Coding Theory and Cryptography:
A Conference in Honor of Joachim Rosenthal's 60th Birthday

The Marginal Distribution of the Lee Channel and its Applications

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Outline

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1. Preliminaries and Motivation
2. The Lee Channel and its Properties
3. Information Set Decoding
4. Information Set Decoding using Restricted Spheres
    - Bounded Minimum Distance Decoding
    - Decoding Beyond the Minimum Distance

Syndrome Decoding Problem

Assume we send a codeword $x \in \mathcal{C}$ and receive a vector $y = x + e \in (\mathbb{Z}/p^s\mathbb{Z})^n$.

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Given an $(n-k) \times n$ parity-check matrix H of $\mathcal{C}$ and a syndrome $s = yH^\top$, find the length-n vector e such that

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  - Is an NP-hard problem (in the Hamming metric, Lee metric, ...)
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  - generic decoding has a large cost in the Lee metric
- Has a unique solution for a relatively small weight (w.r.t. the GV bound)

Ring-Linear Codes

Let p a prime number and s and n two positive integers.

Definition

A *linear code* $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ is a $\mathbb{Z}/p^s\mathbb{Z}$-submodule of $(\mathbb{Z}/p^s\mathbb{Z})^n$.

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Parameters:

- n is called the *length* of $\mathcal{C}$
- $k := \log_{p^s} |\mathcal{C}|$ is the $\mathbb{Z}/p^s\mathbb{Z}$-*dimension* of $\mathcal{C}$
- $R := k/n$ denotes the *rate* of $\mathcal{C}$.

The Lee Metric

Definition

For $a \in \mathbb{Z}/p^s\mathbb{Z}$ and $e = (e_1, \ldots, e_n) \in (\mathbb{Z}/p^s\mathbb{Z})^n$ we define their *Lee weight*, respectively, by

$$\mathrm{wt_L}(a) := \min(a, \left| p^s - a \right|),$$

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Example over $\mathbb{Z}/5\mathbb{Z}$

- $0: \quad \mathrm{wt}_\mathrm{L}(0) = 0$
- $1: \quad \mathrm{wt}_\mathrm{L}(1) = 1$
- $2: \quad \mathrm{wt}_\mathrm{L}(2) = 2$
- $3: \quad \mathrm{wt}_\mathrm{L}(3) = 2$
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- 4 : $\mathrm{wt_L}(4) = 1$

Properties:

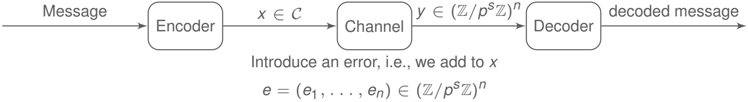
For every $a \in \mathbb{Z}/p^s\mathbb{Z}$ and $e \in (\mathbb{Z}/p^s\mathbb{Z})^n$

- $\mathrm{wt_L}(a) = \mathrm{wt_L}(|\, p^s - a \,|)$
- $\mathrm{wt_H}(a) \leq \mathrm{wt_L}(a) \leq \lfloor p^s/2 \rfloor =: M$
- $\mathrm{wt_H}(e) \leq \mathrm{wt_L}(e) \leq nM$

Outline

The Constant-Weight Lee Channel

Take a linear code $\mathcal{C} \subset (\mathbb{Z}/p^s\mathbb{Z})^n$.



Here: Take e uniformly at random from $e \in \mathcal{S}_{t,p^s}^{(n)} := \left\{ z \in (\mathbb{Z}/p^s\mathbb{Z})^n \;\middle|\; \mathrm{wt}_\mathrm{L}(z) = t \right\}$.

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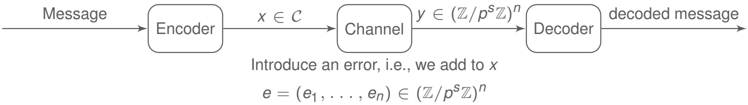

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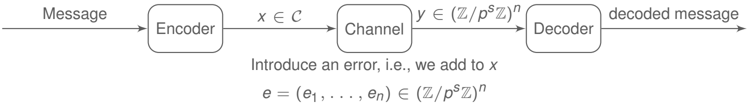

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Lemma

Let $a \in \mathbb{Z}/p^s\mathbb{Z}$ be chosen uniformly at random. Then

$$\delta_{p^s} := \mathbb{E}(\mathrm{wt}_\mathrm{L}(a)) = \begin{cases} \frac{(p^s)^2-1}{4p^s} & \text{if } p^s \text{ is odd,} \\ \frac{p^s}{4} & \text{if } p^s \text{ is even.} \end{cases}$$

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Let E be the random variable corresponding to the realization of a random entry of e.

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Theorem [1]

Assume that the asymptotic relative Lee weight is $T := \lim_{n\to\infty} \frac{t(n)}{n}$. For every $i \in \mathbb{Z}/p^s\mathbb{Z}$ the marginal distribution of E is given by

$$p_i := \mathbb{P}(E = i) = \frac{1}{\sum_{j=0}^{p^s-1} \exp(-\beta\,\mathrm{wt}_\mathrm{L}(j))} \exp(-\beta i)$$

where β is the solution to $T = \sum_{i=0}^{M} \mathrm{wt}_\mathrm{L}(i) p_i$.

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[1] "On the Properties of Error Patterns in the Constant Lee Weight Channel". In: *International Zurich Seminar on Information and Communication (IZS)*. 2022, pp. 44–48.

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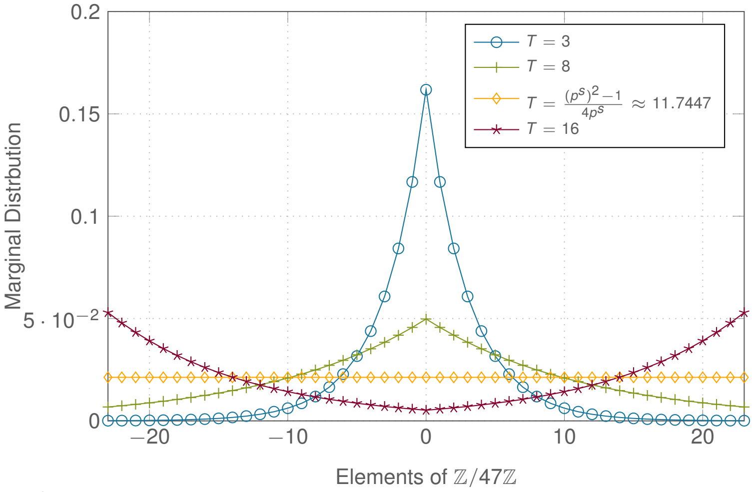
Note $T < \delta_{p^s} \iff \beta > 0$

The Marginal Distribution - Example over $\mathbb{Z}/47\mathbb{Z}$



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Information Set Decoding in the Lee Metric

Consider an instance of the Lee Syndrome Decoding Problem (LSDP):

$$\text{Given } H \in (\mathbb{Z}/p^s\mathbb{Z})^{(n-k)\times n},\ s \in (\mathbb{Z}/p^s\mathbb{Z})^{n-k} \text{ and } t \in \mathbb{N},$$
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  - ⟶ Recent improvements: using partial Gaussian elimination[1]

[1] Matthieu Finiasz and Nicolas Sendrier. "Security bounds for the design of code-based cryptosystems". In: *International Conference on the Theory and Application of Cryptology and Information Security*. Springer. 2009, pp. 88–105.

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[1] Anja Becker et al. "Decoding random binary linear codes in $2^{n/20}$: How 1+ 1= 0 improves information set decoding". In: *Annual international conference on the theory and applications of cryptographic techniques.* Springer. 2012, pp. 520–536.

[2] Alexander May, Alexander Meurer, and Enrico Thomae. "Decoding Random Linear Codes in $\tilde{\mathcal{O}}(2^{0.054n})$". In: *International Conference on the Theory and Application of Cryptology and Information Security.* Springer. 2011

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[1] Violetta Weger et al. "On the hardness of the Lee syndrome decoding problem". In: *Advances in Mathematics of Communications* (2019). DOI: 10.3934/amc.2022029.

[2] André Chailloux, Thomas Debris-Alazard, and Simona Etinski. "Classical and Quantum algorithms for generic Syndrome Decoding problems and applications to the Lee metric". In: *International Conference on Post-Quantum Cryptography*. Springer. 2021, pp. 44–62.

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- The cost of an ISD algorithm is given by

$$\underbrace{\text{nr. of iterations}}_{\frac{1}{\text{success probability per iter.}}} \times \text{cost per iteration}$$

General Framework

We use the idea of partial Gaussian elimination to solve the problem:

1. Find $U \in \mathrm{GL}_{n-k}(\mathbb{Z}/p^s\mathbb{Z})$ such that

$$UH^\top = \begin{pmatrix} \mathbb{I}_{n-k-\ell} & 0 \\ A^\top & B^\top \end{pmatrix}$$

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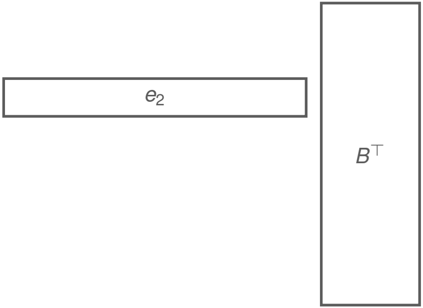
4. Solve the smaller instance of the LSDP. Immediately check whether $e_1 = s_1 - e_2 A^\top$ has Lee weight $t - v$.

Solving the Smaller Instance - Finding e_2

Focus on $e_2 B^\top = s_2$, with $\mathrm{wt}_\mathrm{L}(e_2) = v$



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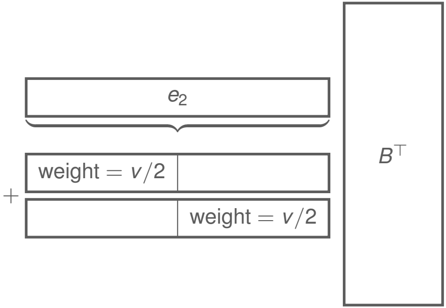

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$$\mathcal{L}_1 := \left\{ y_1 B_1^\top \mid \mathrm{wt}(y_1) = v/2 \right\}$$
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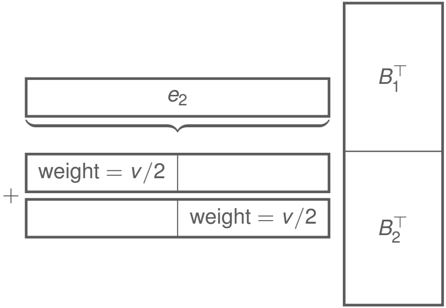

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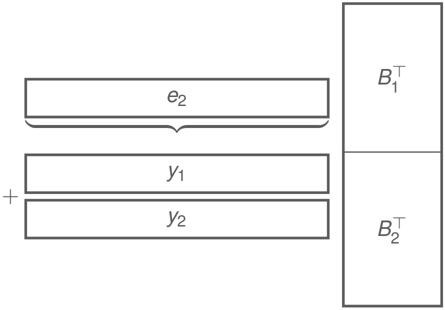

Note: The two vectors $y_1 \in \mathcal{L}_1$ and $y_2 \in \mathcal{L}_2$ share ε nonzero positions. The expected weight of $y_1 + y_2$ is still v.

Outline

New Idea: Using Restricted Spheres

Focus on the small instance of the Lee syndrome decoding problem.

$$\text{Given } B \in (\mathbb{Z}/p^s\mathbb{Z})^{\ell \times (k+\ell)},\ s_2 \in (\mathbb{Z}/p^s\mathbb{Z})^{\ell} \text{ and } v, t \in \mathbb{N}$$
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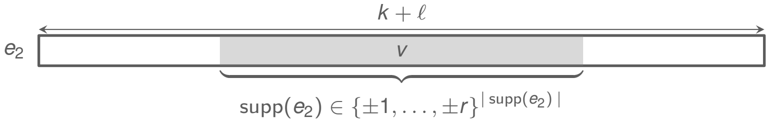
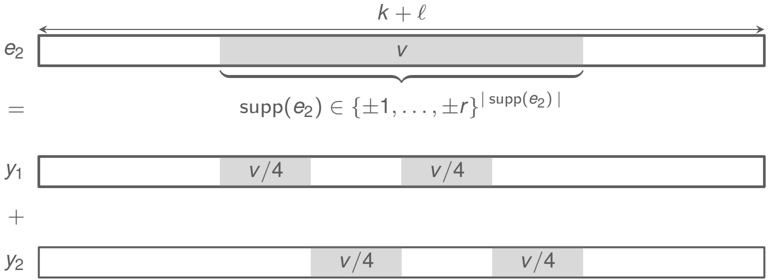
  - for $t/n > M/2$ the contrary is true
- With high probability the least probable entries of e lie **outside** the information set, hence are not in e_2.
- We will restrict e_2 to live either in $\{0, \pm 1, \dots, \pm r\}^{k+\ell}$ or in $\{\pm r, \dots, \pm M\}^{k+\ell}$, respectively.

Bounded Minimum Distance Decoding - Representation of e_2



Bounded Minimum Distance Decoding - Representation of e_2



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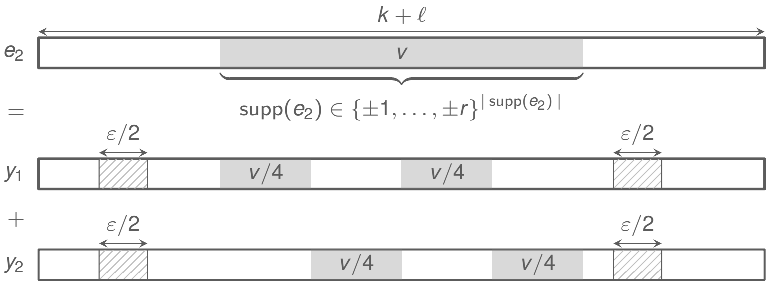

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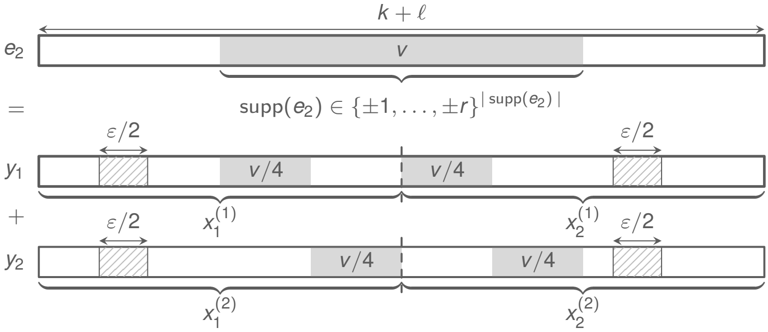
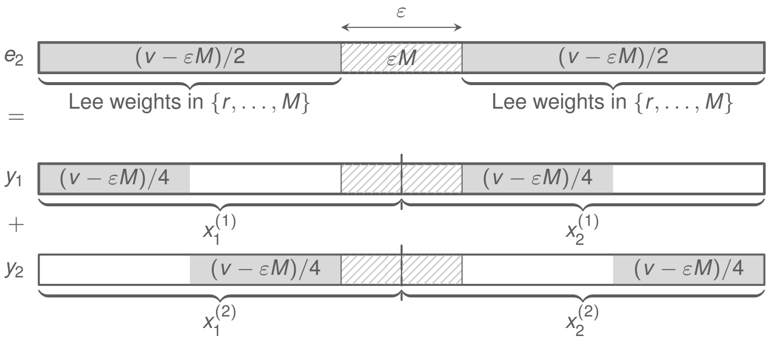


$$\mathcal{B}_i = \left\{ \nu(x) \mid x_{\mathcal{E}_i^c} \in \{0, \dots, \pm r\}^{(k+\ell-\varepsilon)/2}, \mathrm{wt}_{\mathrm{L}}(x_{\mathcal{E}_i^c}) = v/4, x_{\mathcal{E}_i} \in \left(\mathbb{Z}/p^s\mathbb{Z}\right)^{\varepsilon/2}, \nu \in S_{(k+\ell)/2} \right\}$$

Decoding Beyond the Minimum Distance



Bounded Minimum Distance Decoding - BJMM Approach

Recall, $s_2 = e_2 B^\top$, where $e_2 = y_1 + y_2 = (x_1^{(1)}, x_2^{(1)}) + (x_1^{(2)}, x_2^{(2)})$.

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1. Splitting $B = (B_1 \ B_2)$, for $i = 1, 2$ concatenate all $x_1^{(i)}, x_2^{(i)} \in \mathcal{B}_i$ satisfying

$$x_1^{(1)} B_1^\top =_u -x_2^{(1)} B_2^\top,$$
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   They imply the syndrome equations for y_1 and y_2, respectively.

$$y_1 B^\top = 0 \ \text{ and } \ y_2 B^\top = s_2$$

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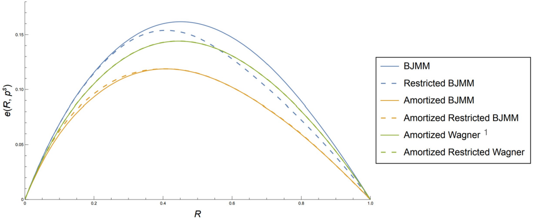
   b) the original LSDP is fulfilled as well
   $$\mathrm{wt}_L(s_1 - (y_1 + y_2) A^\top) = t - v$$

Comparison - Bounded Minimum Distance Decoding in $\mathbb{Z}/47\mathbb{Z}$



1

Algorithm	$e(R^*, p^s)$	R^*
Lee-BJMM	0.1618	0.451
Restricted Lee-BJMM for $r = 5$	0.1539	0.408
Amortized Lee-BJMM	0.1205	0.396
Amortized Restricted Lee-BJMM	0.1189	0.406
Amortized Lee-Wagner	0.1441	0.445
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[1] André Chailloux, Thomas Debris-Alazard, and Simona Etinski. "Classical and Quantum algorithms for generic Syndrome Decoding problems and applications to the Lee metric". In: *International Conference on Post-Quantum Cryptography*. Springer. 2021, pp. 44–62.

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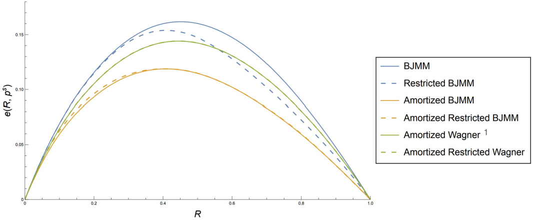

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Thank you for your attention!

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