Injection locking and synchronization in Josephson photonics devices

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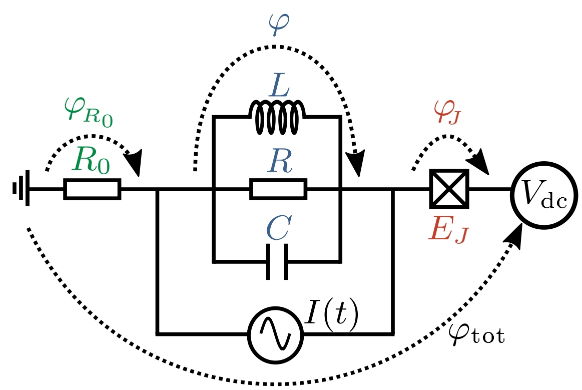
SUPPLEMENTAL MATERIAL

In this supplemental material to our article we present further details on (*i.*) the theory of locking by direct ac-current injection into the cavity, deriving an effective Adler-equation similar to that presented in Sec. III; (*ii.*) the derivation of the effective Kuramoto equations that supplements the results presented in Sec. VI; and (*iii.*) notes on the numerical calculations, relevant to all sections.

LOCKING FROM DIRECT AC-CURRENT INJECTION

A similar locking scenario to the one described in the main text arises when directly injecting a locking signal into the cavity. Experimentally, this can be realized by feeding the cavity with an oscillating current through the transmission line. We now proceed to derive the locking equation for the circuit model of the Josephson photonics device shown in Fig. S.1.



$$I(t) = \epsilon \tfrac{\hbar\Omega}{2eR}\cos(\Omega t + \phi_\epsilon), \quad \epsilon = \tfrac{2e}{\hbar\Omega} I_{\rm ac} R$$

FIG. S.1. (Color online.) Sketch of a dc-voltage biased Josephson photonics circuit with an in-series resistance and a single resonance (with frequency $\omega_0 = 1/\sqrt{LC}$ and width $\gamma = 1/RC$). A small ac-signal is injected by an external current source.

Analogously to the main text, the classical equations of motions are the Kirchhoff equations

$$\dot\varphi_J = \frac{2e}{\hbar} V_{\rm dc} - \frac{2e}{\hbar} I_c R_0 \sin(\varphi_J) - \dot\varphi; \tag{S.1a}$$

$$\ddot\varphi + \gamma\dot\varphi + \omega_0^2 \varphi = \frac{2eI_cR}{\hbar\omega_0}\gamma\omega_0 \sin\varphi_J - \frac{2e}{\hbar} I(t). \tag{S.1b}$$

Here we assume the device is dc-biased by a fixed voltage $V_{\rm dc}$. The injection locking signal is provided by the ac-current $I(t) = I_{\rm ac}\cos(\Omega t + \phi_\epsilon)$ with amplitude assumed small, $\epsilon = (2e/\hbar)(I_{\rm ac}R/\Omega) < 1$. For the other quantities, we use the same parametrization as in Table I. The locking equation of Adler-type is derived using the same time scale separation arguments as in the main text. We use a similar ansatz for the degrees of freedom $\varphi(t)$ and $\varphi_J(t)$ in terms of slowly-varying quantities

$$\varphi_J(t) = \omega_J t + \theta_J(t) + a_J \sin[\omega_J t + \phi_J(t)]; \tag{S.2a}$$

$$\varphi(t) = a \sin[\omega_J t + \phi(t)] \;. \tag{S.2b}$$

The unknown functions, $\theta_J(t)$, $\phi_J(t)$ and $\phi(t)$, are slowly-varying in time, with $\dot\theta_J$, $\dot\phi_J$, $\dot\phi \simeq \nu \ll \omega_J$, while the 'fast' frequencies are only slightly detuned from each other, $\omega_J \simeq \omega_{\rm dc} \simeq \omega_0$.

Separating each equation of motion into slow and rapidly oscillating parts, we find the same equation for the slow component $\theta_J(t)$ as Eq. (8), namely

$$\dot\theta_J = \omega_{\rm dc} - \omega_J - v_{R_0}\frac{a_J}{2}\sin(\phi_J - \theta_J), \tag{S.3}$$

The components oscillating with frequencies close to ω_J are modified in this biasing condition compared to the main text,

$$-\omega_J a_J \cos(\omega_J t + \phi_J) = v_{R_0}\sin(\omega_J t + \theta_J) + a\omega_J\cos(\omega_J t + \phi), \tag{S.4a}$$

$$\tilde I_c \omega_0 \sin(\omega_J t + \theta_J) - \epsilon\Omega\cos(\Omega t + \phi_\epsilon) = a\;\left\{\frac{1}{\gamma}\left[\omega_0^2 - \left(\omega_J + \dot\phi\right)^2\right]\sin(\omega_J t + \phi) + \omega_J\cos(\omega_J t + \phi)\right\}. \tag{S.4b}$$

These equations are the counterparts of Eqs.(9a) and (9b) in the main text. A transformation of eqs. (S.4) to a frame rotating with ω_J yields

$$a_J e^{i\phi_J} = i\frac{v_{R_0}}{\Omega}e^{i\theta_J} - ae^{i\phi}, \tag{S.5a}$$

$$ae^{i\phi} = \frac{1}{z\left(\nu_j\right)}\left(\tilde I_c e^{i\theta_J} - i\epsilon e^{i\phi_\epsilon}e^{i(\Omega-\omega_J)t}\right), \tag{S.5b}$$

where we have defined the complex dimensionless impedance $z(\nu_j) \equiv |z|e^{i\chi} = \left(2\nu_j + i\gamma\right)/\gamma$, in analogy to Sec. VI, that here is always evaluated at the detuning $\nu_j \equiv \left(\omega_0 - \omega_J - \dot\phi\right)$ between the effective Josephson frequency and the resonance. Outside the locking region ν_j is a function of the injected signal through $\omega_J(\epsilon)$ and $\dot\phi(\epsilon, t)$ and imprints its dependence onto the dimensionless impedance, both on its absolute value $|z|(\epsilon)$ and its phase $\chi(\epsilon)$. In the locking region $\dot\phi = 0$ and $\omega_J = \Omega$, such that $\nu_j = \omega_0 - \Omega$ becomes independent of ϵ.

As in the main text, we find the locking equation by substituting Eq. (S.5b) into Eq. (S.5a) and taking the imaginary part to find an expression for $a_J\sin(\phi_J - \theta_J)$. The expression is then substituted into Eq. (S.3), yielding

$$\dot\theta_J = (\omega_{\rm dc} - \omega_J) - \frac{v_{R_0}}{2}\left[\epsilon\,\frac{1}{|z|}\sin\left[(\Omega - \omega_J)t - \theta_J + \phi_\epsilon - \chi + \frac{\pi}{2}\right] + \frac{v_{R_0}}{\Omega} + \frac{\tilde I_c}{|z|}\sin(\chi)\right]. \tag{S.6}$$

which is analogous to Eq.(11) in the main text.

Defining the Adler phase similarly to Eq. (12),

$$\psi(t) = \omega_J t + \theta_J - \Omega t - \phi_\epsilon - \frac{\pi}{2}, \tag{S.7}$$

we obtain the locking equation for a direct ac-current injection. Here too the locking equation has the form of a generalized Adler equation,

$$\dot\psi = \nu(\epsilon, \Omega) - \nu_c(\epsilon, \Omega)\,\frac{\epsilon}{2}\sin\left[\psi + \chi(\epsilon, \Omega)\right]. \tag{S.8}$$

The parameters are given by,

$$\nu(\epsilon, \Omega) = \left(\omega_{\rm dc} - \frac{v_{R_0}}{\Omega}\,\frac{v_{R_0}}{2} - \frac{\tilde I_c}{|z|(\epsilon, \Omega)}\,\frac{v_{R_0}}{2}\right) - \Omega. \tag{S.9a}$$

$$\nu_c(\epsilon, \Omega) = \frac{v_{R_0}}{|z|(\epsilon, \Omega)}, \quad |z|(\epsilon, \Omega) = \frac{1}{\gamma}\sqrt{4\left[\omega_0 - \omega_J(\epsilon, \Omega) - \dot\phi(\epsilon, \Omega)\right]^2 + \gamma^2}. \tag{S.9b}$$

$$e^{i\chi(\epsilon, \Omega)} = \frac{z(\epsilon, \Omega)}{|z|(\epsilon, \Omega)} = \frac{2\left[\omega_0 - \omega_J(\epsilon, \Omega) - \dot\phi(\epsilon, \Omega)\right] + i\gamma}{\sqrt{4\left[\omega_0 - \omega_J(\epsilon, \Omega) - \dot\phi(\epsilon, \Omega)\right]^2 + \gamma^2}} \tag{S.9c}$$

Compared to Eq. (13), here not only the effective detuning $\nu(\epsilon, \Omega)$, but also the effective width of the locking region $\nu_c(\epsilon, \Omega)$ and the effective locked phase $\chi(\epsilon, \Omega)$ acquire dependence on the injection parameters Ω and ϵ through the dimensionless impedance $z(\nu_j)$.

DERIVATION OF THE EFFECTIVE KURAMOTO-TYPE EQUATIONS FOR SYNCHRONIZATION

The derivation of the Kuramoto-type equations for synchronization starts from the full circuit equations, Eq. (25), and the ansatz for the dominant oscillations, Eqs. (27) and (28) consistent with the limit of weak Josephson coupling $\tilde{I}_c^{(\sigma)} \ll 1$ and small low-frequency impedance $R_0^{(\sigma)}$, i.e. $v_{R_0}^{(\sigma)} \ll \omega_0^{(\sigma)}$. We further assume time scale separation $\dot{\theta}_J^{(\sigma)}, \dot{\phi}_J^{(\sigma)}, \dot{\phi}^{(\sigma)} \ll \omega_{\rm dc}^{(\sigma)}, \omega_0^{(\sigma)}, \omega_J^{(\sigma)}$, as well as $(\omega_{\rm dc}^{(\sigma)} - \omega_0^{(\sigma)})$, $\left(\omega_{\rm dc}^{(2)} - \omega_{\rm dc}^{(1)}\right) \ll \omega_{\rm dc}^{(\sigma)}, \omega_0^{(\sigma)}, \omega_J^{(\sigma)}$. The slowly-varying functions obey the following system of coupled equations

$$\dot{\theta}_J^{(1)} = \left(\omega_{\rm dc}^{(1)} - \omega_J^{(1)}\right) - v_{R_0}^{(1)} \frac{a^{(1)}}{2} \sin\left(\phi_J^{(1)} - \theta_J^{(1)}\right), \tag{S.10a}$$

$$\dot{\theta}_J^{(2)} = \left(\omega_{\rm dc}^{(2)} - \omega_J^{(2)}\right) - v_{R_0}^{(2)} \frac{a^{(2)}}{2} \sin\left(\phi_J^{(2)} - \theta_J^{(2)}\right), \tag{S.10b}$$

$$z^{(1)} b^{(1)} e^{i\phi^{(1)}} = \tilde{I}_c^{(1)} e^{i\theta_J^{(1)}} - \epsilon^{(1)} \frac{\omega_J^{(2)}}{\gamma^{(1)}} b^{(2)} e^{i\phi^{(2)}} e^{i\nu_J t}, \tag{S.10c}$$

$$z^{(2)} b^{(2)} e^{i\phi^{(2)}} = \tilde{I}_c^{(2)} e^{i\theta_J^{(2)}} - \epsilon^{(2)} \frac{\omega_J^{(1)}}{\gamma^{(2)}} b^{(1)} e^{i\phi^{(1)}} e^{-i\nu_J t}, \tag{S.10d}$$

$$a^{(1)} e^{i\phi_J^{(1)}} = i \frac{v_{R_0}^{(1)}}{\omega_0^{(1)}} e^{i\theta_J^{(1)}} - b^{(1)} e^{i\phi^{(1)}}, \tag{S.10e}$$

$$a^{(2)} e^{i\phi_J^{(2)}} = i \frac{v_{R_0}^{(2)}}{\omega_0^{(2)}} e^{i\theta_J^{(2)}} - b^{(2)} e^{i\phi^{(2)}}. \tag{S.10f}$$

where $\nu_J = \left(\omega_J^{(2)} - \omega_J^{(1)}\right)$ is the detuning between the Josephson oscillations of the two devices. We have also introduced the complex dimensionless impedance $z^{(\sigma)} = \left(2\nu^{(\sigma)} + i\gamma^{(\sigma)}\right)/\gamma^{(\sigma)}$, that is obtained from the Fourier transform $\tilde{Z}^{(\sigma)}(\omega)$ of the response $Z^{(\sigma)}$. The impedance $z^{(\sigma)} = \tilde{Z}^{(\sigma)}(\omega = \omega_0^{(\sigma)} - \nu^{(\sigma)})$ is evaluated at the detuning $\nu^{(\sigma)}$ given by $\nu^{(\sigma)} \equiv \left(\omega_0^{(\sigma)} - \omega_J^{(\sigma)} - \dot{\phi}^{(\sigma)}\right)$.

Eqs. (S.10c) and (S.10d) can be written as a matrix equation for the vector $v \equiv \left[b^{(1)} e^{i\phi^{(1)}}, b^{(2)} e^{i\phi^{(2)}}\right]^T$ containing the cavity oscillation amplitudes,

$$Mv = v_J, \quad \text{with} \quad M \equiv \begin{bmatrix} z^{(1)} & \epsilon^{(1)} \frac{\omega_J^{(2)}}{\gamma^{(1)}} e^{i\nu_J t} \\ \epsilon^{(2)} \frac{\omega_J^{(1)}}{\gamma^{(2)}} e^{-i\nu_J t} & z^{(2)} \end{bmatrix}, \quad \text{and} \quad v_J \equiv \left[\tilde{I}_c^{(1)} e^{i\theta_J^{(1)}}, \tilde{I}_c^{(2)} e^{i\theta_J^{(2)}}\right]^T.$$

The matrix M can be inverted analytically to obtain the following expressions for the cavity amplitudes

$$b^{(1)} e^{i\phi^{(1)}} = \frac{\tilde{I}_c^{(1)} e^{i\theta_J^{(1)}} z^{(2)} - \epsilon^{(1)} \left(\omega_J^{(2)}/\gamma^{(1)}\right) \tilde{I}_c^{(2)} e^{i\theta_J^{(2)}} e^{i\nu_J t}}{z^{(1)} z^{(2)} - \epsilon^{(1)} \epsilon^{(2)} \left(\omega_J^{(1)}/\gamma^{(1)}\right) \left(\omega_J^{(2)}/\gamma^{(2)}\right)}, \tag{S.11a}$$

$$b^{(2)} e^{i\phi^{(2)}} = \frac{\tilde{I}_c^{(2)} e^{i\theta_J^{(2)}} z^{(1)} - \epsilon^{(2)} \left(\omega_J^{(1)}/\gamma^{(2)}\right) \tilde{I}_c^{(1)} e^{i\theta_J^{(1)}} e^{-i\nu_J t}}{z^{(1)} z^{(2)} - \epsilon^{(1)} \epsilon^{(2)} \left(\omega_J^{(1)}/\gamma^{(1)}\right) \left(\omega_J^{(2)}/\gamma^{(2)}\right)}. \tag{S.11b}$$

The above expressions for the amplitudes of the two cavity oscillations can be inserted into the original set of coupled equations, specifically Eqns. (S.10e) and (S.10f), to obtain expressions for the relative phases $(\theta_J^{(\sigma)} - \phi_J^{(\sigma)})$, that can then be inserted back into Eqns. (S.10a) and (S.10b), in analogy to the procedure used in Sec. III. We arrive at the following two equations for the slow components of the Josephson phases,

$$\dot{\theta}_J^{(1)} = \left(\omega_{\rm dc}^{(1)} - \omega_J^{(1)}\right) - v_{R_0}^{(1)} \frac{1}{2} \mathrm{Im} \left\{ i \frac{v_{R_0}^{(1)}}{\omega_0^{(1)}} - \frac{\tilde{I}_c^{(1)} z^{(2)} - \epsilon^{(1)} \left(\omega_J^{(2)}/\gamma^{(1)}\right) \tilde{I}_c^{(2)} e^{i\nu_J t} e^{i\left(\theta_J^{(2)} - \theta_J^{(1)}\right)}}{z^{(1)} z^{(2)} - \epsilon^{(1)} \epsilon^{(2)} \left(\omega_J^{(1)}/\gamma^{(1)}\right) \left(\omega_J^{(2)}/\gamma^{(2)}\right)} \right\}, \tag{S.12a}$$

$$\dot{\theta}_J^{(2)} = \left(\omega_{\rm dc}^{(2)} - \omega_J^{(2)}\right) - v_{R_0}^{(2)} \omega_0^{(2)} \frac{1}{2} \mathrm{Im} \left\{ i \frac{v_{R_0}^{(2)}}{\omega_0^{(2)}} - \frac{\tilde{I}_c^{(2)} z^{(1)} - \epsilon^{(2)} \left(\omega_J^{(1)}/\gamma^{(2)}\right) \tilde{I}_c^{(1)} e^{-i\nu_J t} e^{i\left(\theta_J^{(1)} - \theta_J^{(2)}\right)}}{z^{(1)} z^{(2)} - \epsilon^{(1)} \epsilon^{(2)} \left(\omega_J^{(1)}/\gamma^{(1)}\right) \left(\omega_J^{(2)}/\gamma^{(2)}\right)} \right\}. \tag{S.12b}$$

The above equations give the non-linear evolution of the slow phases $\theta_J^{(\sigma)}$ as a function of the couplings $\epsilon^{(\sigma)}$. These equations reduce to the Kuramoto model in the limit $\epsilon^{(1)}\left(\omega_J^{(2)}/\gamma_{(1)}\right) \ll 1$ and $\epsilon^{(2)}\left(\omega_J^{(1)}/\gamma_{(2)}\right) \ll 1$, where after linearizing with respect to the couplings, the equations become

$$\dot{\theta}_J^{(1)} = \tilde{\nu}^{(1)} + \epsilon^{(1)}\ \frac{1}{2} v_{R_0}^{(1)} \frac{\omega_J^{(1)}}{\gamma^{(2)}} \frac{\tilde{I}_c^{(2)}}{|z_1||z_2|} \sin\left[\nu_J t + \theta_J^{(2)} - \theta_J^{(1)} - \chi_1 - \chi_2\right], \tag{S.13a}$$

$$\dot{\theta}_J^{(2)} = \tilde{\nu}^{(2)} - \epsilon^{(2)}\ \frac{1}{2} v_{R_0}^{(2)} \frac{\omega_J^{(2)}}{\gamma^{(1)}} \frac{\tilde{I}_c^{(1)}}{|z_1||z_2|} \sin\left[\nu_J t + \theta_J^{(2)} - \theta_J^{(1)} + \chi_1 + \chi_2\right]. \tag{S.13b}$$

with $\tilde{\nu}^{(\sigma)} \equiv \omega_J^{(\sigma)}(\epsilon = 0) - \omega_J^{(\sigma)}$ and $z^{(\sigma)} \equiv |z^{(\sigma)}|e^{\chi^{(\sigma)}}$. These equations are equivalent to Eq. (29) of the main text.

SOME NOTES ON NUMERICAL IMPLEMENTATION

For numerical results the coupled equations of motions, Eq. (1), were solved using a real-valued variable-coefficient ordinary differential equation (VODE) solver with a BDF method implemented in the Python library SciPy. Typically solutions were calculated for time intervals of $0 \le t\omega_0 \le 10^5$. Spectra were calculated using standard FFT routines from SciPy with a frequency resolution of $\delta\omega/\omega_0 \approx 8 \cdot 10^{-5}$ given by a time interval $2.5 \cdot 10^4 < t\omega_0 \le 10^5$ after reaching the steady state. To regularize the spectra we used a Kaiser-Bessel window with a shape parametrized by $\alpha = 3$. The Josephson frequency was numerically computed by a time average of the solution for $\dot{\varphi}_J$ in a time interval $7.5 \cdot 10^4 \le t\omega_0 \le 10^5$.

Simulations including noise use a lower-order Euler-Maruyama algorithm to solve the full equations of motions with an included auxiliary equation creating colored noise by an Ornstein-Uhlenbeck process. Wiener increments are drawn from a Gaussian distribution with random seed by a NumPy random number generator. Here were used a rectangular window function to calculate spectra.

Phase space distributions were calculated in a rotating frame with 200 bins in both coordinate directions. We used a larger steady state interval $2.5 \cdot 10^5 \le t\omega_0 \le 10^6$ with $7.5 \cdot 10^7$ time steps ($2 \cdot 10^5 \le t\omega_0 \le 7.2 \cdot 10^5$ with $14 \cdot 10^7$ time steps for Fig. 6(c) respectively).

With these parameters plots can be easily created without extensive optimization to reduce numerical costs. Single runs on standard PCs, or two-parameters sweeps and multi-runs for noise averaging on a Baden-Württemberg Cluster JUSTUS2 require typical runtimes ranging from few minutes to a few days on ~ 100 cores.