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Poisson's Summation Formula in Radar Imaging

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Abstract

Poisson's Summation Formula (PSF) has numerous applications, not only in radar imaging. It is the bridge between continuous (infinite integrals) and discrete settings (finite sums) and, hence, links the Fourier transform, Fourier series, the Discrete-Time Fourier Transform (DTFT) and the Discrete Fourier Transform (DFT) to one another. Most importantly, however, is the fact that Poisson's Summation Formula is the actual core statement hidden in *any* sampling theorem. In this paper, we show that two operations, *discretization* and *periodization*, are needed to come from infinite integrals to finite sums. These operations are moreover involved in *any* imaging process, in particular, in radar imaging. We use two operations, $comb$ and rep, to better understand Poisson's Summation Formula, both introduced by Woodward who is also known for introducing the $sinc$ function and *Woodward's ambiguity* function. With these operations applied to functions it can be shown that "sampling a function with 1/T" means to "periodize its Fourier transform with T" and, vice versa, "periodizing a function with T" means to "sample its Fourier transform with 1/T". This simple statement turns out to be both, the sampling theorem and Poisson's summation formula. However, there are functions which cannot be sampled and functions which cannot be periodized. Constant functions or constantly growing functions, for example, can be sampled but cannot be periodized. Vice versa, a Dirac delta can be periodized, it yields a Dirac comb, but cannot be sampled. So, another outcome in this paper is an easy-to-apply rule for distinguishing those functions which can be sampled *or* periodized or sampled *and* periodized and those which cannot.

1 Introduction

The fact that Poisson's Summation Formula is not even mentioned in many textbooks on Fourier Analysis or Signal Processing is an indication for the fact that it is not yet fully understood today. However, Poisson's summation formula (PSF) is most important, not only in engineering, in quantum mechanics or in mathematics in general because it bridges the gap between calculus (integration vs. differentiation) and linear algebra (sums vs. differences). It is therewith the actual bridge between our natural world, which appears to be smooth, and the world in our computers, which appears to be discrete. The Whittaker-Kotel'nikov-Shannon sampling theorem is just one expression of the Poisson Summation Formula [1] but there are many more. The actual difficulty to understand versions of the Poisson Summation Formula arises from at least three circumstances. First, it does not describe, as formulas usually do, an equivalence between function *values* or, more generally, between *functions* but an equivalence between *operations* on functions. These operations are discretization (sampling) and periodization (replicating). The second difficulty is the fact that this equivalence is still true on functions which do not even have any function values, such as the "Dirac delta" function. It is a so-called "generalized function". Now, the third difficulty is the fact that "generalized functions" are not yet fully understood today. An open question is, for example, whether a Dirac delta can be squared and if it can be squared then in what sense can it be squared? It is clear, on one hand, a square of the Dirac delta does not exist in a sense we originally expected it to exist. On the other hand, its existence is *required*, in some sense, from a physical point of view and, after all, also from a pure mathematical point of view. However, the treatment of generalized functions in this paper is based on a rigorous mathematical approach, called the theory of generalized functions. It allows, for example, using an integral, say the "Dirac integral", that is wider than both Riemann's and Lebesgue's integral [2, 3]. Using it, the Dirac delta integrated yields 1 instead of zero [4] and this, in turn, is needed in order to explain *sampling theory*. It also means that the Fourier transform preserves a *volume of unity*, also on generalized functions. We proceed as follows. Section 2 introduces to the Poisson Summation Formula, Section 3 explains "reciprocity", Section 4 reviews "imaging" which is part of the applications explained in Section 5. Section 6, finally, concludes this study.

2 Poisson's Summation Formula

The simplest form of Poisson's Summation Formula is [5]

$$\sum_{k=-\infty}^{+\infty} g(k) = \sum_{m=-\infty}^{+\infty} \hat{g}(m) \tag{1}$$

where k and m are integer positions on the real axis in time and frequency domain and g and $\hat{g}$ are functions on the real axis in time and frequency domain—they form a Fourier transform pair. The PSF expresses an "energy balance" or "volume balance" between time and frequency domain. Applying the Fourier transform rule $\mathcal{F}(g(xT)) = 1/T\,\mathcal{F}(g(y/T))$, note this is a volume preservation rule, to the simplest appearance of Poisson's Summation Formula yields

$$\sum_{k=-\infty}^{+\infty} g(kT) = 1/T \sum_{m=-\infty}^{+\infty} \hat{g}(m/T) \qquad (2)$$

which means that one function stretched means its counterpart is compressed. So, clearly, one summation is now *faster* than the other, although both yield the same value. One may observe, the left-hand side, multiplied by T, is approximately equal to the Riemann integral of $g(t)$ when T is small (but greater than zero), see [10], p.36. This property is a well-appreciated feature of Poisson's Summation Formula. Applying now the rule $\mathcal{F}(g(t+x)) = (\mathcal{F}g)(y)\,e^{2\pi ity}$, which is a "volume displacement", yields

$$\sum_{k=-\infty}^{+\infty} g(t+kT) = 1/T \sum_{m=-\infty}^{+\infty} \hat{g}(m/T)\,e^{2\pi it(m/T)} \qquad (3)$$

a periodic function on the left and its *Fourier series* on the right. It is clear that (1), (2), (3) are fully equivalent to one another. Putting $t = 0$ in (3) and $T = 1$ in (2) we come back to (1). Expressed in operations, (3) means we have

$$\triangle\!\!\triangle\!\!\triangle_T g \stackrel{\text{def}}{=} \mathrm{III}_{\mathrm{T}} * g \qquad (4)$$

periodization applied to g on the left and

$$\perp\!\!\perp\!\!\perp_T g \stackrel{\text{def}}{=} \mathrm{III}_{\mathrm{T}} \cdot g \qquad (5)$$

discretization applied to $\hat{g}$ on the right followed by an inverse Fourier transform where

$$e^{2\pi it(m/T)} = \mathcal{F}^{-1}\{\delta(f - \frac{m}{T})\}$$

denote pure frequencies. The symbol

$$\mathrm{III}_{\mathrm{T}}(t) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{+\infty} \delta(t-kT) \qquad (6)$$

denotes a Dirac comb, k are integers and $\delta(t-kT)$ denote Dirac impulses at kT, the symbol $*$ is convolution and $\cdot$ is multiplication. More explicitly,

$$(\triangle\!\!\triangle\!\!\triangle_T g)(t) = \sum_{k=-\infty}^{+\infty} g(t+kT) \qquad (7)$$

is a generalized periodic function and

$$(\perp\!\!\perp\!\!\perp_T g)(t) = \sum_{k=-\infty}^{+\infty} g(kT)\,\delta(t-kT) \qquad (8)$$

is a generalized discrete function. The dependence of a generalized function g on t is always meant *symbolically* and *does not mean* that the value $g(t)$ exists at t. Coefficients $g(kT)$ of discrete functions, as in (8), are usually denoted in finite or infinite sequences $[.., g(-1), g(0), g(1), ..]$. If the summation is finite, they become N-tuples $[g(0), g(1), .., g(N-1)]$ where N is a positive integer greater or equal to 1. The case $N = 1$ is the so-called Gabor-limit. In contrast, n in $t, T, N \in \mathbb{R}^n$ is the dimension of the space. Hence, $n = 2$ means we have matricies (e.g. digital images) and n greater than two means we have tensors, e.g. stacks of images, $n = 3$ in SAR tomography. Everything done in this paper, extends to n dimensions straight forwardly, see e.g. **Figure 5** where $T = [T_1, T_2]$, $B = [B_1, B_2]$ and $N = [N_1, N_2]$. Using these definitions, variant (3) of the Poisson Summation Formula reduces to the much simpler form

$$\triangle\!\!\triangle\!\!\triangle_T g = \frac{1}{T}\,\mathcal{F}^{-1}(\perp\!\!\perp\!\!\perp_{\frac{1}{T}}(\mathcal{F}g)). \qquad (9)$$

Thus, "any periodic function (left) equals its Fourier series (right)". Any Fourier series, in turn, equals an "inverse Fourier transform applied to sampled frequencies". We also see in (9) that periodizing g with T, means to sample its spectrum with $1/T$. Fourier transforming both sides of (9) yields the Fourier transform rule

$$\mathcal{F}(\triangle\!\!\triangle\!\!\triangle_T g) = \frac{1}{T} \perp\!\!\perp\!\!\perp_{\frac{1}{T}}(\mathcal{F}g) \qquad (10)$$

and swapping the roles of g and $\hat{g}$ in (2) and (3) yields

$$\mathcal{F}(\perp\!\!\perp\!\!\perp_{\frac{1}{T}} g) = T\,\triangle\!\!\triangle\!\!\triangle_T(\mathcal{F}g) \qquad (11)$$

which is known as the "dual" of Poisson's Summation Formula [6, 7, 8]. The two scaling factors $1/T$ and T in front of the operators in (10) and (11) balance stretchings and squeezings applied to ordinary or generalized functions such that their overall volume is maintained during the Fourier transform. We only assume positive T here, which is no restriction of generality. In higher dimensional spaces ($n > 1$), they become $1/T \equiv (1/T_1)(1/T_2)\dots(1/T_n)$ and $T \equiv T_1 T_2 \dots T_n$, respectively, which means they collect squeezing factors from all dimensions [8]. Now, if $T = 1$, we have the following beautiful, fully symmetric variants

$$\mathcal{F}(\triangle\!\!\triangle\!\!\triangle g) = \perp\!\!\perp\!\!\perp(\mathcal{F}g) \qquad (12)$$

$$\mathcal{F}(\perp\!\!\perp\!\!\perp g) = \triangle\!\!\triangle\!\!\triangle(\mathcal{F}g) \qquad (13)$$

of Poisson's Summation Formula. They are at least known since 1953, found by the British mathematician and radar pioneer P.M. Woodward (1953), who used them to describe radar imaging [10]. The only difference is that he used symbols *comb* and *rep* instead of $\perp\!\!\perp\!\!\perp$ and $\triangle\!\!\triangle\!\!\triangle$, respectively. However, their applications go far beyond radar applications and it is also known that they *cannot be true* in general. This is because there are functions which cannot be periodized (e.g. 1) and functions which cannot be sampled (e.g. δ). Note that 1 is already a periodic function and δ is already a discrete function. The actual *range of validity* of Poisson's Summation Formula is, in fact, a mystery until today. Knowing this, it led Woodward to write (12) and (13) are "known by heart by most circuit mathematicians" [10]. However, in 1950/51, a new theory was born, the theory of "generalized functions" [11, 12]. It puts us in the position to prove his formulas today including the finding of a simple rule for the validity of Poisson's Summation Formula [8].

3 Reciprocity

3.1 Fourier Transform

A prerequisite for understanding the Poisson Summation Formula is to know that it depends on the *definition* of the Fourier transform. There is, however, only one definition which is the correct one [13]. All other definitions disregard the requirement that a volume of 1 must be maintained in any, forward or backward, Fourier transform. We do therefore not accept any Fourier transform that uses scaling factors, such as 2π, $\sqrt{2\pi}$, $(2\pi)^{-1}$, in order to compensate misbalances between time and frequency domain. What we need instead is $\mathcal{F}1 = \delta$ and $\mathcal{F}\delta = 1$ because the volume of 1 is trivially 1 and the volume of δ is 1 according to Paul Dirac's definition [4]. The only Fourier transform that guarantees this is the so-called "normalized" Fourier transform given by

$$(\mathcal{F}g)(f) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} g(t)\, e^{-2\pi i\, f t}\, dt \tag{14}$$

and

$$g(t) \;=\; \int_{-\infty}^{+\infty} (\mathcal{F}g)(f)\, e^{\,2\pi i\, t f}\, df \tag{15}$$

is its inverse. Using this definition, we exploit the fact that $e^{2\pi i f t}$ are (periodic) *functions* and (discrete) *numbers*, simultaneously [14]. They project the space of complex numbers $\mathbb{C}$ onto two real axes $\mathbb{R}$, time and frequency, and integers $\mathbb{Z}$ mark their "full revolutions" around 1.

3.2 Complementary Variables

Choosing this Fourier transform, PSF variants (12), (13) appear fully *scalar-factor free*, this is not the case using other Fourier transform definitions, and (10), (11) merely express that time t and frequency $f \stackrel{\text{def}}{=} 1/t$ are complementary variables in

$$tf = 1$$

which expresses their *reciprocity*, the same as "time span" T and "frequency span" $B \stackrel{\text{def}}{=} 1/T$ are complementary in $TB = 1$. The latter is known as "time-frequency product" in engineering [15]. Below we will see $TB = 1$ denotes the time-frequency "continuum" and $TB = N$ denotes its discretization. The fact that a continuum cannot be smaller than 1 is known as Gabor's limit and the fact that T and B cannot both be small is Heisenberg's uncertainty principle.

4 Imaging

4.1 Ideal Imaging

Using an ideal imaging system, we may either send an infinitely extended ($T = \infty$) plane wave 1 towards the scenery ψ being imaged such that

$$1 \cdot \psi \;=\; \psi \tag{16}$$

is the outcome (**Figure 1**), in this way we imaged all values of ψ *simultaneously*, or we may send an infinitely fine ($T = 0$) pulse δ in order to scan all values of ψ, one after another, and

$$\delta * \psi \;=\; \psi \tag{17}$$

is the outcome (**Figure 2**) such that we imaged all values of ψ *sequentially*. So, multiplication $\cdot$ means "simultaneously" and convolution means "sequentially". In both cases, our imaging system must be capable to acquire, process and store infinitely extended, infinitely resolved data.

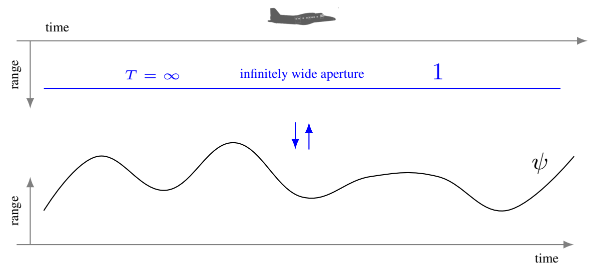

Figure 1 Ideal imaging, simultaneously ($T = \infty$)

However, imaging in generalized functions theory is nothing else than applying *operators* to *functions*. Here, we applied $1\cdot$ and $\delta *$ to the scene being imaged ψ. Note that in (4) and (5), we already encountered operations $\text{III}\cdot$ and $\text{III}*$ applied to functions. It turns out that $\delta *$ and $1\cdot$ represent *ideal* imaging (*no* information loss) while $\text{III}*$ and $\text{III}\cdot$ represent *real* imaging (*including* an information loss). They limit two things, the spatial extent (time span) and the spatial resolution (frequency span), respectively. *This is the connection of imaging to Poisson's Summation Formula.*

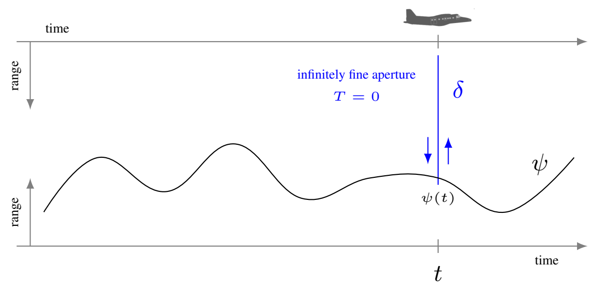

Figure 2 Ideal imaging, sequentially ($T = 0$)

Let us now denote δ_t the sensor δ positioned at t then

$$\langle \delta_t, \psi \rangle \;=\; \psi(t) \tag{18}$$

describes the acquisition of "one pixel" in equation (17) where $*$ expresses a motion of δ with respect to ψ. We may think of $\langle \cdot, \cdot \rangle$ as a generalized integration of ψ over one pixel which would be zero according to Riemann and Lebesgue. An alternative notation of (18) is

$$\delta_t \cdot \psi \;=\; \psi(t) \tag{19}$$

which means $\delta_t\cdot$ applied to ψ yields one pixel. Note that every t is infinitely precise ($T = 0$), hence, the system bandwidth is $1/T = \infty$. It means, for example, if $t = \pi$, we know *all* decimal places of π. This, of course, is impossible in real systems where numbers t are *finitely* precise.

4.2 Real Imaging

4.2.1 Real Aperture Radar

In practice, an ideal imaging system δ is replaced by a real imaging system ϕ. It will be such that (17) is replaced by

$$\phi * \psi = \bar{\psi} \tag{20}$$

where every pixel is now blurred by ϕ (**Figure 3**). The function ϕ is called *point spread function* or *system transfer function*. It is important to see that ϕ, in contrast to δ, introduces the "width of one pixel" [14] sized $1/B$, into an image $\bar{\psi}$ taken of ψ. Because $1/B$ is now the lowest possible wavelength in time domain, it follows that B is the largest possible frequency (band-limit).

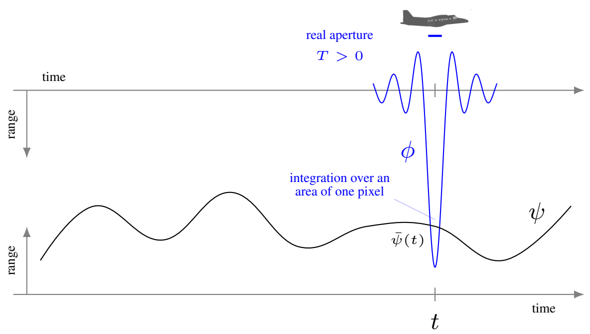

Figure 3 Real aperture radar imaging

In contrast to (18), which is an infinitely exact value, one "pixel" has now the meaning of an *average* over one resolution cell, depicted in **Figure 3**, as

$$\langle \phi_t, \psi \rangle = \int_{-\infty}^{\infty} \phi(\tau - t)\, \psi(\tau)\, d\tau = \bar{\psi}(t) \tag{21}$$

given by the "footprint" of ϕ_t. In this way, the total volume (integral) of ψ is *maintained* in any data acquisition although we acquire $\bar{\psi}$ instead of ψ. We actually "pack" the energy of each resolution cell into the value of an acquired pixel. This process of energy packing is actually called *quantization* in quantum physics. Because resolution cells are of size $\boldsymbol{\tau} = 1/B$, we are now free to pick pixels at a maximum distance of the size of one resolution cell which amounts to a quantization

$$t_k = \boldsymbol{\tau}\, k \tag{22}$$

in time t where $0 < \boldsymbol{\tau} < 1/B$ is one quant. The condition to pick pixels at distances which are no longer than one resolution cell is known as the *Nyquist criterion* or *Raabe's condition*. However, this condition, also known as the *sampling theorem*, must be obeyed *twice* in any imaging process. We need it another time when the infinite sequence of acquired values

$$\ldots,\ \bar{\psi}(-t_2),\ \bar{\psi}(-t_1),\ \bar{\psi}(t_0),\ \bar{\psi}(t_1),\ \bar{\psi}(t_2),\ \ldots \tag{23}$$

is cut down to only *finitely* many values (see **Figure 5**)

$$\bar{\psi}(t_0),\ \bar{\psi}(t_1),\ \bar{\psi}(t_2),\ \ldots,\ \bar{\psi}(t_{N-1}) \tag{24}$$

where N needs to be chosen *sufficiently* large. One may recall, periodization (finitization) in time is discretization (sampling) in frequency domain, and vice versa [8, 9].

4.2.2 Double-sided Sampling Theorem

While the distance between pixels $1/B$ stands for the lowest possible wavelength, it turns out that $T = (1/B)N$ stands for the largest possible wavelength in acquired data. In Fourier transformed data it is the other way around, while $1/T$ stands for the lowest possible wavelength, $B = (1/T)N$ stands for the largest possible wavelength. Note that this is the same N. Both conditions together now, form a double-sided time-frequency sampling theorem

TB = N

where TB, or N respectively, must be chosen larger than the time-frequency product of the point spread function ϕ. Choosing larger N introduces redundancy, it is known as *oversampling* or *zero padding*, choosing smaller N introduces an information loss, known as *aliasing*. It should be mentioned, however, that both, oversampling and aliasing, preserve the total signal energy. So, N is a measure of energy concentration. If $N = 1$, then this is Gabor's limit. It means that all energy is "packed" onto just one value (pixel), a total average of the entire space (image), and if N gets larger and larger then this means to "unpack" energy, to spread it across a larger and larger space and to allow it to be finer and finer structured [17]. If N is large, it looks like it's going to be a continuum again.

4.2.3 Synthetic Aperture Radar

Because the "footprint" of ϕ in (21) is usually very small, it is dominated by noise. A trick in Synthetic Aperture Radar (SAR) imaging, see **Figure 4**, is therefore to increase the footprint of ϕ and therewith to raise the useful signal, via averaging, well above the imaging system noise level.

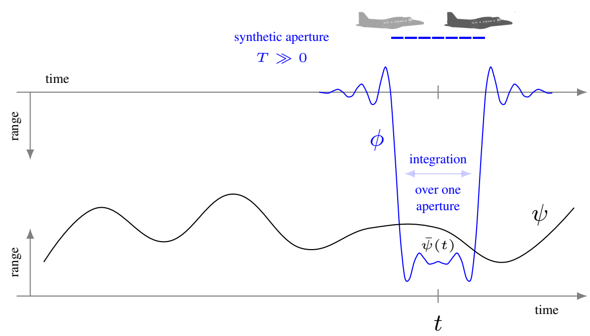

Figure 4 Synthetic aperture radar imaging

The "double-sided" sampling theorem described above is now nothing else than acquiring data following, first, the classical sampling theorem and, second, to acquire images no shorter than $T_1 = (1/B_1)N_1$ and $T_2 = (1/B_2)N_2$ one "footprint" of the used point spread function ϕ. Here, 1 means flight direction (azimuth) and 2 means radar range direction, $1/B_1$ and $1/B_2$ being the sampling rates in azimuth and range, respectively.

4.3 Poisson's Summation Formula

In generalized functions theory, (23) is the function

$$\ldots + \bar{\psi}(-t_1)\, \delta_{-t_1} + \bar{\psi}(t_0)\, \delta_{t_0} + \bar{\psi}(t_1)\, \delta_{t_1} + \ldots \tag{25}$$

denoted as $(1/B)$-*discrete* function $Ш_{\frac{1}{B}}\bar{\psi}$ and (24) is

$$\bar{\psi}(0)\,\delta_0 \;+\; \bar{\psi}(t_1)\,\delta_{t_1} \;+\; \ldots \;+\; \bar{\psi}(t_{N-1})\,\delta_{t_{N-1}} \qquad (26)$$

denoted as T-periodic discrete function $\Delta\!\!\Delta\!\!\Delta_T Ш_{\frac{1}{B}}\bar{\psi}$ where $T = (1/B)N$. Fourier transformed, (25) is

$$\mathcal{F}\Big(\sum_{k=-\infty}^{+\infty} \bar{\psi}(t_k)\,\delta_{t_k}\Big) \;=\; \sum_{k=-\infty}^{+\infty} \bar{\psi}(t_k)\,e^{-2\pi i\,t_k f} \qquad (27)$$

an *infinite* Fourier series (periodic function) and (26) is

$$\mathcal{F}\Big(\sum_{k=0}^{N-1} \bar{\psi}(t_k)\,\delta_{t_k}\Big) \;=\; \sum_{k=0}^{N-1} \bar{\psi}(t_k)\,e^{-2\pi i\,t_k f} \qquad (28)$$

a *finite* Fourier series (discrete periodic function), respectively. Equation (27) is moreover known as the Discrete-Time Fourier Transform (DTFT) and (28) is known as the Discrete Fourier Transform (DFT). A more detailed derivation of these relationships is given in [9].

4.3.1 Nested PSF Variants

In order to see that the DFT is (11) nested into (10) or (10) nested into (11), we recall that $B \overset{\text{def}}{=} 1/T$ in (11) and let $g = Ш_{\frac{1}{B}}\bar{\psi}$ be a function that has already been sampled. Inserting it into (10) yields

$$\mathcal{F}(\Delta\!\!\Delta\!\!\Delta_T Ш_{\frac{1}{B}}\bar{\psi}) = \frac{B}{T}\; Ш_{\frac{1}{T}}\Delta\!\!\Delta\!\!\Delta_B(\mathcal{F}\bar{\psi}). \qquad (29)$$

Vice versa, let $g = \Delta\!\!\Delta\!\!\Delta_B\bar{\psi}$ be a function that has already been truncated. Inserting it into (11) yields the dual

$$\mathcal{F}(Ш_{\frac{1}{T}}\Delta\!\!\Delta\!\!\Delta_B\bar{\psi}) = \frac{T}{B}\; \Delta\!\!\Delta\!\!\Delta_T Ш_{\frac{1}{B}}(\mathcal{F}\bar{\psi}) \qquad (30)$$

with respect to (29). Without restriction of generality [9], one may either set $T = 1$ or $B = 1$ such that $N = 1 \times N$ and $1 = (1/N) \times N$ yield the two standard decompositions of the DFT where sampling rates of 1 and $1/N$ are used.

4.3.2 Validity of PSF Variants

It remains to explain the *validity* of Poisson's Summation Formula. Above, we have seen that (1), (2) and (3) are fully equivalent. This is because, in one direction they are particular cases $t = 0$ and $T = 1$ of one another and, vice versa, we may derive them via the Fourier transform rules of *translation* and *dilation*, respectively. We have moreover then seen that (3) is nothing else than (10) and (11) which involve operations of III· and III∗ applied to functions. In other words, the validity of Poisson's Summation Formula hinges on the question *when* and *when not* we are allowed to apply III to a function, either via multiplication or via convolution. A theorem of Larent Schwartz [11] helps, it states that

- functions whose Fourier transform vanish at infinity may be sampled,
- functions which vanish at infinity may be periodized,
- functions which fulfill both conditions may be sampled *and* periodized

where *vanish at infinity* means that $\psi(t)$ is zero as t tends to be $-\infty$ or $+\infty$. For further details, one may refer to Lemma 1 and Lemma 2 in [14]. Functions which are known to fulfill both conditions are *Schwartz functions*. This is why Schwartz functions play a special role in generalized functions theory, to define the Fourier transform on tempered distributions [9], and in quantum mechanics [18] to define operators applied to (local) *wave functions*.

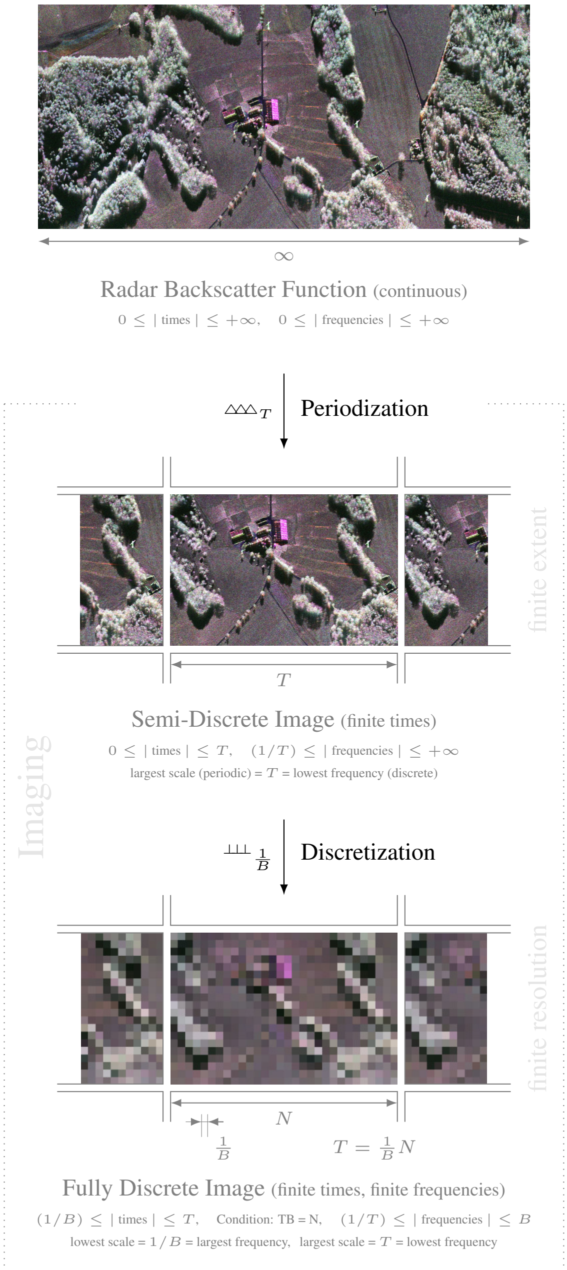

Figure 5 Imaging is discretization applied after periodization or, vice versa.

5 Applications

5.1 Imaging

A most important application of Poisson's Summation Formula is its role in *any* imaging process. Here, we have two effects, we limit, first, the spatial extent of the imaged scene and second, the spatial resolution of the imaged scene. See **Figure 5** for an example. It is clear that this does even apply to *analogue* imaging techniques, for example, whenever we take a photo.

5.2 Measuring

More generally, Poisson's Summation Formula is involved in any *measuring* process in physics. This is because measuring devices can only capture finitely extended, finitely resolved data. Whenever we acquire numbers in any way, we usually chop decimal places. This, in turn, determines their "resolution". It determines how small acquired numbers can become. Beside that, acquired numbers cannot be infinitely large, so there will be a largest number such that every acquired number is a number modulo that number.

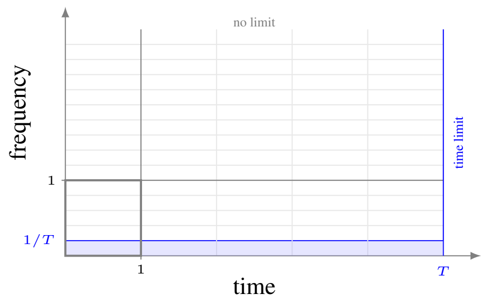

Figure 6 Periodization in time implies discretization in frequency domain, because TB = 1 where B=1/T.

The phenomenon of having a limit in one direction, say time, is depicted in **Figure 6**. Here, time has the effect of repeating itself as soon as T is exceeded. This in turn has the effect of forming frequency "quants", sized $1/T$. In other words, every frequency is now an integral multiple of $1/T$. The deeper reason is that a *volume of unity* is maintained by $(1/T) \times T = 1$, the "continuum", which may be understood as a generalization of Parseval's theorem on Fourier series and Plancherel's theorem on Fourier transforms [16].

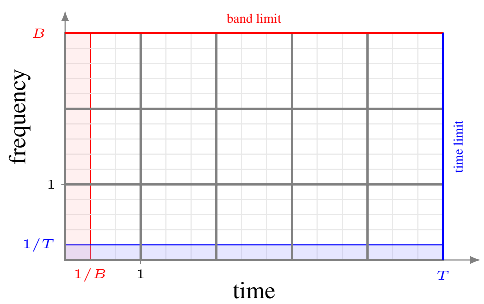

Figure 7 Time-Frequency discretization, TB = N is an integer such that (1/B) N = T and (1/T) N = B

Introducing now an upper limit in frequency direction, see **Figure 7**, eventually forces the time-frequency plane to become periodic in both directions. Periodicity, in turn, means discreteness in its dual domain. So, if both domains are periodic they are both discrete.

6 Conclusions

Poisson's Summation Formula plays a very important role, not only in radar imaging. Understanding it, is the key to many different phenomena, including the effect of quantization in physics, Gabor's limit and Heisenberg's uncertainty principle.

7 Literature

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