Analysis of Sliding-Mode Control Systems with Relative Degree Altering Disturbances ⋆

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Abstract

We consider sliding-mode control systems subject to unmatched disturbances. Classical first-order sliding-mode techniques are capable to compensate unmatched disturbances if differentiations of the output of sufficiently high order are included in the sliding variable. For such disturbances it is commonly assumed that they do not affect the relative degree of the system. In this contribution we consider disturbances that alter the relative degree of the process and study their impact on the closed-loop control system with a classical first-order sliding-mode design. We analyse the reaching and sliding phase of the resulting closed-loop system and analyse its stability properties. It turns out that the sliding-manifold is not of reduced dimension and the uniqueness of the solution may be lost. Also attractivity of the sliding-manifold and global stability of the origin may be lost whereas the disturbance rejection properties of the sliding-mode control are not impaired. We present a necessary and sufficient condition for the existence of unique solutions for the closed-loop system. The second-order case is studied in great detail and allows to parametrically specify the conditions obtained before. We derive a necessary condition for the global asymptotic stability of the closed-loop system. Further we present a constructive condition for the global asymptotic stability of the closed-loop system using a piece-wise linear Lyapunov function. Each of the prominent results is illustrated by an numerical example.

Key words: Sliding-mode control, Unmatched disturbances, Relative degree

1 Introduction

This paper is dedicated to systems that are affected by model uncertainties that reduce the relative degree of the process. Such uncertainties may be due to model simplifications where minor physical effects are chosen to be disregarded, e.g. see [19] for an example. But relative degree-altering uncertainties may also be induced by a standard input-output linearisation where the transformation depends on an uncertain model, see e.g. [11] or the example in Section 2.2.

In this manuscript we consider the standard first-order sliding-mode controller (SMC) in particular. Sliding-mode control techniques are well-known for their robustness properties with regard to model uncertainties and external disturbances. In particular disturbances that enter the system via the same input-space as the control signal, so-called matched disturbances, may be completely rejected on the sliding manifold. Moreover, if the sliding-manifold includes derivatives of the output (of sufficiently high order) also unmatched disturbances may be compensated. There are several propositions that exploit this approach, see e.g. [2, 3, 8, 9, 22]. All these methods consider system structures ensuring that the relative degree of the system is not changed by the disturbance. However, model uncertainties may change the relative degree of the system as demonstrated e.g. by [11].

Systems with uncertain relative degree have been subject to various research in the recent past. The concept of ill-defined relative degree has been studied e.g. in [5, 6, 7, 17, 18]. Basically, a system with ill-defined relative degree has states x for which the relative degree is larger than at some nominal point x_0, i.e. $\mathcal{L}_g\mathcal{L}_f^{r-1}h(x) = 0$, where r denotes the relative degree at x_0. Another line of research assumes that an upper bound of the uncertain

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relative degree is known. These kind of systems are analysed in various works such as [10, 13, 14, 21, 23]. In [13] a direct adaptive tracking and disturbance rejection algorithm for single-input, single-output minimum-phase linear systems is developed. [23] use a state observer to define a linear control law to reject disturbances on an integrator chain system. Conditions are given for which the stability of the closed loop is ensured.

In this contribution we analyse systems subject to disturbances that alter the relative degree by exactly one and study its impact on a closed-loop system with classically designed first-order sliding-mode controller. It turns out that global attractivity of the sliding-manifold may be lost in some cases while disturbance compensation in sliding-mode is retained. More severely, the uncertainty may render the system unstable and uniqueness of the solution may be lost. We shall study a generic second-order system in great detail and analyse attractivity properties, existence of equilibria as well as their stability properties. In particular, we derive constructive stability conditions for the disturbed system using a piece-wise linear Lyapunov function.

The paper is structured as follows. The next section introduces the system class considered and defines the concept of relative degree altering disturbances. We show how such disturbance may naturally occur in a standard transformation into Byrnes-Isidori form and give an illustrating example. Section 3 gives a precise problem definition for this contribution where we consider a classical first-order sliding-mode controller. Section 4 contains the formal analysis of the resulting closed-loop system in reaching and sliding-phase. In Section 5 we study the second-order case in full detail. We derive parametric conditions for the existence of a relative degree altering disturbance as well as the loss of a unique solution. In case of the existence of a unique solution we analyse in particular the existence and location of equilibria in the reaching phase and develop a constructive condition for the global asymptotic stability of the closed-loop system. Several numerical examples illustrate the effects of the relative-degree altering disturbance as well as the construction of the Lyapunov function in Section 6.

This manuscript is based on the previously published work [19] on the effects of relative-degree altering disturbances in the context of sliding-mode control. For completeness of exposition the analysis in Section 4 is included in this manuscript.

Alongside with the common mathematical notation, we shall use $\mathcal{L}_f h(x)$ for the Lie derivative of h with respect to the vector field f, i.e. $\mathcal{L}_f h(x) = \frac{\partial h(x)}{\partial x} f(x)$. The Lie derivative of h with respect to the sum of vector fields $f(x) + \phi(x)$ shall be denoted by

$$\mathcal{L}_{f+\phi} h(x) = \frac{\partial h(x)}{\partial x}\big(f(x) + \phi(x)\big).$$

The k-th Lie derivative is denoted by

$$\mathcal{L}_f^k h(x) = \frac{\partial \mathcal{L}_f^{k-1} h(x)}{\partial x} f(x).$$

We denote with e_i the i-th unitvector and with I and 0 the identity and zero matrix of appropriate dimension.

2 System Class

We consider process dynamics of the form

$$\dot{x} = f(x) + g(x)u + \phi(x) \tag{1a}$$
$$y = h(x)\,, \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}$ the control input and $y(t) \in \mathbb{R}$ the output of interest. The vector fields f and g are sufficiently smooth and of matching dimensions, where $f(0) = 0$ and $g(x) \neq 0$ for all x. The output function h is uniformly continuous and the state vector x as well as the output y and its derivatives with respect to time are assumed to be known. The function ϕ is an unknown bounded disturbance. Without loss of generality but for ease of exposition we restrain our analysis to time-invariant disturbances. Note however that the results in Section 4 can be expanded to time-varying disturbances.

The disturbance can be divided into a matched and an unmatched disturbance, ϕ_m and ϕ_u, respectively, with

$$\phi_\mathrm{m}(x) = g(x)g^+(x)\phi(x) \tag{2a}$$
$$\phi_\mathrm{u}(x) = g^\perp(x)g^{\perp+}(x)\phi(x)\,, \tag{2b}$$

where $g^\perp(x)$ is a full-rank left annihilator of $g(x)$, i.e. a matrix with independent columns that spans the null space of $g(x)$. It satisfies $g^\perp(x)g^\top(x) = 0$ and $\mathrm{rk}(g^\perp) = n-1$. Moreover, we denote with $g^+(x)$ the left pseudo-inverse of $g(x)$, i.e. $g^+(x) = (g^\top(x)g(x))^{-1}g^\top(x)$.

We denote system (1) without disturbance, i.e. $\phi \equiv 0$, as the *nominal system*. Accordingly, we denote by r the relative degree of the nominal system. More precisely, for $\phi \equiv 0$, the relative degree of the output $y = h(x)$ with respect to the input u at the point $x \in \mathbb{R}^n$ is identical to r, i.e.

$$\mathcal{L}_g \mathcal{L}_f^k h(x) = 0\,, \quad \text{for} \quad k \in \{0, \dots r-2\} \tag{3a}$$
$$\mathcal{L}_g \mathcal{L}_f^{r-1} h(x) \neq 0\,, \tag{3b}$$

where $\mathcal{L}$ denotes the Lie-derivative. Note that the relative degree is a local property. If not stated otherwise, we consider the relative degree at the origin $x = 0$.

2.1 *Relative degree altering disturbance*

The disturbance ϕ may have an impact on the relative degree of system (1). Therefore, we shall distinguish disturbances that retain the nominal relative degree from disturbances that change the relative degree with respect to the nominal case.

Definition 1 *Consider system* (1) *with nominal relative degree r. The disturbance ϕ is called* (relative degree) preserving *if*

$$\mathcal{L}_g \mathcal{L}_{f+\phi}^k h(x) = 0\,, \quad \textit{for} \quad k \in \{0, \dots r-2\} \tag{4a}$$
$$\mathcal{L}_g \mathcal{L}_{f+\phi}^{r-1} h(x) \neq 0\,. \tag{4b}$$

Otherwise ϕ is called (relative degree) altering.

Note that any matched uncertainty is relative degree preserving. Typically, this property is also required for unmatched disturbances as many common control techniques for non-linear systems, such as input-output linearisation or sliding mode control, rely on the knowledge of the relative degree to divide the states into a set of (controlled) external states and a set of (uncontrolled) internal states. This is achieved using the transformation into Byrnes-Isidori Form. In this context it is worth noting that the property of a relative degree preserving (altering) disturbance is invariant with respect to regular state transformations if the input and output remain the same.

2.2 *Transformation into Byrnes-Isidori Form*

Consider the state transformation [15]

$$\tau(x) = \left[h(x)\ \mathcal{L}_f h(x) \dots \mathcal{L}_f^{r-1} h(x)\ \ \tau_{r+1}(x) \dots \tau_n(x)\right]^\top \tag{5}$$

where $\tau_j(x)$ is chosen such that τ is a diffeomorphism and $\mathcal{L}_g \tau_j(x) = 0$ for $j \in \{r+1, \dots, n\}$. This is the state transformation for the nominal system into Byrnes-Isidori Form. The dynamics of the external states of the nominal system $\xi = [\tau_1(x)\ \dots\ \tau_r(x)]^\top$ have the form

$$\dot{\xi} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ \mathcal{L}_g \mathcal{L}_f^{r-1} h(x) \end{bmatrix} u + \\ + \begin{bmatrix} 0 \\ \mathcal{L}_f^r h(x) \end{bmatrix} + \phi^{\text{ext}}(\xi, \eta) \tag{6}$$

where x is evaluated at $x = \tau^{-1}(\xi, \eta)$ and the disturbance acting on the external dynamics is given by

$$\phi^{\text{ext}}(\xi, \eta) := \left[\mathcal{L}_\phi h(x)\ \mathcal{L}_\phi \mathcal{L}_f h(x)\ \cdots\ \mathcal{L}_\phi \mathcal{L}_f^{r-1} h(x)\right]^\top .$$

If the first $r-1$ entries of the disturbance vector $\phi^{\text{ext}}(\xi, \eta)$ are identical to zero then $\phi^{\text{ext}}(\xi, \eta)$, and therefore also ϕ, is a relative degree preserving disturbance as introduced in Definition 1. This would allow to employ standard control laws v for integrator systems.

In general, however, ϕ may not be relative degree preserving as illustrated by the following example. In such case, ξ in (6) does not represent the external dynamics.

2.3 *Example*

We consider a single link manipulator with a flexible joint and neglected damping as discussed in [20]. The system has the form (1) with the vector fields

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{MgL}{J_\text{e}} \sin(x_1) - \frac{k}{J_\text{e}}(x_1 - x_3) \\ x_4 \\ \frac{k}{J_\text{a}}(x_1 - x_3) \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_\text{a}} \end{bmatrix}$$
$$h(x) = x_1\,,$$

where M, J_e denote the mass and inertia of the effector with centre of gravity at distance L from the point of attack. k denotes the spring constant and J_a the inertia of the actuator.

If we consider the neglected damping as model uncertainty, the disturbance takes the form

$$\phi(x) = \left[0\ \ -\frac{d}{J_\text{e}}(x_2 - x_4)\ \ 0\ \ \frac{d}{J_\text{a}}(x_2 - x_4)\right]^\top ,$$

with d as unknown small damping constant.

The output y of the nominal system has full relative degree n with respect to the input u. This allows the transformation (5) to yield only external states ξ. The transformation (5) into these nominal external states takes the form

$$\tau(x) = \begin{bmatrix} x_1 \\ x_2 \\ -\frac{MgL}{J_\text{e}} \sin(x_1) - \frac{k}{J_\text{e}}(x_1 - x_3) \\ -\frac{MgL}{J_\text{e}} \cos(x_1) x_2 - \frac{k}{J_\text{e}}(x_2 - x_4) \end{bmatrix}. \tag{7}$$

Then the transformed system has the form (6) with the disturbance $\phi^{\text{ext}} := [\phi_1^{\text{ext}}\ \phi_2^{\text{ext}}\ \phi_3^{\text{ext}}\ \phi_4^{\text{ext}}]^\top$ having the form $\phi_1^{\text{ext}}(\xi) = \phi_3^{\text{ext}}(\xi) = 0$ and

$$\phi_2^{\text{ext}}(\xi) = \frac{d}{k}\xi_4 + \frac{MgLd}{kJ_\text{e}} \cos(\xi_1)\xi_2$$
$$\phi_4^{\text{ext}}(\xi) = -\left(\frac{MgL}{J_\text{e}^2} \cos(\xi_1) + 2\frac{k}{J_\text{e}^2}\right) \phi_2^{\text{ext}}(\xi)\,.$$

The disturbance $\phi_2^{\text{ext}}(\xi)$ impairs the integrator structure of the transformed system and counteracts the benefits of the transformation τ for the design of a control law.

Remark 2 *Note that for this example the relative degree altering uncertainty is introduced by a parameter uncertainty. A more involved example where the uncertainty does not depend on a parameter approximation can be found in [11].*

As an expedient of this situation the transformation (5) may be adapted to contain the disturbances in addition to the nominal dynamics. While these are naturally not explicitly available, information about them is contained in the derivatives of the output which are used in common sliding mode control laws.

3 Problem definition

In this paper we consider system (1) with full relative degree and unmatched disturbances that reduce the relative degree of the nominal system by exactly one, i.e.

$$\mathcal{L}_g\mathcal{L}_{f+\phi}^k h(x) = 0\,, \quad \text{for} \quad k \in \{0, \dots n-3\} \tag{8a}$$

$$\mathcal{L}_g\mathcal{L}_{f+\phi}^{n-2} h(x) \neq 0\,. \tag{8b}$$

Thus the disturbed system (1) has relative degree $n-1$, whereas the nominal system has relative degree $r = n$.

We shall consider the Byrnes-Isidori form of the disturbed system, where the state-space is decomposed into external ξ and internal states η. In contrast to Section 2.2, the external states are defined using the exact derivative of the output including the disturbance

$$\xi_1 := \tau_1(x) = y = h(x)\,, \tag{9a}$$

$$\xi_i := \tau_i(x) = y^{(i-1)} = \mathcal{L}_{f+\phi}^{i-1} h(x), \quad i = 2 \dots n-2\,, \tag{9b}$$

$$\xi_{n-1} := \tau_{n-1}(x) = y^{(n-2)} = \mathcal{L}_{f+\phi}^{n-2} h(x)\,. \tag{9c}$$

The remaining component

$$\eta := \tau_n(x), \tag{10}$$

is scalar and chosen such that τ is a diffeomorphism and $\mathcal{L}_g\tau_n(x) = 0$. The resulting dynamics are given by

$$\dot{\xi} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ \mathcal{L}_g\mathcal{L}_{f+\phi}^{n-2} h(x) \end{bmatrix} u + \begin{bmatrix} 0 \\ \mathcal{L}_{f+\phi}^{n-1} h(x) \end{bmatrix} \tag{11a}$$

$$\dot{\eta} = \mathcal{L}_{f+\phi}\tau_n(x)|_{x=\tau^{-1}(\xi,\eta)}\,. \tag{11b}$$

Compare these dynamics with (6) of the previous section, where the transformation (5) is based on the *nominal* system dynamics. Note that, strictly speaking, the system (6) is not in Byrnes-Isidori form if ϕ^{ext} is relative degree altering. Due to the full relative degree of the nominal system, the dimension of these nominal external dynamics is n and there are no internal dynamics. However, the external dynamics (11) resulting from the transformation (7) of the *disturbed system*, are of dimension $n-1$ with first-order internal dynamics. Assuming that the external dynamics are locally asymptotically stable by design, we require that the so-called zero dynamics, i.e. (11b) with $\xi \equiv 0$, are locally asymptotically stable. Since $\eta(t) \in \mathbb{R}$, local asymptotic stability of the zero dynamics is ensured if and only if

$$\eta\, \mathcal{L}_{f+\phi}\tau_n(x)|_{x=\tau^{-1}(\xi,\eta)} < 0 \tag{12}$$

for all $\eta \neq 0$ within the considered neighbourhood.

The disturbance ϕ^{ext} in (6) is an unmatched disturbance and affects the desired integrator chain of the system. This makes the analysis as well as the control design difficult. In comparison to that, the external dynamics (11) form an integrator chain of length $n-1$. The disturbance $\mathcal{L}_{f+\phi}^{n-1} h(x)$ acting on the $n-1$st state is matched and can be compensated using conventional sliding mode techniques. The term $\mathcal{L}_g\mathcal{L}_{f+\phi}^{n-2} h(x)$, however, is much less benign. As ϕ is relative degree altering, $\mathcal{L}_g\mathcal{L}_{f+\phi}^{n-2} h(x)$ dependents on the disturbance ϕ, c.f. (8b), and acts as an unmatched disturbance. Note if ϕ is relative degree preserving, this term is identical to zero, c.f. (4a), and the internal dynamics vanish producing an additional integrator state. This would render all uncertainties matched and would allow to compensate them by a classical first-order sliding-mode control considered in the following.

3.1 First-order sliding-mode control law

We apply a standard first-order sliding-mode control, incorporating derivatives of the output, which compensates matched as well as unmatched, relative degree preserving uncertainties. We choose the switching function based on the nominal relative degree as

$$\sigma(y, \dot{y}, \dots y^{(n-1)}) = y^{(n-1)} - \gamma\big(y, \dot{y}, \dots, y^{(n-2)}\big) \tag{13}$$

with the function γ designed such that the system

$$y^{(n-1)} = \gamma\big(y, \dot{y}, \dots, y^{(n-2)}\big) \tag{14}$$

is asymptotically stable at 0. Since the system (1) has relative degree $n-1$, Equation (13) can be written as a function of the state x and the input u

$$\sigma(x, u) = s_\phi(x) + \varsigma_\phi(x)u \tag{15}$$

with the non-trivial functions

$$\varsigma_\phi(x) := \mathcal{L}_g \mathcal{L}_{f+\phi}^{n-2} h(x) \tag{16}$$

$$s_\phi(x) := \mathcal{L}_{f+\phi}^{n-1} h(x) - \gamma\big(h(x), \mathcal{L}_{f+\phi}^{1} h(x), \dots \mathcal{L}_{f+\phi}^{n-2} h(x)\big). \tag{17}$$

These incorporate the influence of the unknown disturbance ϕ. Note that (13) is the implemented switching function, while (15) is usually unknown and is used for analysis purposes only. We write $\sigma(y, \dot{y}, \dots y^{(n-1)})$ to emphasise that the derivatives are obtained by differentiating the output signal y, and we write $\sigma(x, u)$ if these derivatives are substituted by their analytical expressions from the right-hand side of (9).

We conclude this section by stating the standard first-order sliding-mode control law with $L > 0$:

$$u = -\frac{\mathcal{L}_f s_0(x) + L \operatorname{sgn}\left(\sigma(y, \dot{y}, \dots y^{(r)})\right)}{\mathcal{L}_g s_0(x)} \tag{18a}$$

$$= \alpha(x) - q(x) \operatorname{sgn}\left(s_\phi(x) + \varsigma_\phi(x) u\right) \tag{18b}$$

with s_0 meaning s_ϕ with $\phi \equiv 0$, and

$$\alpha(x) = -\frac{\mathcal{L}_f s_0(x)}{\mathcal{L}_g s_0(x)}, \quad q(x) = \frac{L}{\mathcal{L}_g s_0(x)}. \tag{19}$$

4 Analysis of the closed loop system

4.1 Reaching phase and sliding phase

Commonly the state space can be divided into subspaces for the reaching phase and a sliding manifold. The reaching phase is defined by all x that fulfil $\sigma(x) \neq 0$ while the sliding phase is defined by $\sigma(x) = 0$. In our case the sliding variable $\sigma(x, u)$ may also depend on u, and therefore this unique division may no longer be possible. In the following we use the sliding variable to define subsets of the state space for which the system can be in reaching or sliding phase, respectively. In case $\sigma(x, u) > 0$, Equation (15) yields $s_\phi(x) + \varsigma_\phi(x) u > 0$. Substituting u from (18) yields the set

$$X_1 := \{x \in \mathbb{R}^n \mid s_\phi(x) + \varsigma_\phi(x)(\alpha(x) - q(x)) > 0\} \tag{20}$$

describing all points in $\mathbb{R}^n$ for which $\sigma > 0$. Similarly, for $\sigma(x, u) < 0$ we obtain the set

$$X_2 := \{x \in \mathbb{R}^n \mid s_\phi(x) + \varsigma_\phi(x)(\alpha(x) + q(x)) < 0\} \tag{21}$$

and for $\sigma(x, u) = 0$ we have

$$X_3 := \left\{x \in \mathbb{R}^n \mid \frac{s_\phi(x)}{\varsigma_\phi(x) q(x)} + \frac{\alpha(x)}{q(x)} \in [-1, 1]\right\}. \tag{22}$$

Note that when eliminating u from (18) by substituting (15) we have an implication (and no equivalence) and thus u may not be uniquely defined by the state x.

Indeed, it turns out that X_1, X_2 and X_3 are not necessarily disjoint. Thus, for every point x with $\sigma(x, u) = 0$ holds $x \in X_3$, but under certain conditions for every point in X_3 may also hold $\sigma(x, u) \neq 0$ depending on u subject to (18).

For our analysis we shall distinguish the boundary and the inner of the set X_3. In this context, we consider the set

$$X_3^\circ = \left\{x \in \mathbb{R}^n \mid \frac{\alpha(x)}{q(x)} + \frac{s_\phi(x)}{\varsigma_\phi(x) q(x)} \in (-1, 1)\right\}. \tag{23}$$

Obviously X_3° is a subset of the inner of X_3. If any inner point of X_3 is part of the set X_3° then X_3° is the inner of X_3. The boundary of X_3 is then described by

$$\partial X_3 = \left\{x \in \mathbb{R}^n \mid \frac{\alpha(x)}{q(x)} + \frac{s_\phi(x)}{\varsigma_\phi(x) q(x)} \in \{-1, 1\}\right\}. \tag{24}$$

Before we discuss various cases for which the three sets take different configurations in the state space, we shall note that the three sets always cover the full state space.

Lemma 3 *It is* $X_1 \cup X_2 \cup X_3 = \mathbb{R}^n$.

PROOF. We rearrange (20), (21) and (22) and obtain for any $x_i \in X_i$ with $i \in \{1, 2, 3\}$ that

$$s_\phi(x_1) + \varsigma_\phi(x_1)\alpha(x_1) > \varsigma_\phi(x_1) q(x_1) \tag{25a}$$

$$s_\phi(x_2) + \varsigma_\phi(x_2)\alpha(x_2) < -\varsigma_\phi(x_2) q(x_2) \tag{25b}$$

$$s_\phi(x_3) + \varsigma_\phi(x_3)\alpha(x_3) \in [-|\varsigma_\phi(x_3) q(x_3)|, |\varsigma_\phi(x_3) q(x_3)|] \tag{25c}$$

We can see that every $x \in \mathbb{R}^n$ fulfils at least one of these three conditions. □

For our analysis we shall distinguish three configurations of the sets X_1, X_2, X_3, see also Fig. 1:

Case 1: $X_1 \cap X_2 = \emptyset \ \wedge \ X_3^\circ \neq \emptyset$,
Case 2: $X_1 \cap X_2 = \emptyset \ \wedge \ X_3^\circ = \emptyset$,
Case 3: $X_1 \cap X_2 \neq \emptyset \ \wedge \ X_3^\circ \neq \emptyset$.

Note Case 2 is the classical first-order sliding-mode control, whereas Case 1 and 3 occur when altering disturbances are present.

First we consider the cases where the three sets are disjoint and thus reaching and sliding phase may be defined via regions in the state-space. The following lemma gives a necessary and sufficient condition for such case.

Lemma 4 (Case 1 and 2) *The sets X_1 and X_2 have an empty intersection, i.e. $X_1 \cap X_2 = \emptyset$ if and only if*

$$q(x)\varsigma_\phi(x) \geq 0 \quad \text{for all } x. \tag{26}$$

Then Lemma 3 yields $X_3 = \mathbb{R}^n \setminus (X_1 \cup X_2)$.

PROOF. Condition (25a) and (25b) ensure that X_1 and X_2 are disjoint if and only if $q(x)\varsigma_\phi(x) \geq 0$ for all x. Further, $q(x)\varsigma_\phi(x) \geq 0$ with condition (25c) gives that for every point in X_3 holds

$$s_\phi(x_3) + \varsigma_\phi(x_3)\alpha(x_3) \in [-\varsigma_\phi(x_3)q(x_3), \varsigma_\phi(x_3)q(x_3)]\,.$$

This makes X_3 by definition of X_1 and X_2 and with (25a) and (25b) the complement of the union of X_1 and X_2. □

Note that the dimension of X_3 may be n. However, for the special case of preserving disturbances, we have $\varsigma_\phi(x) = 0$ and obtain a conventional sliding manifold of dimension $n-1$. This finding is summarised in the following corollary.

Corollary 5 (Case 2) *If $\varsigma_\phi(x) = 0$, it is $X_1 \cap X_2 = \emptyset$ and $X_3 = \{x \in \mathbb{R}^n \mid s_\phi(x) = 0\}$ and $X_3^\circ = \emptyset$.*

PROOF. For $\varsigma_\phi(x) = 0$ the disjointness of X_1 and X_2 follows directly from its definition in (20) and (21). The set X_3 is directly obtained using Equation (25c). The set X_3° is calculated analogously. □

The following lemma characterises Case 3.

Lemma 6 (Case 3) *The sets X_1 and X_2 have a non-empty intersection, i.e. $X_1 \cap X_2 \neq \emptyset$, if and only if*

$$q(x)\varsigma_\phi(x) < 0$$

for all $x \in X_1 \cap X_2$. Then $X_1 \cap X_2 = X_3^\circ$.

PROOF. With (25a) and (25b) for every point in $X_1 \cap X_2$ holds

$$\varsigma_\phi(x)q(x) < s_\phi(x) - \alpha(x)q(x) < -\varsigma_\phi(x)q(x)\,. \tag{27}$$

This is equivalent to $\varsigma_\phi(x)q(x) < 0$. Then the set (23) is defined by all x that fulfil

$$s_\phi(x) + \varsigma_\phi(x)\alpha(x) \in (\varsigma_\phi(x)q(x), -\varsigma_\phi(x)q(x))\,.$$

This is an equivalent notation for points fulfilling (27). Thus, it is $X_1 \cap X_2 = X_3^\circ$. □

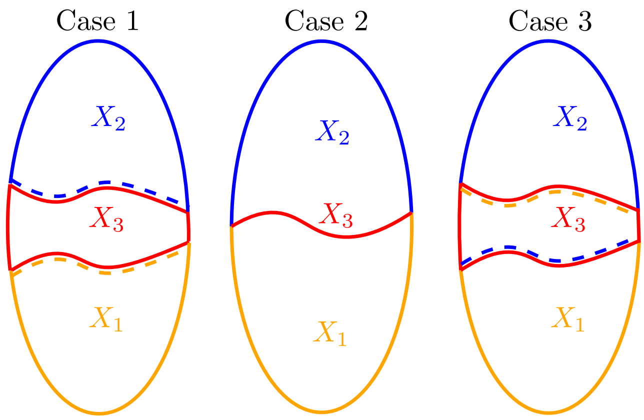

Fig. 1. Three possible configurations of the sets X_i.

In Case 3, X_3° is the intersection of X_1 and X_2. Then, for states $x \in X_3$ all three phases are possible, depending on the choice of the control u or equivalently $\sigma > 0$, $\sigma < 0$ or $\sigma = 0$. This shows that it is not sufficient, to define the sliding phase and reaching phase solely via the state sets X_1, X_2 and X_3. Thus, we say that the system is in sliding phase if $\sigma(x,u) = 0$ and in reaching phase if $\sigma(x,u) \neq 0$.

4.2 Control signal and its continuity

The control law is the solution of the implicit equation (18b) which takes one of the three forms

$$u(x) \in \{u^-, u^+, u^\circ\} \tag{28}$$

with

$$u^- := \alpha(x) - q(x) \tag{29a}$$
$$u^+ := \alpha(x) + q(x) \tag{29b}$$
$$u^\circ := -\frac{s_\phi(x)}{\varsigma_\phi(x)}\,, \qquad \varsigma_\phi \neq 0\,. \tag{29c}$$

Remark 7 *The introduced control law is uniquely defined by x if and only if $X_1 \cap X_2 \cap X_3 = \emptyset$, i.e. in Case 1 and 2; equivalently condition (26) holds.*

For the case of preserving disturbances (Case 2) we obtain the conventional first-order sliding-mode control law.

Theorem 8 *For $\varsigma_\phi = 0$ the control given by (18) yields*

$$u(x) = \begin{cases} u^- & x \in X_1, \\ u^+ & x \in X_2, \\ -\frac{\mathcal{L}_f s_\phi(x) - \mathcal{L}_\phi s_\phi(x)}{\mathcal{L}_g s_\phi(x)} & x \in X_3\,, \end{cases} \tag{30}$$

resembling the conventional first-order sliding-mode control law. Notably, for $\phi = 0$ it is $u(x) = \alpha(x)$ for $x \in X_3$.

PROOF. The equality for $x \in X_1$ and $x \in X_2$ is clear. The control law for $x \in X_3$ is the equivalent control law resulting of (15) by having

$$\dot{\sigma} = \mathcal{L}_f s_\phi(x) + \mathcal{L}_g s_\phi(x) u^\circ + \mathcal{L}_\phi s_\phi(x) \,.$$

Requiring $\dot{\sigma} = 0$ leads to

$$u^\circ = -\frac{\mathcal{L}_f s_\phi(x) - \mathcal{L}_\phi s_\phi(x)}{\mathcal{L}_g s_\phi(x)} \,.$$

Remark 9 *For preserving disturbances the control u is discontinuous.*

For relative degree altering disturbances ϕ, i.e. $\varsigma_\phi \neq 0$, the sets X_1, X_2, X_3 can take the configuration of Case 1 or Case 3. As we show in the following, in Case 1 the control signal is continuous in sliding-mode, whereas in Case 3 neither the sliding-variable nor the control signal is guaranteed continuous.

Theorem 10 *If $X_1 \cap X_2 = \emptyset$ and $X_3^\circ \neq \emptyset$, i.e. Case 1, the control law u is continuous in x.*

PROOF. Continuity of u^-, u^+ and u^0 is ensured by the continuity of α, q, s_ϕ and φ_ϕ. We show continuity at the transitions of u within the set (28). Note for Case 1, we only have transitions at the boundary of X_1 and X_2. For $\hat{x} \in \partial X_3 \cap \partial X_1$ holds, (c.f. (20))

$$s_\phi(\hat{x}) + \varsigma_\phi(\hat{x})(\alpha(\hat{x}) - q(\hat{x})) = 0.$$

For any sequence (x_n) with only elements in X_1 and $\lim_{n\to\infty} x_n = \hat{x}$ we have

$$\lim_{n\to\infty} u^-(x_n) = \lim_{n\to\infty} \alpha(x_n) - q(x_n) = \alpha(\hat{x}) - q(\hat{x}) \quad (31a)$$
$$= \frac{-s_\phi(\hat{x})}{\varsigma(\hat{x})} = \lim_{n\to\infty} \frac{s_\phi(x_n)}{\varsigma(x_n)} = u^0(\hat{x}) \,. \quad (31b)$$

Continuity at the boundary of X_2 can be shown analogously. □

If the sets take the configuration of Case 3, the control law is not uniquely defined. While the following implications always hold:

$$u(x) = u^- \Rightarrow x \in X_1 \,, \quad (32)$$
$$u(x) = u^+ \Rightarrow x \in X_2 \,, \quad (33)$$
$$u(x) = u^\circ \Rightarrow x \in X_3 \,, \quad (34)$$

whereas the opposite implication does not hold in general. In fact, for Case 3 where the sets overlap, the control signal may take any value in $\{u^-, u^+, u^\circ\}$ for $x \in X_3^\circ$.

Lemma 11 *If $X_1 \cap X_2 \neq \emptyset$, i.e. Case 3, u and σ are not unique in X_3 and thus may be discontinuous in x.*

PROOF. For Case 3 we have $q(x) \neq 0$. For $\hat{x} \in \partial X_1$ also holds $\hat{x} \in X_2 \cap X_3$ and thus we may choose $u(\hat{x}) = u^+ = \alpha(\hat{x}) + q(\hat{x})$. But for $\hat{x} \in X_3$ we may choose $u(\hat{x}) = u^\circ = -\frac{s_\phi(\hat{x})}{\varsigma_\phi(\hat{x})} = \alpha(\hat{x}) - q(\hat{x}) \neq u^+$. The first choice of u yields $\sigma(\hat{x}, u^+) \neq 0$ while the second gives $\sigma(\hat{x}, u^\circ) = 0$.

Remark 12 *Note that the continuity of u can be retained if the sliding variable σ is continuous. Considering* (15) *yields the control law as a function of σ*

$$u(x, \sigma) = -\frac{\sigma - s_\phi(x)}{\varsigma_\phi(x)} \,.$$

This function is unique and continuous in σ, in particular at $\sigma = 0$. This remarkable property for a sliding-mode control is obtained for both cases with altering disturbance, i.e. Case 1 and 3.

4.3 Closed loop system

For the system (1) with sliding-variable (15) and control (28) obtained from the sliding-mode control law (18), the closed-loop dynamics may take the form:

$$\dot{x}^- = f(x) + g(x)(\alpha(x) - q(x)) + \phi \,, \quad (35a)$$
$$\dot{x}^+ = f(x) + g(x)(\alpha(x) + q(x)) + \phi \,, \quad (35b)$$
$$\text{or} \quad \dot{x}^0 = f(x) - g(x)\frac{s_\phi(x)}{\varsigma_\phi(x)} \,. \quad (35c)$$

For the cases of non-overlapping sets X_i (Case 1 and 2) we obtain the following closed-loop dynamics.

Theorem 13 *If $X_1 \cap X_2 = \emptyset$ then the closed-loop system* (1) *and* (18) *takes the form*

$$\dot{x} = \begin{cases} \dot{x}^- & \text{for } x \in X_1 \\ \dot{x}^+ & \text{for } x \in X_2 \\ \dot{x}^\circ & \text{for } x \in X_3 \end{cases} \quad (36a)$$
$$y = h(x) \quad (36b)$$

with $\dot{x}^-$, $\dot{x}^+$,$\dot{x}^\circ$ given in (35).

Note that for Case 2 we have the typical Fillipov solutions on the sliding manifold, whereas in Case 1 we obtain classical Caratheodory solutions. For overlapping sets X_i, i.e. Case 3, uniqueness of the solution is lost.

Theorem 14 *If $X_1 \cap X_2 \neq \emptyset$ then the closed loop system* (1) *and* (18) *has the form*

$$
\begin{aligned}
\dot{x} &= \dot{x}^- && x \in X_1 && \text{(37a)}\\
\dot{x} &= \dot{x}^+ && x \in X_2 && \text{(37b)}\\
\dot{x} &\in \{\dot{x}^-, \dot{x}^\circ, \dot{x}^+\} && x \in X_3^\circ && \text{(37c)}
\end{aligned}
$$

with $\dot{x}^-$, $\dot{x}^+$,$\dot{x}^\circ$ given in (35).

In this case the dynamics on X_3° are given by a differential inclusion and the solution on X_3° is not well-defined (not even in the sense of Fillipov, since there is no guiding manifold available).

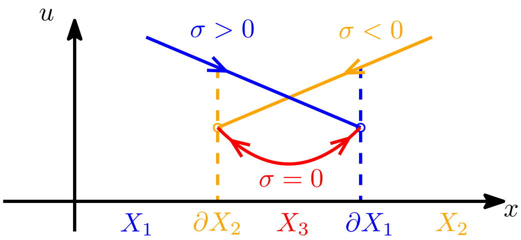

Fig. 2. Possible composition for control laws in Case 3.

Figure 2 illustrates a possible scenario in Case 3. For the set X_i, the possible input u is displayed. It can be seen that for $x \in X_3$ three different u and σ are possible. Further, the continuity of u for $\sigma > 0$ and $\sigma = 0$ at the boundary of X_1 is illustrated.

For the displayed scenario the state x is moving towards the boundary of X_1 for $\sigma > 0$. For $\sigma < 0$ it moves to the boundary of X_2. For $\sigma = 0$ the vector fields point to the boundary of X_3 as well. Thus, choosing $\sigma = 0$ whenever $x \in X_3$ leads to a chattering of the solution at the boundary of X_3.

However, depending on the choice u and σ on X_3 various solutions are possible. Arbitrary switching between the three different values of the sliding variable in the interior of X_3 may lead to a complex manifold of solutions. While this manifold might include all possible paths in the one-dimensional case, its structure is more complex for higher dimensional systems.

4.4 Sliding-mode dynamics and disturbance compensation

In this section we study the internal dynamics induced by the sliding-mode and the relative-degree altering disturbance. In the spirit of [22] we shall distinguish the states in Byrnes-Isidori form as external states ξ, designed internal states ζ of the dynamics in sliding-mode and inherited internal states η of the open-loop system.

For the nominal design with no altering disturbance, we choose the state transformation

$$
\begin{aligned}
\xi_1 &= \sigma && \text{(38a)}\\
\zeta_1 &:= \tau_1(x) = h(x)\,, && \text{(38b)}\\
\zeta_i &:= \tau_i(x) = \mathcal{L}_f^{i-1} h(x)\,, \quad i \in \{2, \dots, n-2\}\,, && \text{(38c)}\\
\zeta_{n-1} &:= \tau_{n-1}(x) = \mathcal{L}_f^{n-2} h(x) && \text{(38d)}
\end{aligned}
$$

and obtain the reduced dynamics for $\sigma \equiv 0$, i.e.

$$
\begin{aligned}
\dot{\zeta}_1 &= \zeta_2\,, && \text{(39a)}\\
\dot{\zeta}_i &= \zeta_{i+1}\,, \quad \text{for} \quad i \in \{2, \dots, n-2\}\,, && \text{(39b)}\\
\dot{\zeta}_{n-1} &= \gamma(\zeta_1, \dots, \zeta_{n-1}) && \text{(39c)}
\end{aligned}
$$

which are stable by design.

In the case of a relative degree altering disturbance, we cannot choose $\sigma(x, u)$ as the external state ξ_1, because $\sigma(x, u)$ depends on the input. Instead, we use the transformation (38b)-(38d) and the internal state $\eta := \tau_n(x)$ as diffeomorphism and with $\mathcal{L}_g \tau_n(x) = 0$. For $\sigma \equiv 0$ we obtain the closed-loop sliding-mode dynamics

$$
\begin{aligned}
\dot{\zeta}_1 &= \zeta_2\,, && \text{(40a)}\\
\dot{\zeta}_i &= \zeta_{i+1}\,, \quad \text{for} \quad i \in \{2, \dots, n-2\}\,, && \text{(40b)}\\
\dot{\zeta}_{n-1} &= \gamma(\zeta_1, \dots, \zeta_{n-1}) && \text{(40c)}\\
\dot{\eta} &= \mathcal{L}_{f+\phi} \tau_n(x)|_{x=\tau^{-1}(\xi,\eta)}\,. && \text{(40d)}
\end{aligned}
$$

First we note that the zero-dynamics (40) in sliding-mode with disturbance are not reduced in dimension as we observe in the nominal case (39). However, the dynamics described by ζ, which represent the designed nominal sliding mode are not affected by the disturbance. Hence, matched and unmatched disturbances are compensated on ζ and thus are invisible at the output y.

Still, the relative degree altering disturbances ϕ introduce additional internal dynamics (40d) and may even render them unstable as we shall illustrate in Section 6. In such case, the unbounded internal state will render the control signal unbounded, a scenario that cannot occur for relative degree preserving disturbances.

In order to analyse the stability properties and obtain constructive results we need to narrow the system class. In the next section we shall study a linear system that may be obtained generically from a feedback linearisation. We analyse the resulting dynamics, existence of equilibria and derive sufficient conditions for which global asymptotic stability is retained in the presence of relative-degree altering disturbances.

5 Second-order Case

In this section we shall study the following class of systems in detail:

$$\dot{x} = Ax + Bu + \phi \tag{41a}$$
$$y = Cx \tag{41b}$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{42}$$

The nominal system shall have the relative degree $r = 2$, i.e. $a_{12}, b_2 \neq 0$. Note that except for the structure of B and C this represents the most general parametrisation for the system class.

In order to alter the relative degree in the second-order case the disturbance has to act on the input. Thus we consider $\phi = \begin{bmatrix} b_1 & 0 \end{bmatrix}^\top u$ and define the input matrix of the disturbed system as

$$\widetilde{B} := \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

It is readily checked that $\mathcal{L}_g \mathcal{L}_f^0 h(x) = b_1 \neq 0$ and thus the disturbed system has a relative degree of 1.

A canonical choice of σ in (13) satisfying (14) is

$$\sigma = y + k_1 \dot{y} = Cx + k_1 CAx + k_1 C\widetilde{B}u, \quad k_1 > 0. \tag{43}$$

Then the variables from (16) (17), and (19) take the form

$$\varsigma_\phi(x) = k_1 C\widetilde{B} \qquad s_\phi(x) = Cx + k_1 CAx$$
$$\alpha(x) = -\frac{(C + k_1 CA)Ax}{(C + k_1 CA)B} \qquad q(x) = \frac{L}{(C + k_1 CA)B}.$$

The first-order control law (18) becomes

$$u = \frac{-1}{(C + k_1 CA)B}\big((C + k_1 CA)Ax + L\,\mathrm{sgn}(\sigma)\big), \tag{44}$$

with $L > 0$.

Substituting (44) into (41) we can write the closed-loop dynamics in the reaching phase (36a) for $x \in X_1 \cup X_2$ as

$$\dot{x} = A_x x + E_x\,\mathrm{sgn}(\sigma) \tag{45}$$

with the matrices

$$A_x = A - \frac{\widetilde{B}(C + k_1 CA)A}{(C + k_1 CA)B}, \quad E_x = \frac{-L\widetilde{B}}{(C + k_1 CA)B} \tag{46}$$

and equilibria

$$x_{\mathrm{R}i} = (-1)^i A_x^{-1} E_x \tag{47}$$

if A_x regular. Existence and location of these equilibria are discussed in Section 5.4.1.

5.1 Zero dynamics with respect to y introduced by the uncertainty

Note that the nominal system with $\phi = 0$ has relative degree 2. The disturbed system has relative degree 1 and thus exhibits internal dynamics.

Define the orthogonal complement $\widetilde{B}^\perp = \begin{bmatrix} -b_2 & b_1 \end{bmatrix}^\top$ such that $\widetilde{B}^\perp \widetilde{B} = 0$. Choosing $\xi := Cx$ and $\eta := \widetilde{B}^\perp x$ we obtain the following internal and external dynamics

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} a_{11} + a_{12}\frac{b_2}{b_1} & \frac{a_{12}}{b_1} \\ a_{21}b_1 - \frac{b_2}{b_1} - a_{11}\frac{b_2}{b_1} & -a_{12}\frac{b_2}{b_1} + a_{22} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} b_1 u \\ 0 \end{bmatrix}.$$

Thus, the zero dynamics introduced by the altering disturbance ϕ take the form

$$\dot{\eta} = \widetilde{B}^\perp \begin{bmatrix} C \\ \widetilde{B}^\perp \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \eta \end{bmatrix} = \left(-a_{12}\frac{b_2}{b_1} + a_{22}\right)\eta.$$

For stable zero dynamics we require, c.f. (12),

$$-a_{12}\frac{b_2}{b_1} + a_{22} < 0. \tag{48}$$

5.2 Control law and state sets

To characterise the sets X_i in (20)-(22), we define

$$w^\top := C + k_1 CA - k_1 C\widetilde{B}\frac{C + k_1 CA}{(C + k_1 CA)B}A, \tag{49}$$
$$d := k_1 C\widetilde{B}\frac{L}{(C + k_1 CA)B} = \frac{b_1}{a_{12}b_2}L, \tag{50}$$

and obtain

$$s_\phi(x) + \varsigma_\phi(x)\alpha(x) = w^\top x$$
$$\varsigma_\phi(x)q(x) = d.$$

Then, the conditions (25) describing the sets X_i take the form

$$w^\top x > d \qquad \text{for } x \in X_1 \tag{51a}$$
$$w^\top x < -d \qquad \text{for } x \in X_2 \tag{51b}$$
$$|w^\top x| \leq |d| \qquad \text{for } x \in X_3. \tag{51c}$$

Thus, the boundary ∂X_3 of the set X_3 is given by the parallel lines $\pm w^\top x$, whose minimal distance is $2|d|$. Moreover, with Lemma 4 and 6 we can determine the occuring cases based on this d as

$$\begin{aligned} d > 0 \quad &\Leftrightarrow \quad X_1 \cap X_2 = \emptyset \text{ (Case 1)},\\ d = 0 \quad &\Leftrightarrow \quad X_3^\circ = \emptyset \text{ (Case 2)},\\ d < 0 \quad &\Leftrightarrow \quad X_1 \cap X_2 \neq \emptyset \text{ (Case 3)}. \end{aligned}$$

Thus with Lemma 6 we have multiple solutions for the closed-loop system if $d < 0$. Accordingly, with $k_1, L > 0$ we obtain Case 3 with (50) for

$$\frac{b_1}{a_{12}b_2} < 0\,, \tag{52}$$

otherwise the control (44) is uniquely defined.

In view of the internal dynamics (48) we observe that a disturbance that yields Case 3 tends to destabilise the internal dynamics while the disturbance in Case 1 tends to stabilise the internal dynamics.

5.3 Sliding-mode dynamics of the closed loop system

As discussed in Section 4.4 the stability of the sliding-mode dynamics is not guaranteed by design due to the impact of the altering disturbance (Case 1 and 3). For $\sigma(x,u) = 0$ the control u given by (29c) has the form

$$u^\circ = -\frac{1}{k_1 C\widetilde{B}}(Cx + k_1 CAx)\,.$$

This yields the sliding-mode dynamics

$$\dot{x} = A_\sigma x \tag{53}$$

with

$$\begin{aligned} A_\sigma &:= A - \frac{\widetilde{B}}{k_1 C\widetilde{B}}(C + k_1 CA) \qquad (54)\\ &= \begin{bmatrix} -\frac{1}{k_1} & 0 \\ a_{21}b_1 - a_{12}\frac{b_2^2}{b_1} - a_{11}b_2 + a_{22}b_2 & -a_{12}\frac{b_2}{b_1} + a_{22} \end{bmatrix}. \end{aligned}$$

Thus the sliding-mode dynamics are stable if and only if $k_1 > 0$ and the zero dynamics of the original system with respect to y and uncertainty ϕ are stable as ensured by condition (48).

Remark 15 *Note that asymptotic stability of* (53) *does not imply the global asymptotic stability of the closed-loop system* (41),(44) *as the analysis in the following section reveals.*

5.4 Analysis of the unique solution (Case 1)

In order to analysis the system properties in full detail, we resort to Case 1, i.e. $d > 0$, for which a unique solution exists. The complete closed-loop dynamics of (41) with control law (44) are given by

$$\dot{x} = \begin{cases} A_x x + E_x, & w^\top x > d \\ A_\sigma x, & |w^\top x| \le d \\ A_x x - E_x, & w^\top x < -d\,. \end{cases} \tag{55}$$

We shall assume that A_x has real, semi-simple eigenvalues. The eigenvalues and eigenvectors of A_σ are real as can be seen in Equation (54).

Note that according to Theorem 10 the system dynamics are continuous at the set boundaries, i.e.

$$A_x x + E_x = A_\sigma x \quad \text{for } x \in \{x \mid w^\top x = d\} \tag{56a}$$
$$A_x x - E_x = A_\sigma x \quad \text{for } x \in \{x \mid w^\top x = -d\}\,. \tag{56b}$$

In order to analyse the effect of the disturbance with respect to attractivity and stability of the overall system we first consider the dynamics in the reaching phase. These are governed by the affine dynamics $\dot{x} = A_x x \pm E_x$. We first discuss the existence and possible location of equilibria. Then we consider the location of the equilibria with respect to the sets X_i and their attractivity properties to formulate conditions for the global asymptotic stability of the closed-loop system (55).

5.4.1 Existence and location of equilibria in the reaching phase

In this section we examine the existence of equilibria for the affine dynamics in the reaching phase, $\dot{x} = A_x x \pm E_x$ with regard to the disturbance b_1. Therefore it is convenient to write $A_x(b_1)$, $E_x(b_1)$ $w(b_1)$ in order to to make the dependence on b_1 explicit.

Throughout this section we assume that any disturbance $b_1 \neq 0$ leads to Case 1., i. e. $\frac{b_1}{a_{12}b_2} > 0$.

We shall also denote the system matrices in the nominal case $b_1 = 0$ by $\bar{A}_x := A_x(0)$, $\bar{E}_x := E_x(0)$. The nominal switching function is denoted by $\sigma(x) = \bar{w}^\top x$ with $\bar{w} = w(0)$. Using (42) and (46) it can be readily verified that

$$\bar{A}_x = \begin{bmatrix} a_{11} & a_{12} \\ \frac{-(1+k_1 a_{11})a_{11}}{k_1 a_{12}} & \frac{-(1+k_1 a_{11})}{k_1} \end{bmatrix}, \quad \bar{E}_x = \frac{-L}{k_1 a_{12}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{57a}$$

$$\bar{w}^\top = \begin{bmatrix} 1 + k_1 a_{11} & k_1 a_{12} \end{bmatrix}, \qquad \bar{d} = 0\,. \tag{57b}$$

The closed loop dynamics of the reaching phase can be written as

$$\dot{x} = A_x(b_1)x + (-1)^{i-1}E_x(b_1) \quad x \in X_i, i \in \{1,2\}\,.$$

In order to determine the equilibria (47), we recall with Equation (57) that $E_x(b_1)$ is not identical to zero if $L \neq 0$. Further, to obtain the inverse of A_x written as $A_x^{-1} = \frac{\mathrm{adj}(A_x(b_1))}{\det(A_x(b_1))}$, we consider the determinant for $A_x(b_1)$, which has the form

$$\det(A_x(b_1)) = -b_1 \frac{a_{11}k_1 + 1}{a_{12}b_2k_1} \det(A) \tag{58}$$

and is non-zero for $b_1 \neq 0$ and $a_{11}k_1 \neq -1$. Then there exist two equilibria $x_{\mathrm{R}i}(b_1), i \in \{1,2\}$, given by

$$x_{\mathrm{R}i}(b_1) = \frac{(-1)^i}{\det(A_x(b_1))} \mathrm{adj}(A_x(b_1))E_x(b_1). \tag{59}$$

For small disturbances $b_1 \to 0$, $x_{\mathrm{R}i}(b_1)$ converges towards the asymptote given by the vector $\mathrm{adj}(\bar{A}_x)\bar{E}_x \in \mathbb{R}^2$. The following proposition presents a condition for which this asymptote does not coincide with the sliding surface $\sigma(x) = 0$ for the nominal case and, in particular, $x_{\mathrm{R}i}(b_1)$ lies outside of $X_3(b_1)$ for small disturbances.

Proposition 16 *If*

$$\bar{w}^\top \mathrm{adj}(\bar{A}_x)\bar{E}_x \neq 0\,, \tag{60}$$

then there exists a disturbance $\tilde{b}_1 \in \mathbb{R}$ such that the equilibria are for all smaller disturbances $b_1 \in \mathbb{B}(0, \tilde{b}_1)$ not part of the sliding manifold, i.e.

$$x_{\mathrm{R}i}(b_1) \notin X_3(b_1)\,, \quad i \in \{1,2\}\,. \tag{61}$$

PROOF. Due to the convergence of d and $\det(A_x)$ to zero and $w^\top(b_1)\,\mathrm{adj}(A_x(b_1))E_x(b_1)$ to a non-zero value, there exists $\tilde{b}_1 \in \mathbb{R}$ such that

$$|w^\top(b_1)\,\mathrm{adj}(A_x(b_1))E_x(b_1)| > |\det(A_x(b_1))d(b_1)| \tag{62}$$

for all $b_1 \in \mathbb{B}(0, \tilde{b}_1)$. Consequently, we obtain for $i \in \{1,2\}$

$$|w^\top(b_1)x_{\mathrm{R}i}(b_1)| \overset{(59)}{=} \frac{|w^\top(b_1)\,\mathrm{adj}(A_x(b_1))E_x(b_1)|}{|\det(A_x(b_1))|} \tag{63}$$

$$\overset{(62)}{>} |d(b_1)| \tag{64}$$

which leads with Equation (51) to $x_{\mathrm{R}i}(b_1) \notin X_3(b_1)$. □

Typically, the SMC is designed to reach the sliding surface $\sigma(x) = 0$ in finite time. Thus we expect that (60) holds for the nominal case. Indeed using (57) for $L, k_1 \neq 0$ leads to the condition

$$\bar{w}^\top \,\mathrm{adj}\, \bar{A}_x\bar{E}_x = \frac{L}{k_1} \neq 0\,.$$

Remark 17 *Symbolic calculations utilising computer algebra have also shown that Equation* (60) *holds up to order 4 if the relative degree condition $CA^{i-1}B = 0$ for $i < n$ is fulfilled. No analytic proof has been found to the knowledge of the authors up to this point for the identities of higher order.*

We conclude that for small b_1 the equilibria in the reaching phase can be in either of the two sets X_1, X_2 but not in X_3. We say that the equilibria are on the *same sides* if $x_{\mathrm{R}i} \in X_j$ for $i = j$ and that they are on the *opposite sides* if $x_{\mathrm{R}i} \in X_j$ for $i \neq j$. For symmetry reasons, the equilibria are on the *same side* if and only if

$$w^\top x_{\mathrm{R}1} > d\,. \tag{65}$$

This allows us to investigate the global asymptotic stability of the closed loop system.

5.4.2 *Global asymptotic stability of the closed loop system*

In this section, we analyse global asymptotic stability of the closed loop system. We start by providing a sufficient condition for global stability and then continue to develop sufficient conditions which ensure the stability. We start with the sufficient condition for global stability.

Proposition 18 *If system* (55) *is globally asymptotic stable at the origin then A_x is asymptotically stable and the equilibria are on the opposing sides, i. e.*

$$-w^\top A_x^{-1}E_x < d.$$

PROOF. Proof per contradiction. If A_x is asymptotically stable and the equilibria (47) are on the same sides, then there exist initial conditions that converge directly into the respective equilibria and thus never enter X_3 that contains the origin. Hence the closed-loop system is not globally asymptotically stable. Similar reasoning yields that the system is not globally asymptotically stable if A_x is unstable and the equilibria are on either side. □

For the remainder of this section we shall study systems with asymptotically stable A_x and equilibria on the opposing sides.

We consider the piecewise linear Lyapunov function, e.g. [16], of the form

$$V(x) = \max\left(|l_1^\top x|, \ldots, |l_p^\top x|\right) \tag{66}$$

with $l_i \in \mathbb{R}^n$ for $i \in \{1, \ldots, p\}$ defining faces of a polygon-shaped level set. We define the set of all indices denoting the active faces for the Lyapunov function for a given point x:

$$I_V(x) := \left\{ i \in (1, \ldots, p) \mid V(x) = |l_i^\top x| \right\} .$$

Note if x is in the inner of a polygon face, the set $I_V(x)$ consists of exactly one element. Otherwise x is on some edge of the polygon and the number of elements of $I_V(x)$ corresponds to the number of adjacent faces to the edge.

As V is a non-differential function, we shall use the Dini-derivative as in [1] given by

$$D^+\big(V(x(t))\big) = \limsup_{\Delta \to 0^+} \frac{V(x(t+\Delta)) - V(x(t))}{\Delta} .$$

Simply speaking, the Dini-derivative takes the value of the limit of the difference quotient defined by the trajectory x. At the edges of the piecewise linear Lyapunov function (66), the Dini-derivative takes a value given by one of the adjacent faces. More precisely:

Lemma 19 *Let x be a differentiable trajectory and V be defined by* (66)*. Then, for all $t > 0$ exists $i \in I_V(x(t))$ such that*

$$D^+(V(x(t))) = l_i^\top \dot{x} . \tag{67}$$

PROOF. It is clear for any sequence $x(t_k)$ that the Lyapunov function V can only take one of the values $|l_i^\top x(t_k)|$ for $i \in \{1, \ldots, p\}$. Therefore we obtain with the continuity of x at least one $j \in \{1, \ldots, p\}$ such that $V(x(t_k)) = l_j x(t_k)$. Then the claim follows with the definition of the Dini-derivative and the lim sup. □

Consequently, if the vector fields are pointing inwards with respect to every active face, then

$$D^+(V(x(t))) < 0 \tag{68}$$

and $V(x)$ is a Lyapunov function for the system generating the trajectory x.

Consider the piecewise affine system (55). Since the vector field is continuous along the set boundaries we can show the following.

Proposition 20 *Let V be a candidate Lyapunov function of the form* (66) *with $n = 2$ for system* (55) *and let for all states $x \in X_1^{\mathrm{cl}} \cup X_2^{\mathrm{cl}}$ hold*

$$l_i^\top (A_x x + E_x) < 0, \quad \forall\, i \in I_V(x) . \tag{69}$$

Then, for all states $x \in X_3 \setminus \{x \mid w^\top x = 0\}$ it holds

$$l_i^\top A_\sigma x < 0, \quad \forall\, i \in I_V(x) . \tag{70}$$

PROOF. With Equation (69) and the fact that the vector field is continuous at the boundary as in (56b), we have at the boundary for all $x \in \partial X_3$ that

$$l_i^\top A_\sigma x < 0, \quad \forall\, i \in I_V(x) .$$

We can describe every point of $X_3 \setminus \{x \mid w^\top x = 0\}$ using a point from the boundary, i.e for all points $y \in X_3 \setminus \{x \mid w^\top x = 0\}$ there is $\mu \in (0, 1]$ and $x \in \partial X_3$, such that $y = \mu x$. This leads to the claim as

$$l_i^\top A_\sigma y = l_i^\top A_\sigma \mu x = \mu l_i^\top A_\sigma x < 0, \quad \forall\, i \in I_V(x) .$$

□

Remark 21 *Proposition 20 can be easily extended to systems* (55) *with $n > 2$.*

Remark 22 *The result also holds in a similar fashion for homogeneous Lyapunov functions. An extension to non-linear system dynamics can be achieved with homogeneity of the vector fields and convexity of the set X_3.*

With Lemma 19 we conclude that (69) and (70) imply that (68) holds globally for $x \in \mathbb{R}^n \setminus \{x \mid w^\top x = 0\}$.

We shall now construct a Lyapunov function of the reaching phase. Denote the eigenvalues and left eigenvectors of A_x by $\lambda_1, \lambda_2 < 0$ and $v_1, v_2 \in \mathbb{R}^2$, respectively, such that $v_i^\top A_x = \lambda_i v_i^\top$, and consider the state transformation $\tilde{x} = T^{-1} x$ with $T^{-1} = [v_1 \;\; v_2]^\top$.

Proposition 23 *Given the system* (55) *such that*

$$w^\top T e_i v_i^\top E_x < 0 \quad \textit{for } i \in \{1, 2\} , \tag{71}$$

and the Lyapunov function (66) *with $l_i^\top = w^\top T e_i v_i^\top$, $i = 1, 2$, such that*

$$D^+(V(x)) < 0 \quad \textit{for } x \in \{x \in \mathbb{R}^2 \mid w^\top x = 0\} . \tag{72}$$

Then the origin of (55) *is globally asymptotically stable.*

PROOF. We consider the state transformation $\tilde{x} = T^{-1}x$ that brings the dynamics of (55) into diagonal form:

$$\dot{\tilde{x}} = \begin{cases} \tilde{D}_x\tilde{x} + \tilde{E}_x, & \tilde{w}^\top\tilde{x} > d\,, \\ T^{-1}A_\sigma T\tilde{x}, & -d < \tilde{w}^\top\tilde{x} < d\,, \\ \tilde{D}_x\tilde{x} - \tilde{E}_x, & \tilde{w}^\top\tilde{x} < -d\,, \end{cases} \tag{73}$$

where $\tilde{D}_x = T^{-1}A_xT = \mathrm{diag}(\lambda_1, \lambda_2)$, $\tilde{E}_x = T^{-1}E_x$ and $\tilde{w}^\top = w^\top T$.

Note that for the i-th element of $\tilde{w}$ we have $\tilde{w}^i = w^\top Te_i$. Thus the Lyapunov function (66) in transformed coordinates is diagonal and reads

$$V(T\tilde{x}) = \max(|\tilde{l}_1^\top\tilde{x}_1|, |\tilde{l}_2^\top\tilde{x}_2|)\,, \tag{74}$$

where $\tilde{l}_1^\top = \begin{bmatrix}\tilde{w}_1 & 0\end{bmatrix}$ and $\tilde{l}_2^\top = \begin{bmatrix}0 & \tilde{w}_2\end{bmatrix}$. Certainly, if $V(T\tilde{x})$ is a Lyapunov function for (73) then $V(x)$ is a Lyapunov function for (55).

In the transformed coordinates condition (71) using the identities $\tilde{w}_1 = w^\top Te_1$ and $v_1^\top E_x = e_1^\top\tilde{E}_x$ reads:

$$\tilde{w}_i e_i^\top\tilde{E}_x < 0 \quad \text{for } i \in \{1,2\}\,. \tag{75}$$

The derivative of V along the faces of the level set are given by

$$\begin{bmatrix}\tilde{l}_1^\top\dot{\tilde{x}} \\ \tilde{l}_2^\top\dot{\tilde{x}}\end{bmatrix} = \begin{bmatrix}\tilde{l}_1^\top(\tilde{D}_x\tilde{x} + \tilde{E}_x) \\ \tilde{l}_2^\top(\tilde{D}_x\tilde{x} + \tilde{E}_x)\end{bmatrix} \tag{76}$$

In the reaching phase X_1^{cl} we have $\tilde{w}^\top\tilde{x} \geq d$. Expanding the scalar product yields

$$\tilde{w}_1\tilde{x}_1 \geq d - \tilde{w}_2\tilde{x}_2 > -\tilde{w}_2\tilde{x}_2\,. \tag{77}$$

Let $\tilde{x}$ be a point on the level set of (74) such that $|\tilde{w}_1\tilde{x}_1| \geq |\tilde{w}_2\tilde{x}_2|$, i.e. the face defined by $\tilde{l}_1^\top$ is active. With (77) it is readily verified that $\tilde{w}_1\tilde{x}_1 > 0$. Using the first component of (76) we obtain

$$\dot{V}(x) = \tilde{l}_1^\top(\tilde{D}_x\tilde{x} + \tilde{E}_x) = \lambda_1\tilde{w}_1\tilde{x}_1 + \tilde{w}_1 e_1^\top\tilde{E}_x\,.$$

With (75) and $\lambda_1 < 0$ we find $\dot{V}(x) < 0$.

The same result is obtained for points $\tilde{x}$ on the other level set defined by the face $\tilde{l}_2^\top$. For symmetry reasons we obtain the same result for the reaching phase X_2^{cl} and with Lemma 19 we have

$$D^+(V(x)) < 0 \quad \text{for } x \in X_i^{\mathrm{cl}}, i = 1,2\,.$$

For the sliding phase we employ Proposition 20 and the condition in (72) and obtain for $i \in \{1,2\}$

$$D^+(V(x)) < 0 \text{ for } x \in X_3\,.$$

□

Note that the Lyapunov function is chosen such that two of the four corners are always in the sliding manifold (see Fig. 7), i.e. it is

$$w^\top x = 0 \quad \forall\, x \in \left\{x \in \mathbb{R}^2 \mid w^\top Te_1v_1x = -w^\top Te_2v_2x\right\}\,.$$

Note further that the condition in (71) is equivalent to

$$w^\top Te_iv_i^\top A_x^{-1}E_x > 0 \tag{78}$$

which makes evident that this is an additional constraint on the location of the equilibria of the reaching phase dynamics. In transformed coordinates $\tilde{x}$, the equilibria must lie in one of the four quadrants. Which quadrant depends on the vector w whose entries as well as the entries of $\tilde{w}$ might be negative. It guarantees in particular, however, that the equilibria are on the opposing sides because we obtain with Condition (78) that

$$\begin{aligned} 0 &> w^\top Te_1v_1^\top E_x + w^\top Te_2v_2^\top E_x & (79)\\ &= w^\top(Te_1v_1^\top + Te_2v_2^\top)A_x^{-1}E_x & (80)\\ &= w^\top x_{\mathrm{R}2}\,. & (81) \end{aligned}$$

Therefore, we have given in this section a constructive way to design a Lyapunov function for a subset of all systems with stable A_x and equilibria on the opposite sides.

6 Numerical Examples

We illustrate the phenomena and results obtained in the previous sections. We start by studying some phenomena that can occur if the system is in Case 3. We continue by discussing the potential loss of global stability even in Case 1 and end with an example for a global asymptotically stable system despite relative degree altering disturbance using the proposed Lyapunov function.

6.1 Oscillations and induced sliding-mode (Case 3)

In Case 3 the reaching phases overlap with the sliding phase and the control as well as the closed-loop dynamics (37) are not well-defined. In this section, we illustrate two possible solutions by a simulation example.

Consider the second-order system (41), (42) with

$$a_{11} = 100, \qquad a_{12} = -1, \qquad a_{21} = 1,$$
$$a_{22} = 0.1, \qquad b_1 = -0.01, \qquad b_2 = 1\,.$$

For these parameters, $d < 0$ in (52) and thus we have in Case 3, where the control law (44) is not well-defined.

Figure 3 shows the phase plane of the system. The red and dark blue solid lines are the boundaries of the reaching sets X_2 and X_1, respectively. Note that the reaching phases overlap and thus the boundaries are on opposing sides of the sliding phase X_3. Two possible solutions x for the identical initial value $x(0)$ in X_3 are depicted. The green solution is obtained by choosing $u = u^\circ$ in (29) within X_3. The resulting trajectory tends towards the boundary of X_3. As the dynamics in the reaching phase X_1 just outside of X_3 point towards X_3, the solution is constructed in the Filippov sense and results in a sliding motion along the boundary of X_3. Note however that this is not the designed sliding-mode $\sigma \equiv 0$. Figures 4 and 5 show the evolution of the control signal and sliding-variable σ, respectively. The control signal in sliding mode shows discretisation chattering (due to discrete-time simulation) with a non-zero average as might be expected for non-vanishing disturbances. However, the sliding-variable also shows discretisation chattering. In a continuous-time analysis we may expect the sliding-variable to converge to a non-zero constant. Thus the originally intended control goal $\sigma \equiv 0$ is not achieved.

The purple line depicts the solution for which u in (29) is chosen as u^+ or u^- only. We switch the control signal between those values at the boundaries $\partial X_1, \partial X_2$, respectively. The result is a solution that oscillates in the inner of X_3 between its boundaries as can be seen in Fig. 3. The sliding variable of this solution in Fig. 5 is piecewise continuous. In each segment the sliding variable converges to zero. After reaching zero, i.e. reaching the boundary of X_3, the sliding variable jumps to a non-zero value. This is due to the change of the dynamics outside of X_3. This behaviour can be repeatedly observed. Of course any other choice of switching the control signal (29) yields another valid solution for the same initial state.

6.2 Loss of global stability (Case 1)

This section illustrates an example where global stability is lost due to the relative degree altering uncertainty. We consider the second-order system (41),(42) with

$$a_{11} = 3, \qquad a_{12} = 1, \qquad a_{21} = 1,$$
$$a_{22} = 2, \qquad b_1 = 0.02, \qquad b_2 = 0.1\,.$$

The sliding-variable is chosen according to (43) with $k_1 = 0.05$ and $L = 1$ is the sliding-mode gain in

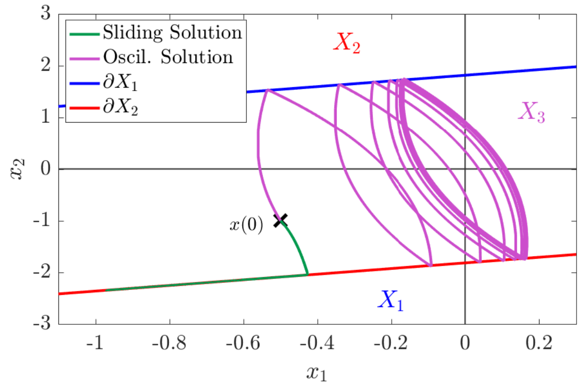

Fig. 3. Two solutions for an initial condition in state-space of Example 6.1.

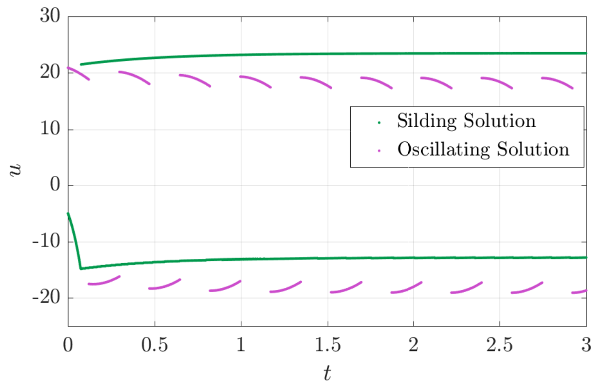

Fig. 4. Evolution of the control signal for the solutions of Example 6.1.

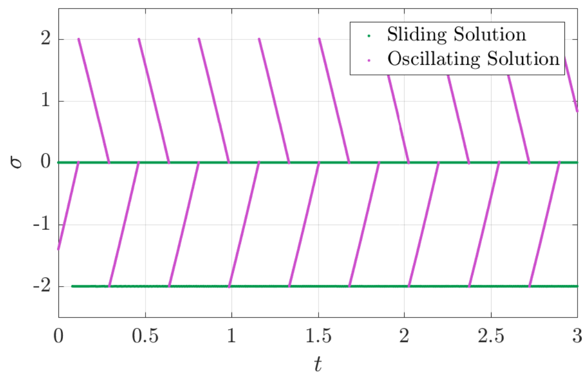

Fig. 5. Evolution of the sliding variable for the solutions of Example 6.1.

(44). With (50) and (49) it is readily verified that $d = 0.2 > 0$ such that the resulting system is in Case 1 and $w^\top = \begin{bmatrix} 0.45 & -0.2 \end{bmatrix}$. The dynamics in the reaching

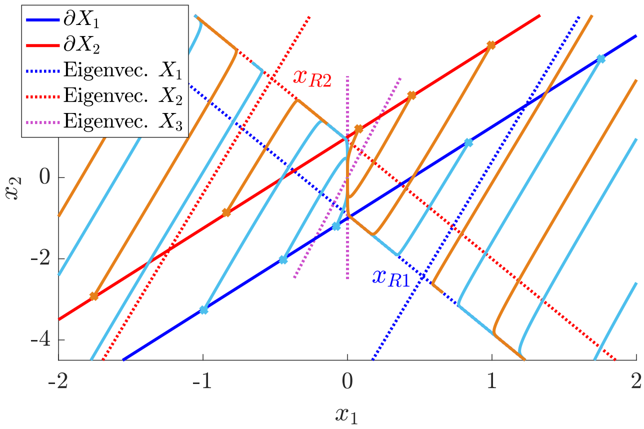

Fig. 6. Solutions for various initial state of Example 6.2.

phase (55) with (46) are given by

$$A_x = \begin{bmatrix} -11 & -4 \\ -69 & -23 \end{bmatrix} \quad \text{and} \quad E_x = \begin{bmatrix} -4 \\ -20 \end{bmatrix}.$$

The equilibria are obtained by (59) and for the reaching phase X_1 we have $x_{R1}^\top = \begin{bmatrix} 0.5217 & -2.4348 \end{bmatrix}$. It is readily verified that $w^\top x_{R1} > d$ and thus the equilibria are within the reaching phase (*same side*). The eigenvalues of A_x are $\lambda_1 = 0.6635$ and $\lambda = -34.6635$, thus the equilibria are unstable.

The plot of the phase plane in Figure 6 shows the boundaries of the sets X_1 and X_2 as solid red and blue line, respectively. The sliding-phase X_3 is in-between these lines. Note that in Case 1 the reaching phases do not overlap. The eigenvectors in each set X_i are depicted as dashed lines. It can be seen that the equilibria of each reaching-set dynamics are within the respective set, i.e. we have the case *same side*. A number of solutions of the closed-loop system are depicted as solid lines with initial states chosen just outside the sliding-phase, orange and light-blue respectively. It can be seen that the resulting solutions are continuous, in particular at the boundary of X_1 and X_2 as stated in (56b). We observe that the sliding-phase is still attractive for some initial states close to the origin. However, trajectories starting east of the blue eigenvectors or west of the red eigenvectors diverge.

6.3 Global stability and Lyapunov function (Case 1)

We consider system (41) with the parameters

$$a_{11} = 3, \qquad a_{12} = 1, \qquad a_{21} = 1,$$
$$a_{22} = 0, \qquad b_1 = 0.02, \qquad b_2 = 1\,.$$

Again the sliding-variable is chosen according to (43) with $k_1 = 0.05$ and $L = 1$. With (50) and (49) it is readily verified that $d = 0.02 > 0$ such that the resulting system is in Case 1 and $w^\top = \begin{bmatrix} 1.08 & 0.027 \end{bmatrix}$. The dynamics in the reaching phase (55) with (46) are given by

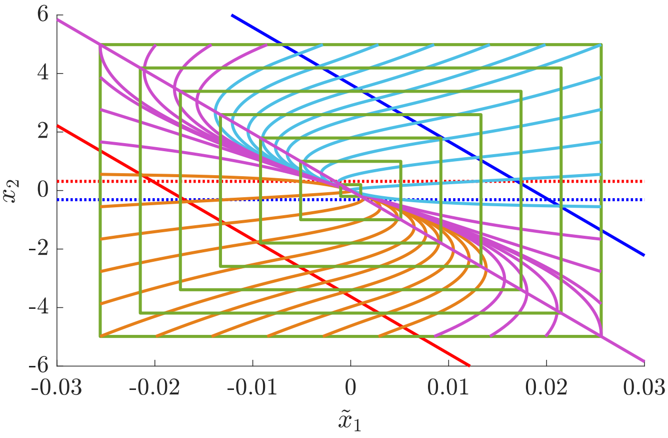

Fig. 7. Solutions in transformed coordinates $\tilde{x}$ with level sets of the Lyapunov function for a globally asymptotically stable system, Example 6.3

$$A_x = \begin{bmatrix} 1.6 & 0.54 \\ -69 & -23 \end{bmatrix} \quad \text{and} \quad E_x = \begin{bmatrix} -0.4 \\ -20 \end{bmatrix},$$

with stable eigenvalues $\lambda_1 = -0.0215$ and $\lambda_2 = -21.3785$. Note that $-w^\top A_x^{-1} E_x < 0$ and thus the equilibria in the reaching phase are on the opposing sides, respectively. Hence $x_R = 0$ is the unique equilibrium for the closed-loop system.

We employ Proposition 23 to establish global asymptotic stability of $x_R = 0$. We construct the transformation matrix $T^{-1} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^\top$ with the left eigenvectors of A_x are given by $v_1^\top = \begin{bmatrix} 0.9997 & 0.0235 \end{bmatrix}$ and $v_2^\top = \begin{bmatrix} 0.9488 & 0.3160 \end{bmatrix}$, and verify condition (71) and (72) using (67) at the point $x = [-w_2 \; w_1]^\top$. Thus the equilibrium is asymptotically stable with piecewise linear Lyapunov function (66).

Figure 7 shows the phase plane in transformed coordinates $\tilde{x} = T^{-1}x$. Note that the eigenvectors in reaching phase (dotted lines) in these coordinates are aligned with the coordinate axis. The remaining two eigenvectors are far outside the displayed range. Multiple level sets of V for $V(\tilde{x}) = 0.0011 + 0.044k$, $k \in \{1, \ldots, 6\}$, are shown in green. A number of trajectories with initial state on the most outer set $V(\tilde{x}) = 0.0275$ are displayed, whose colours correspond to the set of the initial state (light blue: X_1, orange X_2, purple X_3). It can be seen that the trajectories point inwards at the level sets confirming that V is indeed a Lyapunov function for the closed-loop system. Note that the points close to the south eastern corners are almost parallel to the eastern edge but still point inside the level set. We can also illustrate that the

Lyapunov function is constructed such that two corners are on the originally designed sliding manifold $\tilde{w}^\top x = 0$ (purple straight line).

7 Conclusion

Disturbances that change the relative degree of the system may have a strong impact on the closed-loop control system even if a sliding-mode controller is applied. Well-definedness and stability of the solution as well as attractivity of the sliding-manifold may be lost. We derive necessary and sufficient conditions for which such scenario is avoided and that ensure stability and disturbance compensation of unmatched uncertainties. For the second-order case we give a thorough analysis of all cases and we obtain a readily checked condition to distinguish the cases of uniquely and non-uniquely defined solutions. For the former case we present a simple constructive condition for the global asymptotic stability of the closed-loop system using a piecewise linear Lyapunov function.

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