On the Computation of Isolated Sublattices

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Isolated Sublattices

Definition

Let $(S, \leq)$ be a lattice. A subset $S' \subseteq S$ is called an *isolated sublattice* if it fulfills the following properties:

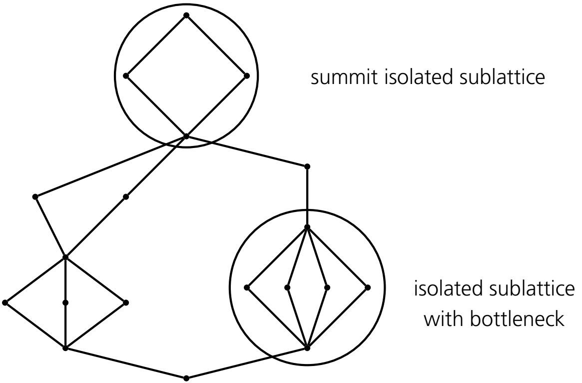
- S' is a sublattice with greatest element $\top_{S'}$ and least element $\bot_{S'}$.
- $\forall x \notin S' \forall y' \in S' : y' \leq x \Rightarrow \top_{S'} \leq x$
- $\forall x \notin S' \forall y' \in S' : x \leq y' \Rightarrow x \leq \bot_{S'}$

Definition

An isolated sublattice is called a *summit isolated sublattice* if $\top_{S'} = \top$ holds.
An isolated sublattice is called an *isolated sublattice with bottleneck* if $\top'_S$ is meet-irreducible.

- nontrivial: $S' \neq S$, useful: $S' \neq \{s'\}$
- Can be used for counting closure operators
- Efficient algorithm?

Examples


summit isolated sublattice
isolated sublattice
with bottleneck

Conventions and Notations

- Lattices are given by their Hasse diagram (S, E)
- $(u, v) \in E \Rightarrow u < v$
- Reverse graph denoted by $G^{\leftarrow} = (S, E^{\leftarrow})$
- Undirected version denoted by $G^{\leftrightarrow}$
- u is called a (v, w)-*separator* if every path from v to w contains u

Summit Isolated Sublattices

- all isolated sublattices have the form $[x, y]$
- summit isolated sublattices have the form $[z, \top]$
- $[z, \top]$ is a lattice
- $x \notin [z, \top] \wedge y' \in [z, \top] \wedge y' \leq x \Rightarrow \texttt{FALSE}$
- hence $\forall x \notin S' \forall y' \in S' : y' \leq x \Rightarrow \top_{S'} \leq x$ superfluous
- so we have:

Lemma

$[z, \top]$ *is a summit isolated sublattice iff the following implication holds:*

- $\forall x \notin [z, \top] \forall y' \in [z, \top] : x \leq y' \Rightarrow x \leq z.$

- still simpler:

Lemma

$[z, \top]$ *is a summit isolated sublattice iff for all* $x \notin [z, \top]$ *the inequality* $x \leq z$ *holds.*

- $\Rightarrow$: $y' =_{def} \top$
- $\Leftarrow$: $x \leq z$ is $\texttt{TRUE}$ by assumption for all $x \notin [z, \top]$

Connection with Separators

Lemma

Let $(S, \leq)$ be a finite lattice with Hasse diagram $G = (S, E)$. Then every summit isolated sublattice of S has the form $[z, \top]$ where z is a $(\top, \bot)$-separator in $G^{\leftarrow}$.

- trivial for $z = \bot$ and $z = \top$
- consider a path $p = v_1 v_2 \ldots v_n$ in $G^{\leftarrow}$ with $v_1 = \top$ and $v_n = \bot$
- consider $i \in [1, n-1]$ with $v_i \in [z, \top]$ and $v_{i+1} \notin [z, \top]$
- assume that $v_i \neq z$ holds
- now $v_i \in [z, \top]$ implies $z < v_i$, and $(v_i, v_{i+1}) \in E^{\leftarrow}$ implies $v_{i+1} < v_i$
- by the previous slide we have $v_{i+1} < z$
- transitivity of $<$ yields $v_{i+1} < v_i$
- so $(v_i, v_{i+1}) \notin p$ ↯
- computation of separators in directed graph via modified flow algorithms

Characterization Using Undirected Graphs

Theorem

Let $(S, \leq)$ be a finite lattice with Hasse diagram $G = (V, E)$. Then every summit isolated sublattice of S has the form $[z, \top]$ where z is a $(\top, \bot)$-separator in $G^{\leftrightarrow}$.

- trivial for $z = \bot$ and $z = \top$
- consider a path $p = v_1 v_2 \ldots v_n$ in $G^{\leftrightarrow}$ with $v_1 = \top$ and $v_n = \bot$
- consider i with $v_i \in [z, \top]$ and $v_{i+1} \notin [z, \top]$
- claim: $v_i = z$
- $v_i \in [z, \top]$ and $v_{i+1} \notin [z, \top]$ imply $(v_i, v_{i+1}) \in E^{\leftarrow}$
- $v_i \in [z, \top] \Rightarrow \exists\, p' = v'_1 v'_2 \ldots v'_{n'}$ in $G^{\leftarrow}$ with $v'_1 = \top$, $v'_{n'} = v_i$ and $v'_{i'} \neq z$
- analogously, $\exists\, p'' = v''_1 v''_2 \ldots v''_{n''}$ in $G^{\leftarrow}$ with $v''_1 = v_{i+1}$, $v''_{n''} = \bot$ and $v''_{i''} \neq z$
- hence $v'_1 v'_2 \ldots v'_{n'} v''_1 v''_2 \ldots v''_{n''}$ is a $\top$-$\bot$-path in $G^{\leftarrow}$
- claim follows from previous slide

Linear Time Algorithm for Summit Isolated Sublattices

- algorithm by Tarjan (1972) computes separators in undirected graphs in linear time

Theorem

Given the Hasse diagram (S, E) of a finite lattice S, it can be determined in $\mathcal{O}(|E|)$ time whether S has a nontrivial useful summit isolated sublattice. In the case of existence, a nontrivial useful summit isolated sublattice can be determined also in $\mathcal{O}(|E|)$ time.

- time bound is asymptoticaly optimal
- uses only a simple DFS, no sophisticated network flow algorithms
- computation (in the case of existence) of an inclusion-maximal nontrivial useful summit isolated sublattice in $\mathcal{O}(|E|)$

General Isolated Sublattices

Lemma

Let $(S, \leq)$ be a finite lattice with Hasse diagram $G = (S, E)$. Then all isolated sublattices of S are exactly the intervals $[x, y]$ where x is a $(\bot, y)$-separator in G and y is an $(x, \top)$-separator in G.

- leads to quadratic time algorithm
- number of sublattices may be quadratic in $|S|$
- (consider a chain)
- however, number of inclusion-maximal isolated sublattices is bounded by $|S|$
- (inclusion-maximal isolated sublattices are disjoint)
- separator property is transitive:
- if v_2 is a (v_1, v)-separator and v_3 is a (v_2, v)-separator then v_3 is a (v_1, v)-separator
- here no use of supremum/infimum properties
- existence of (v_1, v_3), (v_2, v_3) and (v_1, v_4) excludes existence of (v_2, v_4) (for distinct v_i)
- hammocks?