jodot_cf = Jacobian_Jodot_xy(  o_cf,xq_ib);
 Jodot_ic = Jacobian_Jodot_xy(frameorigin_o_xy(H_ic),xq_ib);

 %% big adjoints A_cy:  A_cy = Adjoint_A_xy(H_cy);
A_cb = Adjoint_A_xy(inv(H_bc));

%%  Com-Foot vectors: b_o_cf, i_o_cf and o_cf
% b_o_cf = o_bf - o_bc;  i_o_cf   = R_ib * b_o_cf

b_o_cf = frameorigin_o_xy(H_bf) - o_bc;
i_o_cf   = R_ib * b_o_cf;

R_bc = H_bc(1:2,1:2);
R_bf = H_bf(1:2,1:2);
R_cf = R_bc.' * R_bf;

o_cf  = R_bc.' * b_o_cf;

%% Jr_ocf, Jp_ocf: Radial and polar Jacobians
r = sqrt(i_o_cf(1)^2 + i_o_cf(2)^2);
rr = r*r;

Jr_iocf = [ i_o_cf(1)/r , i_o_cf(2)/r];
Jp_iocf = [-i_o_cf(2)/rr, i_o_cf(1)/rr];

r = sqrt(o_cf(1)^2 + o_cf(2)^2);
rr = r*r;

Jr_ocf = [ o_cf(1)/r , o_cf(2)/r];
Jp_ocf = [-o_cf(2)/rr, o_cf(1)/rr];

%% Momentum hc: h_b = Mbb*nu_ib + Mbq * qdot;  hc = A_cb^(-T) *
 ↪  h_b
```

```
h_b = simplify(Mb(1:3,:) * nuq_ib);
h_c = simplify(inv(A_cb).' * h_b);

A_ic_inert = Adjoint_A_xy(framepose_H_xy(0,o_ic));
h_inertc = simplify(inv(A_ic_inert).' * h_b);

%% o_xy:  o_xy = frameorigin_o_xy(H_xy);
o_i1 = frameorigin_o_xy(H_i1);
o_if = frameorigin_o_xy(H_if);
o_ic = frameorigin_o_xy(H_ic);

%% Jv_xy:  Jv_xy = row12(Jnu_xy)
Jv_if = row12(Jnu_if);

%% Jacobians for controllers

% Internal velocity Jacobian: Jvfint = Jv_if * P_int;
P_int   = simplify([zeros(4,3),[-inv(Mb(1:3,1:3))*Mb(1:3,4);1]]);

% Jwf_ext:  wf_ext = Jwf_ext * nuib; odot_cf = Jp_ocf * wf_ext
alpha_c     = symmat(theta_ib + theta_bc + pi/2);
Jalphadot_c = Jacobian_Jydot(alpha_c,xq_ib);

       Jrdot_cf = simplify(Jr_ocf * R_cf * Jv_if * P_int);
Jalpha_iodot_cf = simplify(Jp_iocf * Jiodot_cf);

%% Consistency tests Mb Cbp CbL
 ↪  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Coriolis matrix
disp('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')
disp('CbPsi + CbPsi^T nust be zero')
simplify(expand(CbPsi + CbPsi.'))
disp('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')

disp('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')
disp('Cbp must be passive Mbdot -CbPsi - CbPsi^T = 0')
simplify(expand(Mbdot-Cbp-Cbp.'))
disp('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')

disp('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')
disp('Coriolis vectors of Euler-Mewton and Lagrange Method must
 ↪  be the same')
```

```
diffCbvec = simplify(expand(Cbpvec-CbLvec))
disp('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')

%% convert sytem matrices to sim
↪  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

printsimmat(Mb,symvarStr,simvarStr)
printsimmat(MbI,symvarStr,simvarStr)
printsimmat(Cbp,symvarStr,simvarStr)
printsimmat(Gb,symvarStr,simvarStr)
printsimmat(E_pot,symvarStr,simvarStr)

printsimmat(o_i1,symvarStr,simvarStr)
printsimmat(o_if,symvarStr,simvarStr)
printsimmat(o_ic,symvarStr,simvarStr)
printsimmat(o_cf,symvarStr,simvarStr)

printsimmat(Jodot_i1,symvarStr,simvarStr)
printsimmat(Jodot_if,symvarStr,simvarStr)
printsimmat(Jodot_ic,symvarStr,simvarStr)
printsimmat(Jodot_cf,symvarStr,simvarStr)

printsimmat(Jalpha_iodot_cf,symvarStr,simvarStr)
printsimmat(Jrdot_cf,symvarStr,simvarStr)
printsimmat(alpha_c,symvarStr,simvarStr)
printsimmat(Jalphadot_c,symvarStr,simvarStr)

printsimmat(h_c,symvarStr,simvarStr)
printsimmat(h_inertc,symvarStr,simvarStr)
```

A.3. Useful Relations

There are some useful identities on transformations of the cross product [LLM17, Chapter 1.1.7]: For any x and $y \in \mathbb{R}^3$ an any invertible matrix $\boldsymbol{M}$

$$\hat{\boldsymbol{x}}\boldsymbol{y} = \boldsymbol{x} \times \boldsymbol{y} \tag{A.6a}$$

$$\hat{\boldsymbol{x}}\boldsymbol{y} = -\hat{\boldsymbol{y}}\boldsymbol{x} \tag{A.6b}$$

$$\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{x}} = (\boldsymbol{x} \times \boldsymbol{y})\hat{} \tag{A.6c}$$

$$(\boldsymbol{x} \times \boldsymbol{y})\boldsymbol{z} = \boldsymbol{x}(\boldsymbol{y} \times \boldsymbol{z}) \tag{A.6d}$$

$$\boldsymbol{R}(\boldsymbol{x} \times \boldsymbol{y}) = (\boldsymbol{R}\boldsymbol{x}) \times (\boldsymbol{R}\boldsymbol{y}) \tag{A.6e}$$

$$\boldsymbol{R}\hat{\boldsymbol{x}}\boldsymbol{R}^T = (\boldsymbol{R}\boldsymbol{x})\hat{} \tag{A.6f}$$

$$\dot{\overline{\boldsymbol{M}^{-1}}}\boldsymbol{M} = -\boldsymbol{M}^{-1}\dot{\boldsymbol{M}} \tag{A.6g}$$

List of Figures

List of Tables

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