On the spectrum of atmospheric motions

The Horizontal Spectrum of Vertical Velocities near the Tropopause from Global to Gravity Wave and Kolmogorov Scales

Ulrich Schumann

With contributions of Thomas Birner, Martina Bramberger, George Craig, Andreas Dörnbrack, Joseph Egger, Andreas Giez, Sonja Gisinger, Jörgen Jensen, Chris Kruse, Qiang Li, Erik Lindborg, Markus Rapp, Andreas Schäfler, Tobias Selz, Robert Sharman, Nils Wedi, the DEEPWAVE and NAWDEX teams and Michael Eckert and Jessika Wichner

Atmospheric Dynamics as an application of Fluid Mechanics
The stably stratified atmosphere, below and above the tropopause

Brunt-Väisälä Frequency $N/s^{-1}$

$N = \frac{g}{\Theta} \frac{\partial \Theta}{\partial z}$

$\Theta = T \left( \frac{p_{sur}}{p} \right)^{\kappa}$

Height $z/km$

Temperature $T/K$, potential Temperature $\Theta/K$

e.g., for ICAO Standard Atmosphere
Ludwig-Prandtl – Memorial Lecture # 63

# 1 (1957): Albert Betz, Göttingen
# 4 (1969): Ernst Schmidt, TU München
# 21 (1978): Jürgen Zierep, Karlsruhe
# 49 (2006): Rainer Friedrich, TU München
LES of Prandtl’s slope boundary layer

Prandtl, “Strömungslehre” (1942)

Meteorological research

“a suitable occupation during peace”
(1945, cited from Eckert, 2019)

Turbulent Boundary Layer

Turbulent kinetic energy

Slope wind

Rotating flows, \( f = 2 \omega \sin(\varphi) \)

Vorticity conservation

Weather Prediction
Vertical velocity \((w)\) – Fundamental for Atmospheric Dynamics

\[\text{p}=200 \text{ hPa} \left( \approx 12 \text{ km} \right), \text{12 UTC 13 Oct 2016}\]

HALO during NAWDEX

\[w\]

Mass, Energy and Momentum transport
Clear Air Turbulence
Adiabatic warming or cooling
Cloud Formation
Formation of stable inversion layers
Vital for Weather Prediction

\[\text{min}=-1.3 \text{ m/s}, \ \text{max}=1.5 \text{ m/s}\]

ECMWF-IFS TCo7999 (Nils Wedi), 1.2 km grid resolution

10 to 100 times smaller than horizontal wind
1. What do we know about spectra of horizontal velocities (h-spectrum)
2. What do we know about spectra of vertical velocities (w-spectrum)
3. A hypothesis relating w-spectra with h-spectra
4. Validation
5. Implications
6. Conclusions
What do we know about spectra of horizontal velocities: from long-distance commercial aircraft data

Nastrom & Gage (1985)

GASP: Global Atmospheric Sampling Program

6000 flights
B747 airliners, Prandtl’s pitot pipe (TAS) and heading, 9–13 km altitude at horizontal scales of 2.6 to 10000 km

FIG. 3. Variance power spectra of wind and potential temperature near the tropopause from GASP aircraft data. The spectra for meridional wind and temperature are shifted one and two decades to the right, respectively; lines with slopes $-3$ and $-\frac{5}{3}$ are entered at the same relative coordinates for each variable for comparison.
The “Canonical Spectrum” (Skamarock et al., 2014) of horizontal velocities from long-distance commercial aircraft data

(GASP: Nastrom & Gage, 1985; MOZAIC: Lindborg, 1999)

Important for:
• Understanding
• Model tests (Skamarock et al., 2014; Wedi 2016)
• NWP predictability (Lorenz, 1969)
• SGS modeling

How to explain?
What determines the transition scale $L$?

$L \approx 1 \text{ km}$

Fig. 1. Schematic of canonical atmospheric kinetic energy spectra.
Large-scale $k^{-3}$ spectrum: Quasi-geostrophic turbulence
(Charney, 1971)

$$f \times u \approx -\frac{1}{\rho} \nabla_z p,$$

**Synoptic motion scales:**
Quasi-2d turbulence, downscale cascade of enstrophy (vorticity)$^2$
at a rate $\sigma$ (units of s$^{-3}$)

$$E = \sigma^{2/3} k^{-3}$$

$E$ in m$^3$ s$^{-2}$

**Unit of k:** 1/m

**Log (energy)**

**Log (wavenumber $k$)**

**1000s km**

**100s km**

**~1 km**

FIG. 1. Schematic of canonical atmospheric kinetic energy spectra.
Small-scale $k^{-5/3}$ spectrum: Inertial-range energy cascade (Kolmogorov, 1941)

Turbulence scales:
Downscale energy transfer with energy dissipation rate $\varepsilon$ (unit: $m^2 s^{-3}$)

$$E = \varepsilon^{2/3} k^{-5/3}$$

Basic, e.g., for LES SGS

FIG. 1. Schematic of canonical atmospheric kinetic energy spectra.
Stably stratified turbulence

DNS, 1024^3 grid cells, homogeneous, stratified turbulence, forced at large scales (Kimura and Herring, JFM, 2012)

Stably stratified turbulence is strongly anisotropic for scales > Obukhov scale 

\[ L_o = (\varepsilon / N^3)^{1/2} \]
Mesoscale $k^{-5/3}$ spectrum: stably stratified turbulence
(e.g., Lilly, 1983; Riley and Lindborg, 2008)

**Mesoscales:**
Downscale energy transfer with energy dissipation rate $\varepsilon$ (unit: m$^2$ s$^{-3}$)

$$E = \varepsilon^{2/3} k^{-5/3}$$

**Downscale energy transfer rate:**
$$\varepsilon = \frac{u'^3}{L}$$

$u'$ = horizontal velocity fluctuation
$L$ = horizontal length scale.

**FIG. 1.** Schematic of canonical atmospheric kinetic energy spectra.
Gravity waves in stratosphere over mountains (vertically upward energy transfer, not downscale)

Waves: disturbances which travel over long distances and transport energy and momentum

Dörnbrack (2019).
Helmholtz decomposition (1858) for horizontal velocity

Hermann von Helmholtz (1821-1894)

\[ \mathbf{v} = \nabla \times \psi + \nabla \chi \]  
(Helmholtz, Crelle J., 1858)

For horizontal velocity \( \mathbf{v} = (u, v) \) on a smooth surface with normal \( \mathbf{k} \):

\[ \mathbf{v} = \mathbf{k} \times \nabla \psi + \nabla \chi \]

\( \mathbf{v} = \mathbf{v}_R + \mathbf{v}_D, \quad \mathbf{v}_R = \mathbf{k} \times \nabla \psi, \quad \mathbf{v}_D = \nabla \chi. \)  
(Wippermann, Beitr. Phys. Atm., 1957)

The stream function and velocity potential follow from solutions of Poisson equations

\[ \nabla^2 \psi = \zeta = \mathbf{k} \cdot (\nabla \times \mathbf{v}) \]

\[ \nabla^2 \chi = \delta = \nabla \cdot \mathbf{v} \]
Illustration of Helmholtz decomposition

$u, v$: horizontal wind components in east and north directions

NAWDEX case: 12 UTC 13 Oct 2016, $p = 300$ hPa $\approx 9$ km height
\[ \delta = \nabla \cdot \nu \]

\text{divg\_300hPa}
\[ \zeta = k \cdot (\nabla \times v) = \frac{1}{R \cos \theta} \left( \frac{\partial v}{\partial \lambda} - \frac{u \cos \varphi}{\partial \varphi} \right) = \nabla^2 \psi \]

\[ \text{STR} = \psi / (\pi R) \]

Solution of Helmholtz equation after spherical harmonics decomposition (Swarztrauber et al. 1993) using SPHEREPACK (2011)
\[ u_R = -\frac{1}{R} \frac{\partial \psi}{\partial \phi'} \]
\[ u_D = \frac{1}{R \cos \varphi} \frac{\partial \chi}{\partial \lambda} \]

\text{uD}_\text{300hPa}

\text{uD}_\text{300hPa (m/s)}

\text{Data MIN = -10,6, MAX = 9,9, Mean = 0,5}
\[ n_R = \frac{1}{R \cos \varphi \partial \lambda} \partial \psi \]
\[ \nu_D = \frac{1}{R} \frac{\partial \chi}{\partial \varphi} \]
\[ \delta = \nabla \cdot \mathbf{v} = \frac{1}{R \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{R \cos \varphi} \frac{\partial (v \cos \varphi)}{\partial \varphi} = \nabla^2 \chi \]

divg 300hPa
\[
\frac{\partial w}{\partial z} = -\left(\frac{\partial u_D}{\partial x} + \frac{\partial v_D}{\partial y}\right) = -\delta, \quad w(z) - w(0) = -\int_0^z \delta \, dz
\]

Note: only the divergent velocity components determine vertical wind

\[w_{300hPa}\]
Exchange between potential and horizontal kinetic energy
Catalyzed by divergent horizontal velocity (w)

\[ v = k \times \nabla \psi + \nabla \chi \]

work of divergent motions against gradients of geopotential

(Wippermann, 1957; see also Lorenz 1960; Wiin-Nielsen, 1968; Chen and Wiin-Nielsen, 1976)
What do we know about horizontal w spectra from simulations from aircraft measurements?
Global numerical simulation spectra

-3 and -5/3-spectrum as in canonical spectrum near -2 and -5/3 divergent spectrum,

\[ d = \frac{E_d}{E_d + E_r} \approx 0.5 \]

at mesoscales different w-spectra in TRO and STR with unexplained maxima

(Skamarock et al., 2014)
COSMO-DE: 3 years of regional numerical weather prediction data

Data courtesy Tobias Selz

Spectrum only slightly different from canonical spectrum

-2 and -5/3 divergent spectrum, \( d \approx 0.5 \) at mesoscales

Flat w-spectrum with maximum possibly due to convection

(Selz, Bierdel, Craig, JAS, 2019)
DEEPWAVE (D, 2014) and NAWDEX (N, 2016) airborne measurements in the upper troposphere/lower stratosphere (UTLS), 180 and 107 legs, 2048 s each

Fritts et al. (2016), NSF-GV (HIAPER) NCAR EOL

Schäfler et al. (2018), HALO DLR-FX
How to explain the $w$ spectra?

Impact of vertical body motions on $w$ measurement?
Hypothesis:

The spectrum of vertical velocities $E_w$, as a function of wavenumber $k$ and height $h$, is related to the spectrum of horizontal velocities $E_h$

1) at large scales: to $E_d$ by continuity,
2) at small scales: to $E_h$ by dynamics towards local isotropy

$$E_w(k, h) = \frac{\alpha (h_{\text{eff}} k)^2}{\alpha + (h_{\text{eff}} k)^2} d \ E_h(k, h), \ h_{\text{eff}} = \beta h$$

$\alpha \approx 0.5$ for stratified turbulence
$\beta \approx 0.1$ to 0.5, depending on vertical variability of horizontal motions
$d \approx 0.5$, divergent energy fraction at mesoscales

For formal derivation, see J. Atmos Sci (2019)
A mesoscale maximum in the w-spectrum occurs if the divergent horizontal velocity spectrum $E_d$ has a slope flatter than $-2$.

$$E_w(k, h) = \frac{\alpha(h_{\text{eff}} k)^2}{\alpha+(h_{\text{eff}} k)^2} E_d(k, h), \quad h_{\text{eff}} = \beta h$$

$h=10 \text{ km}, \alpha = 1/2, \beta = 0.11, d=1/2$
Comparison of the model spectrum with

global simulations

aircraft measurements
Comparison of the **w-model** to the **global simulation w-spectra**

The proposed model is consistent with the MPAS results.

Some deviations were to be expected because of
- non-2d-isotropy
- surface orography
Again: Mean measured velocity spectra

How to explain the \( w \) spectra?

Impact of vertical body motions?
Mean measured and modelled velocity spectra

How to explain the spectra?

Impact of vertical body motions?

Model for w-spectrum
Comparison of measured and modeled $w$-spectra

Model and observations

similar at high and low $\varepsilon$ in both experiments

$h$ as measured $d=1/2, \beta=0.11, \alpha=1/2$
Ratio of measured and modeled w-spectra

About 30% deviations
Implications

• Transition scale $L$ increases for stronger $w$-variance
• Potential energy follows $w$ spectrum at small scales
• The scales carrying $w$-variance are of order height
A shift in the transition scale $L$ has strong impact on $w$-spectra.

This suggests dynamical importance of vertical motions for this slope transition.

\[ E_w(k, h) = \frac{\alpha (h_{\text{eff}} k)^2}{\alpha + (h_{\text{eff}} k)^2} E_d(k, h), \quad h_{\text{eff}} = \beta h \]
Schematic sketch of $E_w$ for two transition scales $L$

A shift in the transition scale $L$ has strong impact on $w$-spectra.

This suggests dynamical importance of vertical motions for this slope transition.

$$E_w(k, h) = \frac{\alpha(h_{\text{eff}} k)^2}{\alpha + (h_{\text{eff}} k)^2} E_d(k, h), \ h_{\text{eff}} = \beta h$$
Measured and **modeled** w-spectra for the longest legs

\[ <w'^2> = \]

\[ 0.024 \text{ m}^2 \text{s}^{-2} \]

L = 200 km

\[ 0.068 \text{ m}^2 \text{s}^{-2} \text{L} \]

L > 1000 km
Temperature spectra are similar to wind spectra.

Spectrum of potential energy

\[ E_p(k) = \frac{1}{2} \frac{g^2}{N^2 T_0^2} E_T(k) \]

\[ \frac{1}{2} \frac{g^2}{N^2 T_0^2} \approx (2 \text{ to } 10) \frac{m^2}{s^2 K^2} \]

\[ E_p(k) \approx E_u(k) \approx E_v(k) \]

\( w \) converts kinetic into potential energy and backward.

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**Fig. 3.** Variance power spectra of wind and potential temperature near the tropopause from GASP aircraft data. The spectra for meridional wind and temperature are shifted one and two decades to the right, respectively; lines with slopes \(-3\) and \(-\frac{7}{3}\) are entered at the same relative coordinates for each variable for comparison.
Ratio of kinetic to potential energies
Comparisons with model spectra reflect contributions from quasi-geostrophic motions and gravity waves

Red and blue lines are models derived from linear gravity theory (e.g., Fritts and Alexander, 2003, Geller and Gong, 2010) for given $E_h$ using our model for $E_w/E_h$.

Consistent with gravity wave theory:
$E_w < E_p < E_h$
for
$f < \omega < N$

But gravity wave models alone underestimate $E_h/E_p$ and $E_w/E_p$. 
Which scales contribute most to w-variance?

0.5 to 80 km: 90%
7 to 17 km: 50%
Key Points

- Spectra of vertical and divergent horizontal velocity are connected kinematically at large scales and dynamically at small scales.
- Model is consistent with global (and regional) models and flight experiments within ~30%.
- Maxima in observed vertical velocity spectra near 4 to 15 km wavelengths occur together with flat divergent horizontal spectra.
- Enhanced vertical motions shift the transition scale L between the -3 and a flatter (~-5/3) spectrum in the canonical spectrum to larger scales.
Enjoy
(e)motions
Measured velocity spectra, mean over all legs