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Turbulent Equilibrium Conditions for Reynolds-Stress Models

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Bernhard Eisfeld

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Abstract

The turbulent equilibrium conditions are derived for the Reynolds-stress transport equations in two-dimensional incompressible mean flow. The most general formulation of the pressure-strain correlation model is employed so that the result holds for virtually any pressure-strain correlation model suggested so far.

Zusammenfassung

Es werden die Bedingungen für das turbulente Gleichgewicht der Reynolds-Spannungsgleichungen für die zweidimensionale, gemittelte Strömung eines inkompressiblen Fluids hergeleitet. Da die allgemeinst mögliche Formulierung des Modells der Druck-Scher-Korrelation verwendet wird, gilt das Ergebnis für praktisch jedes bisher vorgeschlagene Modell der Druck-Scher-Korrelation.

Symbols

Latin Symbols

Symbol	Dimension	Explanation
b_{ij}	-	Component of Reynolds-stress anisotropy tensor
$C_\alpha^{(s)}$	-	Coefficient of slow term of pressure-strain correlation model, $\alpha = 1, 2$
k	m^2/s^2	Specific kinetic turbulence energy
$P^{(k)}$	m^2/s^3	Production term of specific kinetic turbulence energy
P_{ij}	m^2/s^3	Component of Reynolds-stress production term
$Q_\alpha^{(r)}$	-	Coefficient of rapid term of pressure-strain correlation model, $\alpha = 1 \dots 9$
R_{ij}	m^2/s^2	Component of specific Reynolds stress tensor
S_{ij}	$1/s$	Component of mean-flow strain-rate tensor
U_i	m/s	Component of mean-flow velocity
x_i	m	Coordinate

Greek Symbols

Symbol	Dimension	Explanation
δ_{ij}	-	Kronecker symbol
ϵ	m^2/s^3	Isotropic dissipation rate
ϵ_{ij}	m^2/s^3	Component of Reynolds-stress dissipation term
Π_{ij}	m^2/s^3	Component of pressure-strain correlation
$\Pi_{ij}^{(r)}$	m^2/s^3	Component of rapid part of pressure-strain correlation
$\Pi_{ij}^{(s)}$	m^2/s^3	Component of slow part of pressure-strain correlation
Ω_{ij}	$1/s$	Component of mean-flow rotation tensor

1 Introduction

The equilibrium between production and dissipation of specific kinetic turbulence energy is a cornerstone in the development of two-equation eddy-viscosity turbulence models [9]. It has been derived by Hinze [4] for the boundary-layer equations and holds also in turbulent channel flow [1].

A generalised equilibrium condition based on the Reynolds-stress transport equations has been exploited recently in the analysis of the characteristics of turbulent free-shear flow [3]. It additionally accounts for the pressure-strain correlation and, in incompressible flow, reduces to the classical equilibrium condition, when taking its trace.

With two-equation models, only the eddy viscosity enters into the equlibrium condition, whereas, with Reynolds-stress models, the model of the pressure-strain correlation needs to be accounted for. Therefore, the equilibrium conditions depend on the details of the respective model.

The foundations of Reynolds-stress modeling have been laid by Chou [2] and, particularly for the pressure-strain correlation, by Rotta [7]. According to Rotta [7], in homogeneous turbulence, the pressure-strain correlation can be decomposed into two contributions. The so-called slow term involves only velocity fluctuations and is, therefore, only affected indirectly by the mean flow. In contrast, the so-called rapid term is proportional to the mean-flow velocity gradients, thus reacting immediately to any changes in the mean flow.

Based on physical considerations, Rotta [7] concludes that, in the absence of mean-velocity gradients, the slow term should drive the Reynolds-stress tensor towards isotropy. This implies a dependence of the slow term on the Reynolds-stress anisotropy. For the rapid term, Rotta [7] derives a generic expression with associated constraints that imply a dependence on the specific Reynolds stresses or, in nondimensional form, the corresponding anisotropies.

Various models for the pressure-strain correlation have been developed along the lines of Rotta's [7] ideas. The most general forms of the slow and rapid terms have been provided by Lumley [6] and by Johansson and Hallbäck [5], respectively, based on tensor series expansions. They involve a number of coefficients that might be functions of the invariants of the Reynolds stress anisotropy tensor. Virtually any model of the pressure-strain correlation suggested so far can be viewed as a subset of these general formulations.

Abid and Speziale [1] provide equilibrium values of the Reynolds-stress anisotropies for various models of the pressure-strain correlation, but without giving any details of the underlying equations. In order to close this gap, the turbulent equilibrium conditions for Reynolds-stress models will be derived subsequently for the most general formulations of the pressure-strain correlation model. From this, the conditions for any particular model can be derived by identifying its respective terms in the general formulation and setting all others to zero.

The equilibrium conditions hold in several canonical flows, in particular in the log-law region of boundary layers. They can therefore be exploited in the calibration of Reynolds-stress models as well as in the analysis of model behavior.

2 Modeled Pressure-Strain Correlation

Following Rotta [7], the pressure-strain correlation Π_{ij} is decomposed into a slow term $\Pi_{ij}^{(s)}$ and a rapid term $\Pi_{ij}^{(r)}$ according to

$$\Pi_{ij} = \Pi_{ij}^{(s)} + \Pi_{ij}^{(r)}. \tag{2.1}$$

To this point, the influence of the domain boundary, particularly the effect of viscous walls, is neglected. The slow as well as the rapid term are symmetric and traceless and are modeled independently.

2.1 Slow Term

According to Lumley [6], the most general model of the slow term reads

$$\Pi_{ij}^{(s)} = -\epsilon \left[C_1^{(s)} b_{ij} - C_2^{(s)} \left(b_{ik} b_{kj} - \frac{1}{3} b_{kl} b_{lk} \delta_{ij} \right) \right], \tag{2.2}$$

in which ϵ is the specific dissipation rate and

$$b_{ij} = \frac{R_{ij}}{2k} - \frac{1}{3}\delta_{ij} \tag{2.3}$$

are the anisotropies associated with the specific Reynolds stresses R_{ij} and the specific kinetic turbulence energy $k = \frac{1}{2} R_{ii}$. The coefficients $C_1^{(s)}$ and $C_2^{(s)}$ might be functions of the invariants of the Reynolds-stress anisotropy tensor,

$$II_b = b_{kl} b_{kl}, \tag{2.4}$$

$$III_b = b_{kl} b_{lm} b_{mk}. \tag{2.5}$$

The slow term according to Eq. (2.2) is symmetric and traceless.

2.2 Rapid Term

According to Johansson and Hallbäck [5], the most general model of the rapid term reads

$$\begin{aligned}\Pi_{ij}^{(r)} &= k\Big[Q_1^{(r)}\delta_{ip}\delta_{jq} + Q_2^{(r)}\left(b_{ip}\delta_{jq} + b_{jp}\delta_{iq} - \frac{2}{3}b_{pq}\delta_{ij}\right) \\ &\quad + Q_3^{(r)}b_{pq}b_{ij} + Q_4^{(r)}\left(b_{iq}b_{jp} - \frac{1}{3}b_{pk}b_{kq}\delta_{ij}\right) \\ &\quad + Q_5^{(r)}b_{pl}b_{lq}b_{ij} + \left(Q_5^{(r)}b_{pq} + Q_6^{(r)}b_{pl}b_{lq}\right)\left(b_{ik}b_{kj} - \frac{1}{3}b_{mn}b_{mn}\delta_{ij}\right)\Big] S_{pq} \\ &\quad + k\Big[Q_7^{(r)}\left(b_{ip}\delta_{jq} + b_{jp}\delta_{iq}\right) + Q_8^{(r)}b_{pk}\left(b_{jk}\delta_{iq} + b_{ik}\delta_{jq}\right) \\ &\quad + Q_9^{(r)}b_{pk}\left(b_{jk}b_{iq} + b_{ik}b_{jq}\right)\Big]\Omega_{pq}, \end{aligned} \tag{2.6}$$

in which

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \tag{2.7}$$

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) \tag{2.8}$$

denote the strain rates and rotation rates of the mean flow, respectively. As with the slow term, the coefficients of the rapid term, $Q_\alpha^{(r)}$, $\alpha = 1 \dots 9$, can be functions of the anisotropy invariants II_b (2.4) and III_b (2.5).

The rapid term according to Eq. (2.6) is also symmetric and traceless.

3 Turbulent Equilibrium Conditions

3.1 General Relations

Consider a turbulent shear flow that is governed by the incompressible boundary-layer equations. At sufficiently high Reynolds number, there exists a sublayer, which is in turbulent equilibrium [3], i.e., a region, in which the production term P_{ij} , the pressure-strain correlation Π_{ij} and the dissipation term ϵ_{ij} of the Reynolds-stress transport equation are in balance,

$$0 = P_{ij} + \Pi_{ij} - \epsilon_{ij}. \tag{3.1}$$

Taking the half of the trace, the contribution of the pressure-strain correlation drops out, and there remains the turbulent equilibrium of the k-equation

$$0 = P^{(k)} - \epsilon, \tag{3.2}$$

with $P^{(k)} = P_{ii}/2$ and $\epsilon = \epsilon_{ii}/2$. These relations also hold in turbulent channel flow when assuming diffusion to be negligible.

The Reynolds-stress production term is defined exactly by

$$P_{ij} = -R_{ik}\frac{\partial U_j}{\partial x_k} - R_{jk}\frac{\partial U_i}{\partial x_k}, \tag{3.3}$$

from which the k-production term

$$P^{(k)} = -R_{ik}\frac{\partial U_i}{\partial x_k} \tag{3.4}$$

follows. Since the Reynolds number is assumed to be high, the dissipation term can be taken as isotropic,

$$\epsilon_{ij} = \frac{2}{3}\epsilon\delta_{ij}. \tag{3.5}$$

For analysis, the pressure-strain correlation $\Pi_{ij} = \Pi_{ij}^{(s)} + \Pi_{ij}^{(r)}$, is substituted by the general models for the slow and the rapid term, Eqs. (2.2) and (2.6).

3.2 Two-Dimensional Flow

Consider a two-dimensional flow in the x_1-x_2-plane, so that U_1 is the principal mean flow velocity and $U_3 = R_{13} = R_{23} \equiv 0$. According to the boundary-layer assumptions, there is only one dominating velocity gradient, $\partial U_1/\partial x_2$, leading to a considerable simplification of the production terms,

$$P_{11} = 2P^{(k)} = -2R_{12}\frac{\partial U_1}{\partial x_2}, \tag{3.6}$$

$$P_{22} = P_{33} = 0, \tag{3.7}$$

$$P_{12} = -R_{22}\frac{\partial U_1}{\partial x_2}. \tag{3.8}$$

Introducing the Reynolds-stress anisotropies according to Eq. (2.3), the nonzero production terms can be written

$$P_{11} = 2P^{(k)} = -4kb_{12}\frac{\partial U_1}{\partial x_2}, \tag{3.9}$$

$$P_{12} = -2k\left(b_{22} - \frac{1}{3}\right)\frac{\partial U_1}{\partial x_2}. \tag{3.10}$$

For turbulent channel flow these conditions hold exactly.

Furthermore, for two-dimensional flow, the general model of the rapid term (2.6) simplifies to

$$\begin{aligned}
\Pi_{ij}^{(r,2D)} = & \left\{ \frac{Q_1^{(r)}}{2}\left(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}\right) + \frac{Q_2^{(r)}}{2}\left(b_{i1}\delta_{j2} + b_{i2}\delta_{j1} + b_{j1}\delta_{i2} + b_{j2}\delta_{i1} - \frac{4}{3}b_{12}\delta_{ij}\right)\right. \\
& + Q_3^{(r)} b_{12} b_{ij} + \frac{Q_4^{(r)}}{2}\left(b_{i1}b_{j2} + b_{i2}b_{j1} - \frac{2}{3}b_{1k}b_{k2}\delta_{ij}\right) \\
& + Q_5^{(r)} b_{1l}b_{l2}b_{ij} + \left(Q_5^{(r)} b_{12} + Q_6^{(r)} b_{1l}b_{l2}\right)\left(b_{ik}b_{kj} - \frac{1}{3}II_b\delta_{ij}\right) \\
& + \frac{Q_7^{(r)}}{2}\left(b_{i1}\delta_{j2} - b_{i2}\delta_{j1} + b_{j1}\delta_{i2} - b_{j2}\delta_{i1}\right) \\
& + \frac{Q_8^{(r)}}{2}\left[b_{ik}\left(b_{k1}\delta_{j2} - b_{k2}\delta_{j1}\right) + b_{jk}\left(b_{k1}\delta_{i2} - b_{k2}\delta_{i1}\right)\right] \\
& \left. + \frac{Q_9^{(r)}}{2}\left[b_{ik}\left(b_{k1}b_{j2} - b_{k2}b_{j1}\right) + b_{jk}\left(b_{k1}b_{i2} - b_{k2}b_{i1}\right)\right]\right\} k\frac{\partial U_1}{\partial x_2}.
\end{aligned} \tag{3.11}$$

Finally, due to the equilibrium condition (3.2), the isotropic dissipation rate can be written as

$$\epsilon = -2kb_{12}\frac{\partial U_1}{\partial x_2}, \tag{3.12}$$

which yields the general model of the slow term for two-dimensional flow (2.2)

$$\Pi_{ij}^{(s,2D)} = 2b_{12}\left[C_1^{(s)} b_{ij} - C_2^{(s)}\left(b_{ik}b_{kj} - \frac{1}{3}II_b\delta_{ij}\right)\right] k\frac{\partial U_1}{\partial x_2} \tag{3.13}$$

With these simplifications for two-dimensional boundary layer flow, one obtains the following equilibrium conditions for the different Reynolds-stress components based on the most general formulations of the pressure-strain correlation model according to Lumley [6] and Johansson and Hallbäck [5]:

- Component 11

$$\begin{aligned} 0 \;=\;& -\frac{8}{3} + 2\left[C_1^{(s)} b_{11} - C_2^{(s)}\left(b_{11}^2 + b_{12}^2 - \frac{1}{3}II_b\right)\right] \\ & + \frac{1}{3}Q_2^{(r)} + Q_3^{(r)} b_{11} + \frac{1}{3}Q_4^{(r)}\left(2b_{11} - b_{22}\right) + Q_5^{(r)} b_{11}\left(b_{11} + b_{22}\right) \\ & + \left[Q_5^{(r)} + Q_6^{(r)}\left(b_{11} + b_{22}\right)\right]\left(b_{11}^2 + b_{12}^2 - \frac{1}{3}II_b\right) \\ & - Q_7^{(r)} - Q_8^{(r)}\left(b_{11} + b_{22}\right) - Q_9^{(r)}\left(b_{11}b_{22} - b_{12}^2\right) \end{aligned} \tag{3.14}$$

- Component 22

$$\begin{aligned} 0 \;=\;& \frac{4}{3} + 2\left[C_1^{(s)} b_{22} - C_2^{(s)}\left(b_{12}^2 + b_{22}^2 - \frac{1}{3}II_b\right)\right] \\ & + \frac{1}{3}Q_2^{(r)} + Q_3^{(r)} b_{22} + \frac{1}{3}Q_4^{(r)}\left(2b_{22} - b_{11}\right) + Q_5^{(r)} b_{22}\left(b_{11} + b_{22}\right) \\ & + \left[Q_5^{(r)} + Q_6^{(r)}\left(b_{11} + b_{22}\right)\right]\left(b_{12}^2 + b_{22}^2 - \frac{1}{3}II_b\right) \\ & + Q_7^{(r)} + Q_8^{(r)}\left(b_{11} + b_{22}\right) + Q_9^{(r)}\left(b_{11}b_{22} - b_{12}^2\right) \end{aligned} \tag{3.15}$$

- Component 33

$$\begin{aligned} 0 \;=\;& \frac{4}{3} + 2\left[C_1^{(s)} b_{33} - C_2^{(s)}\left(b_{33}^2 - \frac{1}{3}II_b\right)\right] \\ & - \frac{2}{3}Q_2^{(r)} + Q_3^{(r)} b_{33} - \frac{1}{3}Q_4^{(r)}\left(b_{11} + b_{22}\right) + Q_5^{(r)} b_{33}\left(b_{11} + b_{22}\right) \\ & + \left[Q_5^{(r)} + Q_6^{(r)}\left(b_{11} + b_{22}\right)\right]\left(b_{33}^2 - \frac{1}{3}II_b\right) \end{aligned} \tag{3.16}$$

- Component 12

$$
\begin{aligned}
0 \;=\; & -2\left(b_{22}+\frac{1}{3}\right)+2b_{12}\left[C_1^{(s)}b_{12}-C_2^{(2)}b_{12}\left(b_{11}+b_{22}\right)\right] \\
& +\frac{1}{2}Q_1^{(r)}+\frac{1}{2}Q_2^{(r)}\left(b_{11}+b_{22}\right)+Q_3^{(r)}b_{12}^2+\frac{1}{2}Q_4^{(r)}\left(b_{11}b_{22}-b_{12}^2\right) \\
& +\left[2Q_5^{(r)}+Q_6^{(r)}\left(b_{11}+b_{22}\right)\right]b_{12}^2\left(b_{11}+b_{22}\right) \\
& +\frac{1}{2}Q_7^{(r)}\left(b_{11}-b_{22}\right)+\frac{1}{2}Q_8^{(r)}\left(b_{11}^2-b_{22}^2\right) \\
& +\frac{1}{2}Q_9^{(r)}\left(b_{11}b_{22}-b_{12}^2\right)\left(b_{11}-b_{22}\right) \qquad (3.17)
\end{aligned}
$$

As is verifyed easily, Eqs. (3.14), (3.15) and (3.16) sum up to zero and are, hence, linearly dependent. Eq. (3.16) can therefore be ignored, and only Eqs. (3.14), (3.15) and (3.17) need to be accounted for. They are valid for any model of the pressure-strain correlation that can be expressed as a subset of the general formulations for the slow term (2.2) according to Lumley [6] and the rapid term (2.6) according to Johansson and Hallbäck [5].

Note that the equilibrium conditions (3.14), (3.15) and (3.17) are independent of the mean-flow velocity gradient. Hence they hold identically for any incompressible two-dimensional flow, in which a turbulent equilibrium is present. They essentially relate the components of the Reynolds-stress anisotropy tensor b_{11}, b_{22} and b_{12} to the model coefficients $C_1^{(s)}$, $C_2^{(s)}$ and $Q_\alpha^{(r)}$, $\alpha = 1 \ldots 9$, independently of whether they are constant or functions of the invariants of the Reynolds-stress anisotropy tensor.

Equations (3.14), (3.15) and (3.17) can be used for calibration of the pressure-strain correlation to a particular equilibrium state. However, different canonical flows are associated with different equilibrium states [3]. Therefore, independent of the complexity of the pressure-strain correlation model considered, a tailored approach is required for coping with different types of flow.

4 Conclusion

The turbulent equilibrium conditions have been derived for the Reynolds-stress transport equations based on two-dimensional incompressible boundary-layer assumptions. The derivation employs the most general model formulations for the slow and the rapid term of the pressure-strain correlation.

Three equations are obtained relating the Reynolds-stress anisotropies to the model coefficients that might be functions of the invariants of the Reynolds-stress anisotropy tensor. The equations are independent of the velocity gradient and therefore hold identically for any two-dimensional incompressible flow, in which a turbulent equilibrium is present. Due to the generality of the formulation, the result can be transferred to virtually any pressure-strain correlation model suggested so far.

The obtained equilibrium conditions can be used for calibrating the respective pressure-strain correlation model to a particular equilibrium state, e.g., in the log-law region of a boundary layer. Since different flows are associated with different equilibrium properties, a tailored approach will be needed for broadening the range of accurate predictions, independent of the complexity of the pressure-strain model that is employed.

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