Analysis of Sliding-Mode Control Systems with Unmatched Disturbances Altering the Relative Degree

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We consider sliding-mode control systems subject to unmatched disturbances. Classical first-order sliding-mode techniques are capable to compensate unmatched disturbances if differentiations of the output of sufficiently high order are included in the sliding variable. Commonly for such disturbances it is assumed that the relative degree of the system is not changed. In this contribution we study the impact of disturbances that alter the relative degree (of the process) on the closed-loop control system with the classical first-order sliding-mode design. We analyse the reaching and sliding phase of the resulting closed-loop system. We show that uniqueness of the solution may be lost and derive conditions for such behaviour. We present conditions for the stability of the sliding-mode dynamics and analyse the disturbance rejection properties. For a second-order example system we evaluate the obtained conditions explicitly and show that attractivity of the sliding manifold may be lost. Moreover, a simulation case study of a two-mass spring-damper system illustrates the various closed-loop behaviours.

Analysis of Sliding-Mode Control Systems with Unmatched Disturbances Altering the Relative Degree ⋆

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Abstract: We consider sliding-mode control systems subject to unmatched disturbances. Classical first-order sliding-mode techniques are capable to compensate unmatched disturbances if differentiations of the output of sufficiently high order are included in the sliding variable. For such disturbances it is commonly assumed that they do not affect the relative degree of the system. In this contribution we consider disturbances that alter the relative degree of the process and study their impact on the closed-loop control system with the classical first-order sliding-mode design. We analyse the reaching and sliding phase of the resulting closed-loop system. We show that uniqueness of the solution may be lost and derive conditions for such behaviour. We present conditions for the stability of the sliding-mode dynamics and analyse the disturbance rejection properties. A simulation case study of a two-mass spring-damper system illustrates the various closed-loop behaviours.

Keywords: Sliding-mode control, Unmatched disturbances, Relative degree

1. INTRODUCTION

Sliding-mode control techniques are well-known for their robustness properties with regard to model uncertainties and external disturbances. In particular so-called matched disturbances, roughly speaking disturbances that enter the system via the same input-space as the control signal, may be completely rejected on the sliding manifold. There are several propositions that extend this property to unmatched disturbances, see e.g. (Cao and Xu, 2004; Castaños and Fridman, 2006; Estrada and Fridman, 2008; Ferreira de Loza et al., 2015). All these methods consider system structures ensuring that the relative degree of the system is not changed by the disturbance. However, model uncertainties may change the relative degree of the system as demonstrated e.g. by Hauser et al. (1992).

Systems with uncertain relative degree have been subject to various research in the recent past. The concept of ill-defined relative degree has been studied e.g. in (Commuri and Lewis, 1995; Leith and Leithead, 2001; Chen, 2001; Chen and Ballance, 2002; Lozada-Castillo et al., 2014). Basically, a system with ill-defined relative degree has states x for which the relative degree is larger than at some nominal point x_0, i.e. $\mathcal{L}_g\mathcal{L}_f^{r-1}h(x) = 0$, where r denotes the relative degree at x_0. Another line of research assumes that an upper bound of the uncertain relative degree is known, see e.g. (Tao and Ioannou, 1993; Hoagg and Bernstein, 2007; Furtat and Tsykunov, 2010; Zhao and Li, 2014; Jo et al., 2014). In (Hoagg and Bernstein, 2007) a direct adaptive tracking and disturbance rejection algorithm for minimum-phase linear systems is developed. Zhao and Li (2014) use a state observer for a linear control law to reject disturbances on an integrator chain system.

In this contribution we consider systems with disturbances (or model uncertainties) that reduce the relative degree of the process. We analyse the impact of such disturbances on the closed-loop control system with a classically designed first-order sliding-mode controller. The motivating example in the next section is followed by a formal problem definition and introduction of the class of disturbances in Section 3. We consider the control of a nonlinear input affine system with full relative degree n and use a sliding-variable with derivatives of the output up to the order $n-1$. This is a well-known approach to render unmatched disturbances matched disturbances on the sliding-manifold, see also (Wulff et al., 2020) for a comparison of such methods. However, the disturbance considered induces additional internal dynamics into the system which may alter the closed-loop behaviour significantly. In Section 4 we analyse the resulting closed-loop system. In particular, we shall characterise the reaching and sliding phase in the state space and discuss the continuity properties of the control signal as well as the sliding variable. We characterise the solution of the closed-loop system and analyse the dynamics in sliding-mode together with its disturbance rejection properties. A case-study of a two-mass system is presented in Section 5 and the conclusions are found in Section 6.

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Alongside with the common mathematical notation, we shall use $\mathcal{L}_f h(x)$ for the Lie derivative of h with respect to the vector field f, i.e. $\mathcal{L}_f h(x) = \frac{\partial h(x)}{\partial x} f(x)$. The Lie derivative of h with respect to the sum of vector fields $f(x) + \phi(x)$ shall be denoted by

$$\mathcal{L}_{f+\phi} h(x) = \frac{\partial h(x)}{\partial x}\big(f(x) + \phi(x)\big).$$

The k-th Lie derivative is defined recursively by

$$\mathcal{L}_f^k h(x) = \frac{\partial \mathcal{L}_f^{k-1} h(x)}{\partial x} f(x)$$

where $\mathcal{L}_f^0 h(x) = h(x)$.

2. MOTIVATION

Consider the third-order system with full relative degree

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u + \phi_1(x) + \phi_2(x) \tag{1a}$$

$$y = x_1 \tag{1b}$$

and the unmatched disturbances

$$\phi_1(x) = \begin{pmatrix} bx_2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \phi_2(x) = \begin{pmatrix} cx_3 \\ 0 \\ 0 \end{pmatrix}.$$

Note that the disturbance ϕ_1 preserves the relative degree of the system while the disturbance ϕ_2 reduces the relative degree of the system by one.

By defining a sliding variable σ including the derivatives of the output, it is possible to use a first-order sliding-mode control law to compensate the unmatched disturbances:

$$\sigma = y + 2\dot{y} + \ddot{y} \tag{2a}$$

$$= x_1 + (2 + 2b)x_2 + (2c + b + 1)x_3 + cu\,. \tag{2b}$$

Note that (2a) is the actual implemented switching variable using the derivatives of the measured output y. Equation (2b) can be used for analysis purposes only as the substituted descriptions of ϕ_1 and ϕ_2 are assumed to be unknown. Accordingly, the description (2b) reveals that the sliding variable σ depends on u, due to the disturbance ϕ_2.

The first-order sliding-mode control law with the sliding variable (2b) reads

$$u = -x_2 - 3x_3 - L\,\mathrm{sgn}\Big(x_1 + (2 + 2b)x_2 + (2c + b + 1)x_3 + c\,u\Big) \tag{3}$$

where $L > 0$ denotes the sliding-mode gain. In the case of $c = 0$ the control law u is directly given by the assignment in (3). In the case of a relative degree altering disturbance ϕ_2 with $c \neq 0$ the control u is the solution of the algebraic equation (3). Due to the fact that the right-hand side of (3) is not continuous, the existence and uniqueness of the solution is not ensured. Three solutions for the initial condition $x(0) = (0.2\ 0\ 0)^\top$ are depicted in Fig. 1. All of these solutions are bounded, however their dynamical and stationary behaviour differ considerably.

3. PROBLEM DEFINITION

We consider process dynamics of the form

$$\dot{x} = f(x) + g(x)u + \phi(x,t) \tag{4a}$$

$$y = h(x)\,, \tag{4b}$$

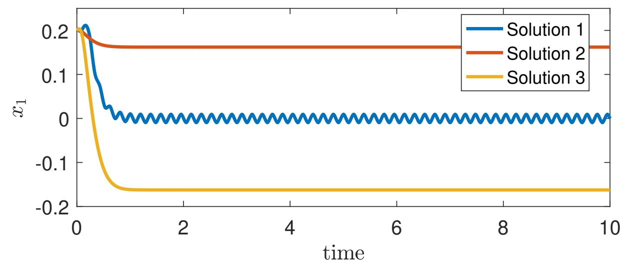

Fig. 1. Three solutions of the closed-loop system (1)-(3) for the identical initial condition $x(0) = (0.2\ \ 0\ \ 0)^\top$.

where $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}$ the control input and $y(t) \in \mathbb{R}$ the output of interest. The vector fields f and g are sufficiently smooth and of matching dimensions, where $f(0) = 0$ and $g(x) \neq 0$ for all x. The output function h is uniformly continuous and the state vector x is assumed to be known. The function ϕ is an unknown bounded disturbance.

The disturbance can be divided into a matched and an unmatched disturbance, ϕ_m and ϕ_u, respectively, with

$$\phi_\mathrm{m}(x,t) = g(x)g^+(x)\phi(x,t) \tag{5a}$$

$$\phi_\mathrm{u}(x,t) = g^\perp(x)g^{\perp+}(x)\phi(x,t)\,, \tag{5b}$$

where $g^\perp(x)$ is a full-rank left annihilator of $g(x)$, i.e. a matrix with independent columns that spans the null space of $g(x)$. It satisfies $g^\perp(x)g^\top(x) = 0$ and $\mathrm{rk}(g^\perp) = n - 1$. Moreover, we denote with $g^+(x)$ the left pseudo-inverse of $g(x)$, i.e. $g^+(x) = (g^\top(x)g(x))^{-1}g^\top(x)$.

By r_0 we denote the relative degree of (4) in the nominal case, i.e. where $\phi \equiv 0$. Throughout this paper we assume $r_0 = n$. More precisely, for $\phi \equiv 0$, the relative degree r_0 of the output $y = h(x)$ with respect to the input u at the point $x \in \mathbb{R}^n$ shall be n, such that

$$\mathcal{L}_g \mathcal{L}_f^k h(x) = 0\,, \quad \text{for} \quad k \in \{0, \dots n-2\} \tag{6a}$$

$$\mathcal{L}_g \mathcal{L}_f^{n-1} h(x) \neq 0\,, \tag{6b}$$

where $\mathcal{L}$ denotes the Lie-derivative. Note that the relative degree is a local property. If not stated otherwise we consider the relative degree at the origin $x = 0$.

3.1 Relative degree altering disturbance

The disturbance ϕ may have an impact on the relative degree of system (4). Therefore, we shall distinguish disturbances that retain the nominal relative degree from disturbances which change the relative degree with respect to the nominal case. [1]

Definition 1. Consider system (4) with nominal relative degree r_0. The disturbance ϕ is called *(relative degree) preserving* if

$$\mathcal{L}_g \mathcal{L}_{f+\phi}^k h(x) = 0\,, \quad \text{for} \quad k \in \{0, \dots r_0 - 2\} \tag{7a}$$

$$\mathcal{L}_g \mathcal{L}_{f+\phi}^{r_0-1} h(x) \neq 0\,. \tag{7b}$$

Otherwise ϕ is called *(relative degree) altering.*

Note that any matched uncertainty is relative degree preserving. Typically, this property is also required for unmatched disturbances.

[1] For ease of exposition and without loss of generality we formulate the definition for time-invariant disturbances only.

In this paper we shall drop this limitation and consider unmatched disturbances that reduce the relative degree of the nominal system by exactly one, i.e.

$$\mathcal{L}_g\mathcal{L}_{f+\phi}^{k}h(x) = 0\,, \quad \text{for} \quad k \in \{0,\dots r_0-3\} \tag{8a}$$

$$\mathcal{L}_g\mathcal{L}_{f+\phi}^{r_0-2}h(x) \neq 0\,. \tag{8b}$$

Thus the disturbed system (4) has relative degree $n-1$, whereas the nominal system has relative degree $r_0 = n$.

We shall consider the Byrnes-Isidori form, where the state-space is decomposed into external and internal states obtained by a suitable state transformation, see Isidori (2013):

$$\xi_1 := \tau_1(x) = y = h(x)\,, \tag{9a}$$

$$\xi_i := \tau_i(x) = y^{(i-1)} = \mathcal{L}_{f+\phi}^{i-1}h(x)\,, i = 2\dots n-2\,, \tag{9b}$$

$$\xi_{n-1} := \tau_{n-1}(x) = y^{(n-2)} = \mathcal{L}_{f+\phi}^{n-2}h(x)\,. \tag{9c}$$

The remaining component

$$\eta := \tau_n(x), \tag{10}$$

is scalar and chosen such that τ is a diffeomorphism and $\mathcal{L}_g\tau_n(x) = 0$. As introduced in Isidori (2013), the states ξ are called external states while η are called internal with respect to the output y and input u.

The resulting internal dynamics are given by

$$\dot{\eta} = w(\xi,\eta) := \mathcal{L}_{f+\phi}\tau_n(x)|_{x=\tau^{-1}(\xi,\eta)}\,. \tag{11}$$

Assuming that the external dynamics are locally asymptotically stable by design, we require that the so-called zero dynamics, i.e. (11) with $\xi \equiv 0$, are locally asymptotically stable. Since $\eta(t) \in \mathbb{R}$, the local asymptotic stability of the zero dynamics is ensured if and only if

$$\eta\, w(0,\eta) < 0 \tag{12}$$

for all $\eta \neq 0$ within the considered neighbourhood.

3.2 First-order sliding-mode control law

We apply a standard first-order sliding-mode control, incorporating derivatives of the output, which compensates matched as well as unmatched, preserving uncertainties. We choose the switching function

$$\sigma(y,\dot{y},\dots y^{(n-1)}) = y^{(n-1)} - \gamma(y,\dot{y},\dots y^{(n-2)}) \tag{13}$$

with the function γ designed such that the system

$$y^{(n-1)} = \gamma(y,\dot{y},\dots y^{(n-2)}) \tag{14}$$

is asymptotically stable at 0. Since the system (4) has relative degree $n-1$, Equation (13) can be written as a function of the state x and the input u

$$\sigma(x,u) = s_\phi(x) + \varsigma_\phi(x)u \tag{15}$$

with the non-trivial functions

$$\varsigma_\phi(x) := \mathcal{L}_g\mathcal{L}_{f+\phi}^{n-2}h(x) \tag{16}$$

$$s_\phi(x) := \mathcal{L}_{f+\phi}^{n-1}h(x) - \gamma(h(x), \mathcal{L}_{f+\phi}^{1}h(x), \dots \mathcal{L}_{f+\phi}^{n-2}h(x)). \tag{17}$$

These incorporate the influence of the unknown disturbance ϕ. Note that (13) is the implemented switching function, while (15) is usually unknown and is used for analysis purposes only. We write $\sigma(y,\dot{y},\dots y^{(n-1)})$ to emphasise that the derivatives are obtained by differentiating the output signal y, and we write $\sigma(x,u)$ if these derivatives are substituted by their analytical expressions from the right-hand side of (9).

We conclude this section by stating the standard first-order sliding-mode control law used:

$$u = -\frac{1}{\mathcal{L}_g s_0(x)}\Big(\mathcal{L}_f s_0(x) + L\,\mathrm{sgn}\left(\sigma(y,\dot{y},\dots y^{(r)})\right)\Big) \tag{18a}$$

$$= \alpha(x) - q(x)\,\mathrm{sgn}(s_\phi(x) + \varsigma_\phi(x)u) \tag{18b}$$

with $s_0(x)$ meaning s_ϕ with $\phi \equiv 0$ and

$$\alpha(x) = -\frac{\mathcal{L}_f s_0(x)}{\mathcal{L}_g s_0(x)}\,, \quad q(x) = \frac{L}{\mathcal{L}_g s_0(x)}\,. \tag{19}$$

4. ANALYSIS OF THE CLOSED LOOP SYSTEM

4.1 Reaching phase and sliding phase

Commonly the state space can be divided into subspaces for the reaching phase and a sliding manifold. The reaching phase is defined by all x that fulfil $\sigma(x) \neq 0$ while the sliding phase is defined by $\sigma(x) = 0$. In our case the sliding variable $\sigma(x,u)$ may also depend on u, and therefore this unique division may no longer be possible. In the following we use the sliding variable to define subsets of the state space for which the system can be in reaching or sliding phase, respectively. In case $\sigma(x,u) > 0$, Equation (15) yields $s_\phi(x) + \varsigma_\phi(x)u > 0$. Substituting u from (18) yields the set

$$X_1 := \{x \in \mathbb{R}^n \mid s_\phi(x) + \varsigma_\phi(x)(\alpha(x) - q(x)) > 0\} \tag{20}$$

describing all points in $\mathbb{R}^n$ for which $\sigma > 0$. Similarly, for $\sigma(x,u) < 0$ we obtain the set

$$X_2 := \{x \in \mathbb{R}^n \mid s_\phi(x) + \varsigma_\phi(x)(\alpha(x) + q(x)) < 0\} \tag{21}$$

and for $\sigma(x,u) = 0$ we have

$$X_3 := \left\{x \in \mathbb{R}^n \mid \frac{s_\phi(x)}{\varsigma_\phi(x)q(x)} + \frac{\alpha(x)}{q(x)} \in [-1,1]\right\}. \tag{22}$$

Note that when eliminating u from (18) by substituting (15) we have an implication (and no equivalence) and thus u may not be uniquely defined by the state x. Indeed, it turns out that X_1, X_2 and X_3 are not necessarily disjoint. Thus, for every point x with $\sigma(x,u) = 0$ holds $x \in X_3$, but under certain conditions for every point in X_3 may also hold $\sigma(x,u) \neq 0$ depending on u subject to (18).

For our analysis we shall distinguish the boundary and the inner of the set X_3. In this context, we consider the set

$$X_3^\circ = \left\{x \in \mathbb{R}^n \mid \frac{\alpha(x)}{q(x)} + \frac{s_\phi(x)}{\varsigma_\phi(x)q(x)} \in (-1,1)\right\}. \tag{23}$$

Obviously X_3° is a subset of the inner of X_3. If any inner point of X_3 is part of the set X_3° then X_3° is the inner of X_3. The boundary of X_3 is then described by

$$\partial X_3 = \left\{x \in \mathbb{R}^n \mid \frac{\alpha(x)}{q(x)} + \frac{s_\phi(x)}{\varsigma_\phi(x)q(x)} \in \{-1,1\}\right\}. \tag{24}$$

Before we discuss various cases for which the three sets take different configurations in the state space, we shall note that the three sets always cover the full state space.

Lemma 2. It is $X_1 \cup X_2 \cup X_3 = \mathbb{R}^n$.

Proof. We rearrange (20), (21) and (22) and obtain for any $x_i \in X_i$ with $i \in \{1,2,3\}$ that

$$s_\phi(x_1) + \varsigma_\phi(x_1)\alpha(x_1) > \varsigma_\phi(x_1)q(x_1) \tag{25a}$$

$$s_\phi(x_2) + \varsigma_\phi(x_2)\alpha(x_2) < -\varsigma_\phi(x_2)q(x_2) \tag{25b}$$

$$s_\phi(x_3) + \varsigma_\phi(x_3)\alpha(x_3) \in [-|\varsigma_\phi(x_3)q(x_3)|, |\varsigma_\phi(x_3)q(x_3)|]. \tag{25c}$$

We can see that every $x \in \mathbb{R}^n$ fulfils at least one of these three conditions.

For our analysis we shall distinguish three configurations of the sets X_1, X_2, X_3, see also Fig. 2:

$$\begin{aligned} \text{Case } 1: X_1 \cap X_2 = \emptyset \ \wedge \ X_3^\circ \neq \emptyset\,, \\ \text{Case } 2: X_1 \cap X_2 = \emptyset \ \wedge \ X_3^\circ = \emptyset\,, \\ \text{Case } 3: X_1 \cap X_2 \neq \emptyset \ \wedge \ X_3^\circ \neq \emptyset\,. \end{aligned}$$

Note, Case 2 is the classical first-order sliding-mode control, whereas Case 1 and 3 occur when altering disturbances are present.

First we consider the cases where the three sets are disjoint and thus reaching and sliding phase may be defined via regions in the state space. The following lemma gives a necessary and sufficient condition for such case.

Lemma 3. (Case 1 and 2). The sets X_1 and X_2 have an empty intersection, i.e. $X_1 \cap X_2 = \emptyset$ if and only if

$$q(x)\varsigma_\phi(x) \geq 0 \quad \text{for all } x. \tag{26}$$

Then Lemma 2 yields $X_3 = \mathbb{R}^n \setminus (X_1 \cup X_2)$.

Proof. Condition (25a) and (25b) ensure that X_1 and X_2 are disjoint if and only if $q(x)\varsigma_\phi(x) \geq 0$ for all x. Further, $q(x)\varsigma_\phi(x) \geq 0$ with condition (25c) gives that for every point in X_3 holds

$$s_\phi(x_3) + \varsigma_\phi(x_3)\alpha(x_3) \in [-\varsigma_\phi(x_3)q(x_3), \varsigma_\phi(x_3)q(x_3)]\,.$$

This makes X_3 by definition of X_1 and X_2 and with (25a) and (25b) the complement of the union of X_1 and X_2.

Note, that the dimension of X_3 may be n. However, for the special case of preserving disturbances, we have $\varsigma_\phi(x) = 0$ and obtain a conventional sliding manifold of dimension $n-1$. This finding is summarised in the following corollary.

Corollary 4. (Case 2). If $\varsigma_\phi(x) = 0$, then $X_1 \cap X_2 = \emptyset$ and $X_3 = \{x \in \mathbb{R}^n \mid s_\phi(x) = 0\}$ and $X_3^\circ = \emptyset$.

Proof. For $\varsigma_\phi(x) = 0$ the disjointness of X_1 and X_2 follows directly from its definition in (20) and (21). The set X_3 is directly obtained using Equation (25c). The set X_3° is calculated analogously.

The following lemma characterises Case 3.

Lemma 5. (Case 3). The set X_1 and X_2 have a non-empty intersection, i.e. $X_1 \cap X_2 \neq \emptyset$, if and only if $q(x)\varsigma_\phi(x) < 0$ for all $x \in X_1 \cap X_2$. Then $X_1 \cap X_2 = X_3^\circ$.

Proof. With (25a) and (25b) for every point in $X_1 \cap X_2$ holds

$$\varsigma_\phi(x)q(x) < s_\phi(x) - \alpha(x)q(x) < -\varsigma_\phi(x)q(x)\,. \tag{27}$$

This is equivalent to $\varsigma_\phi(x)q(x) < 0$. Then the set (23) is defined by all x that fulfil

$$s_\phi(x) + \varsigma_\phi(x)\alpha(x) \in (\varsigma_\phi(x)q(x), -\varsigma_\phi(x)q(x))\,.$$

This is an equivalent notation for points fulfilling (27). Thus, it is $X_1 \cap X_2 = X_3^\circ$.

In Case 3, X_3° is the intersection of X_1 and X_2. Then, for states $x \in X_3$ all three phases are possible, depending on the choice of the control u or equivalently $\sigma > 0$, $\sigma < 0$ or $\sigma = 0$. This shows that it is not sufficient, to define the sliding phase and reaching phase solely via the state sets X_1, X_2 and X_3. Thus, we say that the system is in sliding phase if $\sigma(x, u) = 0$ and in reaching phase if $\sigma(x, u) \neq 0$.

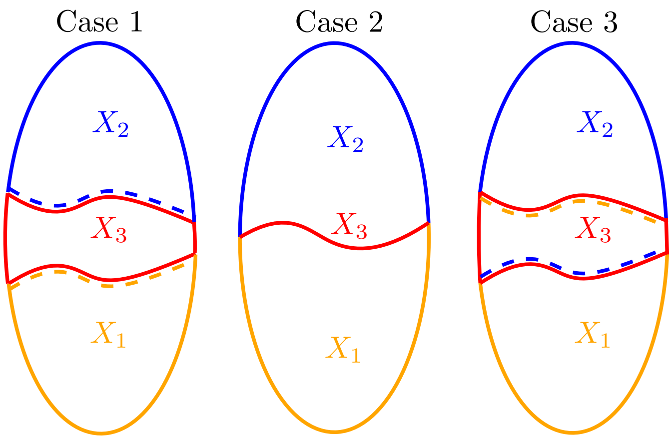

Fig. 2. Three possible configurations of the sets X_i.

4.2 Control signal and its continuity

The control law is the solution of the implicit equation (18b) which takes one of the three forms

$$u(x) \in \{u^-, u^+, u^\circ\} \tag{28}$$

with

$$u^- := \alpha(x) - q(x) \tag{29a}$$

$$u^+ := \alpha(x) + q(x) \tag{29b}$$

$$u^\circ := -\frac{s_\phi(x)}{\varsigma_\phi(x)}\,, \qquad \varsigma_\phi \neq 0\,. \tag{29c}$$

Remark 6. The introduced control law is uniquely defined by x if and only if $X_1 \cap X_2 \cap X_3 = \emptyset$, i.e. in Case 1 and 2; equivalently condition (26) holds.

For the case of preserving disturbances (Case 2) we obtain the conventional first-order sliding-mode control law.

Theorem 7. For $\varsigma_\phi = 0$ the control given by (18) yields

$$u(x) = \begin{cases} u^- & x \in X_1, \\ u^+ & x \in X_2, \\ -\frac{\mathcal{L}_f s_\phi(x) - \mathcal{L}_\phi s_\phi(x)}{\mathcal{L}_g s_\phi(x)} & x \in X_3\,, \end{cases} \tag{30}$$

resembling the conventional first-order sliding-mode control law. Notably, for $\phi = 0$ it is $u(x) = \alpha(x)$ for $x \in X_3$.

Proof. The equality for $x \in X_1$ and $x \in X_2$ is clear. The control law for $x \in X_3$ is the equivalent control law resulting from (15) by having

$$\dot{\sigma} = \mathcal{L}_f s_\phi(x) + \mathcal{L}_g s_\phi(x) u^\circ + \mathcal{L}_\phi s_\phi(x)\,.$$

Requiring $\dot{\sigma} = 0$ leads to

$$u^\circ = -\frac{\mathcal{L}_f s_\phi(x) - \mathcal{L}_\phi s_\phi(x)}{\mathcal{L}_g s_\phi(x)}\,.$$

Remark 8. For preserving disturbances the control u is discontinuous.

For relative degree altering disturbances ϕ, i.e. $\varsigma_\phi \neq 0$, the sets X_1, X_2, X_3 can take the configuration of Case 1 or Case 3. As we show in the following, in Case 1 the control signal is continuous in sliding-mode, whereas in Case 3 neither the sliding-variable nor the control signal is guaranteed continuous.

Theorem 9. If $X_1 \cap X_2 = \emptyset$ and $X_3^\circ \neq \emptyset$, i.e. Case 1, the control law u is continuous in x.

Proof. Continuity of u^-, u^+ and u^0 is ensured by the continuity of α, q, s_ϕ and φ_ϕ. We show continuity at the transitions of u within the set (28). Note for Case 1, we only have transitions at the boundary of X_1 and X_2. For $\hat{x} \in \partial X_3 \cap \partial X_1$ holds, (c.f. (20))

$$s_\phi(\hat{x}) + \varsigma_\phi(\hat{x})(\alpha(\hat{x}) - q(\hat{x})) = 0.$$

For any sequence (x_n) with only elements in X_1 and $\lim_{n\to\infty} x_n = \hat{x}$ we have

$$\lim_{n\to\infty} u^-(x_n) = \lim_{n\to\infty} \alpha(x_n) - q(x_n) = \alpha(\hat{x}) - q(\hat{x}) \quad (31a)$$

$$= \frac{-s_\phi(\hat{x})}{\varsigma(\hat{x})} = \lim_{n\to\infty} \frac{s_\phi(x_n)}{\varsigma(x_n)} = u^0(\hat{x}) \,. \quad (31b)$$

Continuity at the boundary of X_2 is shown analogously.

If the sets take the configuration of Case 3, the control law is not uniquely defined. While the following implications always hold:

$$u(x) = u^- \Rightarrow x \in X_1 \,, \quad (32)$$

$$u(x) = u^+ \Rightarrow x \in X_2 \,, \quad (33)$$

$$u(x) = u^\circ \Rightarrow x \in X_3 \,, \quad (34)$$

whereas the opposite implication does not hold in general. In fact, for Case 3 where the sets overlap the control signal may take any value in $\{u^-, u^+, u^\circ\}$ for $x \in X_3^\circ$.

Lemma 10. If $X_1 \cap X_2 \neq \emptyset$, i.e. Case 3, u and σ are not unique in X_3 and thus may be discontinuous in x.

Proof. For Case 3 we have $q(x) \neq 0$. For $\hat{x} \in \partial X_1$ also holds $\hat{x} \in X_2 \cap X_3$ and thus we may choose $u(\hat{x}) = u^+ = \alpha(\hat{x}) + q(\hat{x})$. But for $\hat{x} \in X_3$ we may choose $u(\hat{x}) = u^\circ = -\frac{s_\phi(\hat{x})}{\varsigma_\phi(\hat{x})} = \alpha(\hat{x}) - q(\hat{x}) \neq u^+$. The first choice of u yields $\sigma(\hat{x}, u^+) \neq 0$ while the second gives $\sigma(\hat{x}, u^\circ) = 0$.

Remark 11. Note, that the continuity of u can be retained if the sliding variable σ is continuous. Considering (15) yields the control law as a function of σ

$$u(x, \sigma) = -\frac{\sigma - s_\phi(x)}{\varsigma_\phi(x)} \,.$$

This function is unique and continuous in σ, in particular at $\sigma = 0$. This remarkable property for a sliding-mode control is obtained for both cases with altering disturbance, i.e. Case 1 and 3.

4.3 Closed loop system

For the system (4) with sliding variable (15) and control (28) obtained from the sliding-mode control law (18), the closed-loop dynamics may take the form:

$$\dot{x}^- = f(x) + g(x)(\alpha(x) - q(x)) + \phi \,,$$

$$\dot{x}^+ = f(x) + g(x)(\alpha(x) + q(x)) + \phi \,,$$

$$\text{or} \quad \dot{x}^0 = f(x) - g(x)\frac{s_\phi(x)}{\varsigma_\phi(x)} \,.$$

For the cases of non-overlapping sets X_i (Case 1 and 2) we obtain the following closed-loop dynamics.

Theorem 12. If $X_1 \cap X_2 = \emptyset$, then the closed-loop system (4) and (18) takes the form

$$\dot{x} = \begin{cases} \dot{x}^- & \text{for } x \in X_1 \\ \dot{x}^+ & \text{for } x \in X_2 \\ \dot{x}^\circ & \text{for } x \in X_3 \end{cases} \quad (35a)$$

$$y = h(x) \,. \quad (35b)$$

Note that for Case 2 we have the typical Fillipov solutions on the sliding manifold, whereas in Case 1 we obtain classical Caratheodory solutions. For overlapping sets X_i, i.e. Case 3, uniqueness of the solution is lost.

Theorem 13. If $X_1 \cap X_2 \neq \emptyset$ then the closed loop system (4) and (18) has the form

$$\dot{x} = \dot{x}^- \qquad x \in X_1 \quad (36)$$

$$\dot{x} = \dot{x}^+ \qquad x \in X_2 \quad (37)$$

$$\dot{x} \in \{\dot{x}^-, \dot{x}^\circ, \dot{x}^+\} \qquad x \in X_3^\circ \,. \quad (38)$$

In this case the dynamics on X_3° are given by a differential inclusion and the solution on X_3° is not well-defined (not even in the sense of Fillipov, since there is no guiding manifold available).

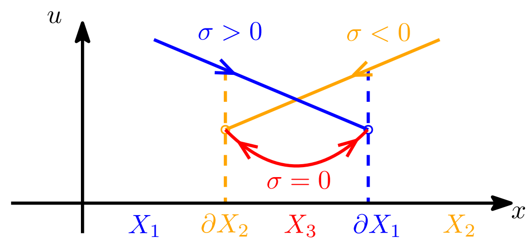

Fig. 3. Possible composition for control laws in Case 3.

Figure 3 illustrates a possible scenario in Case 3. For the set X_i, the possible input u is displayed. It can be seen that for $x \in X_3$ three different u and σ are possible. Further, the continuity of u for $\sigma > 0$ and $\sigma = 0$ at the boundary of X_1 is illustrated. For the displayed scenario the state x is moving towards the boundary of X_1 for $\sigma > 0$. For $\sigma < 0$ it moves to the boundary of X_2. For $\sigma = 0$ the vector fields point to the boundary of X_3 as well. Thus, choosing $\sigma = 0$ whenever $x \in X_3$ leads to a chattering of the solution at the boundary of X_3.

However, depending on the choice u and σ on X_3 various solutions are possible. Arbitrary switching between the three different values of the sliding variable in the interior of X_3 may lead to a complex manifold of solutions. While this manifold might include all possible paths in the one-dimensional case, its structure is more complex for higher dimensional systems.

4.4 Sliding-mode dynamics and disturbance compensation

In the spirit of Wulff et al. (2020) we shall distinguish the states in Byrnes-Isidori form as external states ξ, designed internal states ζ of the dynamics in sliding-mode and inherited internal states η. For the nominal design with no altering disturbance, we choose the state-transformation

$$\xi_1 = \sigma \quad (39a)$$

$$\zeta_1 := \tau_1(x) = h(x) \,, \quad (39b)$$

$$\zeta_i := \tau_i(x) = \mathcal{L}_f^{i-1} h(x) \,, \quad i \in \{2, \ldots, n-2\} \,, \quad (39c)$$

$$\zeta_{n-1} := \tau_{n-1}(x) = \mathcal{L}_f^{n-2} h(x) \quad (39d)$$

and obtain the reduced dynamics for $\sigma \equiv 0$:

$$\dot{\zeta}_1 = \zeta_2 \,, \quad (40a)$$

$$\dot{\zeta}_i = \zeta_{i+1} \,, \quad \text{for} \quad i \in \{2, \ldots, n-2\} \,, \quad (40b)$$

$$\dot{\zeta}_{n-1} = \gamma(\zeta_1, \ldots, \zeta_{n-1}) \quad (40c)$$

which are stable by design.

In the case of a relative degree altering disturbance, we cannot choose $\sigma(x,u)$ as the external state ξ_1, because $\sigma(x,u)$ depends on the input. Instead, we use the transformation (39b)-(39d) and internal state $\eta := \tau_n(x)$ as diffeomorphism and $\mathcal{L}_g \tau_n(x) = 0$. For $\sigma \equiv 0$ we obtain the closed-loop sliding-mode dynamics

$$\dot{\zeta}_1 = \zeta_2 \,, \tag{41a}$$
$$\dot{\zeta}_i = \zeta_{i+1} \,, \quad \text{for} \quad i \in \{2, \dots, n-2\} \,, \tag{41b}$$
$$\dot{\zeta}_{n-1} = \gamma(\zeta_1, \dots, \zeta_{n-1}) \tag{41c}$$
$$\dot{\eta} = \mathcal{L}_{f+\phi} \tau_n(x)|_{x=\tau^{-1}(\xi,\eta)} \,. \tag{41d}$$

The dynamics in sliding-mode are not reduced in dimension as we observe in the nominal case. However, the dynamics described by ζ are not affected by the disturbance. Hence, matched and unmatched disturbances are compensated on ζ and thus are invisible at the output y. However, the relative degree altering disturbances ϕ introduce additional internal dynamics (41d) and may even render them unstable. In such case, the unbounded internal state will render the control signal unbounded, a scenario that cannot occur for relative degree preserving disturbances. Indeed these internal dynamics are the same as in (11).

5. CASE STUDY: MASS-SPRING-DAMPER SYSTEM

Consider a standard system consisting of two connected mass-spring-damper systems

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1-k_2}{m_1} & \frac{-d_1}{m_1} & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & \frac{-k_1-k_2}{m_2} & \frac{-d_1}{m_2} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{pmatrix} u + \phi_1 + \phi_2$$
$$y = x_3$$

with masses m_i, spring stiffness k_i and damping coefficient d_1. We consider the altering disturbance ϕ_1 and the preserving disturbance ϕ_2 with

$$\phi_1 = \begin{pmatrix} 0 \\ -\frac{d_2}{m_1} x_2 + \frac{d_2}{m_1} x_4 \\ 0 \\ \frac{d_2}{m_2} x_2 - \frac{d_2}{m_1} x_4 \end{pmatrix} , \qquad \phi_2 = \begin{pmatrix} 3 \\ \cos(4t) \\ \sin(t) \\ 0 \end{pmatrix} .$$

The relative degree altering disturbance ϕ_1 is the result of an additional damper between two masses with damping coefficient d_2 which is neglected in the nominal model. The output y has relative degree 4 for the nominal case and relative degree 3 if ϕ_1 acts on the system and $d_2 \neq 0$. The relative degree preserving disturbance ϕ_2 is the result of some exogenous excitation. Both disturbances have an unmatched component. The parameters are given by

$$m_1 = 70 \qquad m_2 = 700 \qquad k_1 = 500$$
$$k_2 = 250 \qquad d_1 = 50 \qquad d_2 \in \{-20, 0, 20\} \,.$$

The implemented sliding-mode controller has the form

$$\sigma = y + 3\dot{y} + 3\ddot{y} + \dddot{y} \tag{42a}$$
$$u = -(\mathcal{L}_g \sigma)^{-1} (\mathcal{L}_f \sigma + L \operatorname{sgn}(\sigma)) \tag{42b}$$

with $L = 100 > |\mathcal{L}_{\phi_2} \sigma|$ chosen to compensate the unmatched relative degree preserving disturbance ϕ_2. For three different parameters of d_2 the conditions of Section 4 shall be analysed. It turns out that for $d_2 = 20$ the internal dynamics are stable and the system is in Case 1. For $d_2 = 0$ no internal dynamics exist and the system is in Case 2, i.e. we use a conventional first-order sliding-mode control law. For $d_2 = -20$ the internal dynamics become unstable and the system is in Case 3. Note that systems with negative damping may occur in some application such as in the modelling of aerodynamics flutters, e.g. in (Tewari, 2016). We simulate the nominal system with $d_2 = 0$ and the system with relative degree altering disturbance $d_2 = 20$ and $d_2 = -20$ and the results with view on their stability and compensation properties. The derivatives of the output y are obtained via simple numerical differentiation using high-pass filters. For the simulations we have chosen the third order Bogacki-Shampine solver with a fixed step size of 10^{-4}s as this has proven to give the best results.

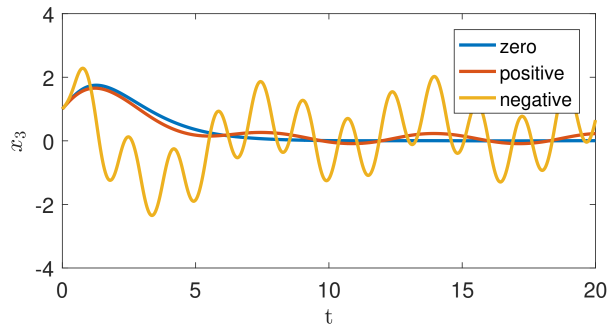

Fig. 4. Closed-loop output evolution.

Fig. 4 shows the evolution of the output $y = x_3$. If $d_2 = 0$, i.e. no altering disturbance ϕ_1, the matched and unmatched preserving disturbance ϕ_2 is completely compensated on the output. For a positive damping $d_2 > 0$ the influence of the disturbances is only attenuated but not completely compensated in the simulation. Indeed, the attenuation depends heavily on the step size of the solver and the approximation of the derivative of the output despite their theoretically perfect compensation behaviour. All tested solvers give a qualitative similar solution, but only differ in some numerical values. For a negative damping $d_2 < 0$ the control law is not well-defined any more. Fig. 4 displays the solution obtained by using a conventional fixed-order integrator. The result is an oscillating solution. As a result, neither of the uncertainties is rejected. However, the solution remains bounded.

Fig. 5 shows the control variable for the three cases. In the nominal case the control variable shows the typical (numerical) chattering. Also for positive damping we observe chattering, although the control signal is expected to be continuous according to Theorem 9. This chattering again is of numerical nature induced by the evaluation of (18b). For negative damping no chattering occurs, but we have a piecewise continuous control signal with discontinuities at the switching instances of the sliding variable.

6. CONCLUSION

Disturbances that change the relative degree of the system may have a strong impact on the closed-loop control system. Well-definedness and stability as well as attractivity of the sliding-manifold may be lost. We derive necessary and sufficient conditions that avoid such scenario and ensure stability and disturbance compensation of unmatched uncertainties.

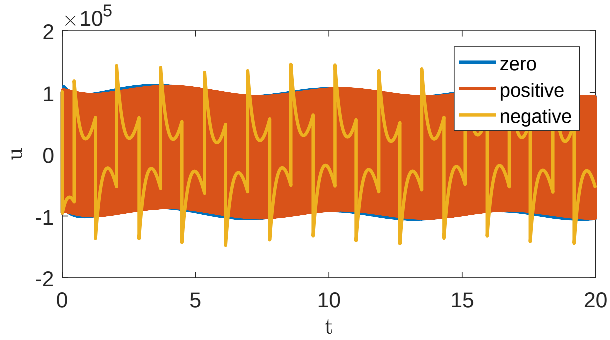

Fig. 5. Closed-loop control variable evolution.

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