On the Computation of Isolated Sublattices

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**Isolated Sublattices**

**Definition**

Let $(S, \leq)$ be a lattice. A subset $S' \subseteq S$ is called an *isolated sublattice* if it fulfills the following properties:

- $S'$ is a sublattice with greatest element $\top_{S'}$ and least element $\bot_{S'}$.
- $\forall x \notin S' \forall y' \in S' : y' \leq x \Rightarrow \top_{S'} \leq x$
- $\forall x \notin S' \forall y' \in S' : x \leq y' \Rightarrow x \leq \bot_{S'}$

**Definition**

An isolated sublattice is called a *summit isolated sublattice* if $\top_{S'} = \top$ holds. An isolated sublattice is called an *isolated sublattice with bottleneck* if $\top'_{S}$ is meet-irreducible.

- nontrivial: $S' \neq S$, useful: $S' \neq \{s'\}$
- Can be used for counting closure operators
- Efficient algorithm?
Examples

- summit isolated sublattice
- isolated sublattice with bottleneck
Conventions and Notations

- Lattices are given by their Hasse diagram \((S, E)\)
- \((u, v) \in E \Rightarrow u < v\)
- Reverse graph denoted by \(G^{-} = (S, E^{-})\)
- Undirected version denoted by \(G^{\leftrightarrow}\)
- \(u\) is called a \((v, w)\)-separator if every path from \(v\) to \(w\) contains \(u\)
Summit Isolated Sublattices

- All isolated sublattices have the form \([x, y]\).
- Summit isolated sublattices have the form \([z, \top]\).
- \([z, \top]\) is a lattice.
- \(x \notin [z, \top] \land y' \in [z, \top] \land y' \leq x \Rightarrow \text{FALSE}\).
- Hence \(\forall x \notin S' \forall y' \in S' : y' \leq x \Rightarrow T_g \leq x\) superfluous.
- So we have:

**Lemma**

\([z, \top]\) is a summit isolated sublattice iff the following implication holds:
- \(\forall x \notin [z, \top] \forall y' \in [z, \top] : x \leq y' \Rightarrow x \leq z.\)

- Still simpler:

**Lemma**

\([z, \top]\) is a summit isolated sublattice iff for all \(x \notin [z, \top]\) the inequality \(x \leq z\) holds.

- \(\Rightarrow : y' =_{\text{def}} \top\)
- \(\Leftarrow : x \leq z\) is TRUE by assumption for all \(x \notin [z, \top]\).
Lemma

Let \((S, \leq)\) be a finite lattice with Hasse diagram \(G = (S, E)\). Then every summit isolated sublattice of \(S\) has the form \([z, \top]\) where \(z\) is a \((\top, \bot)\)-separator in \(G^{\leftarrow}\).

- trivial for \(z = \bot\) and \(z = \top\)
- consider a path \(p = v_1v_2 \ldots v_n\) in \(G^{\leftarrow}\) with \(v_1 = \top\) and \(v_n = \bot\)
- consider \(i \in [1, n-1]\) with \(v_i \in [z, \top]\) and \(v_{i+1} \notin [z, \top]\)
- assume that \(v_i \neq z\) holds
- now \(v_i \in [z, \top]\) implies \(z < v_i\), and \((v_i, v_{i+1}) \in E^{\leftarrow}\) implies \(v_{i+1} < v_i\)
- by the previous slide we have \(v_{i+1} < z\)
- transitivity of \(<\) yields \(v_{i+1} < v_i\)
- so \((v_i, v_{i+1}) \notin p^{\leftrightarrow}\)
- computation of separators in directed graph via modified flow algorithms
Characterization Using Undirected Graphs

Theorem

Let $(S, \leq)$ be a finite lattice with Hasse diagram $G = (V, E)$. Then every summit isolated sublattice of $S$ has the form $[z, \top]$ where $z$ is a $(\top, \bot)$-separator in $G^{\leftrightarrow}$.

- trivial for $z = \bot$ and $z = \top$
- consider a path $p = v_1v_2 \ldots v_n$ in $G^{\leftrightarrow}$ with $v_1 = \top$ and $v_n = \bot$
- consider $i$ with $v_i \in [z, \top]$ and $v_{i+1} \notin [z, \top]$
- claim: $v_i = z$
- $v_i \in [z, \top]$ and $v_{i+1} \notin [z, \top]$ imply $(v_i, v_{i+1}) \in E^{\leftarrow}$
- $v_i \in [z, \top] \Rightarrow \exists p' = v'_1v'_2 \ldots v'_{n'}$ in $G^{\leftarrow}$ with $v'_1 = \top$, $v'_{n'} = v_i$ and $v'_i \neq z$
- analogously, $\exists p'' = v''_1v''_2 \ldots v''_{n''}$ in $G^{\leftarrow}$ with $v''_1 = v_{i+1}$, $v''_{n''} = \bot$ and $v''_{i'} \neq z$
- hence $v'_1v'_2 \ldots v'_{n'}v'_1v'_2 \ldots v'_{n''}$ is a $\top$-$\bot$-path in $G^{\leftarrow}$
- claim follows from previous slide
Linear Time Algorithm for Summit Isolated Sublattices

- Algorithm by Tarjan (1972) computes separators in undirected graphs in linear time.

**Theorem**

*Given the Hasse diagram $(S, E)$ of a finite lattice $S$, it can be determined in $O(|E|)$ time whether $S$ has a nontrivial useful summit isolated sublattice. In the case of existence, a nontrivial useful summit isolated sublattice can be determined also in $O(|E|)$ time.*

- Time bound is asymptotically optimal.
- Uses only a simple DFS, no sophisticated network flow algorithms.
- Computation (in the case of existence) of an inclusion-maximal nontrivial useful summit isolated sublattice in $O(|E|)$.
Lemma

Let \((S, \leq)\) be a finite lattice with Hasse diagram \(G = (S, E)\). Then all isolated sublattices of \(S\) are exactly the intervals \([x, y]\) where \(x\) is a \((\perp, y)\)-separator in \(G\) and \(y\) is an \((x, \top)\)-separator in \(G\).

- leads to quadratic time algorithm
- number of sublattices may be quadratic in \(|S|\)
- (consider a chain)
- however, number of inclusion-maximal isolated sublattices is bounded by \(|S|\)
- (inclusion-maximal isolated sublattices are disjoint)
- separator property is transitive:
  - if \(v_2\) is a \((v_1, v)\)-separator and \(v_3\) is a \((v_2, v)\)-separator then \(v_3\) is a \((v_1, v)\)-separator
  - here no use of supremum/infimum properties
- existence of \((v_1, v_3), (v_2, v_3)\) and \((v_1, v_4)\) excludes existence of \((v_2, v_4)\) (for distinct \(v_i\))
- hammocks?