Global Asymptotic Stabilization with Smooth High-gain/Low-gain Transitions: AVA - Adaptive Variance Algorithm

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Abstract: This paper presents a state-feedback algorithm with adaptive gains, designed to solve the typical gain tuning trade-off between accurate tracking in a neighborhood of the working points and large control inputs far from their proximity. The main idea is to use a Gaussian function to specify a "trust" region around the working point. For values outside this region, the gain decays exponentially and therefore the actuation input is limited. On the other hand, the variance of the Gaussian is constantly adapted, so that the attractive region around the working point will expand and eventually allow the convergence to the desired value. The stability of the algorithm is analyzed and simulations are used to validate the theoretical results.

Keywords: Dynamic State Feedback; Adaptive Gains; Global Asymptotic Stability.

1. INTRODUCTION

A proportional state-feedback action is one of the most fundamental and ubiquitous control scheme used in industry and research. In the robotics community, for example, many different and sophisticated control schemes are in some sense related to a classic proportional-differential (PD) control law, i.e., to a proportional feedback of the robot state. This is the case in the seminal work of Takegaki and Arimoto (1981) and many of the schemes in Siciliano et al. (2008). A typical trade-off that control engineers are faced with is to design the control gains leading to a good compromise between tracking performance and noise rejection. Moreover, the gains are typically tuned to guarantee the desired performance in a neighborhood of the working point and configurations starting far away from it could often lead to a control variable that saturates the actuators. This issue is so pressing in applications, that many works can be found in the robotics literature, like Burkov and Freidovich (1997); Kelly et al. (1997); Dixon et al. (1999); Zavala-Rio and Santibanez (2007) or the results of Teel (1996) in the more general contest of input-to-state stability (ISS). All of these references analyze the presence of saturation in the control scheme, which can otherwise lead to undesirable effects in the closed loop.

A relative common procedure is also to interpolate a smooth trajectory between the starting and final points, that the controller is later required to track. This is typically realized using some kind of low-pass filter, similar to those in Farrell et al. (2009). The drawback of such approach is that the interpolation requires additional tuning and might decrease the performance of the system when tracking rapid trajectories. In the worst scenario, one ends up tuning the parameters of the low-pass filter depending on each specific desired behavior of the system. Most importantly, the low-pass filter will affect the behavior of the system both in the proximity of the working point and far away from it, even though it is not necessary (and actually undesirable) to modify the behavior of the system near the equilibrium point. In other words, only when large control errors are amplified by the controller gains, the behavior should be altered.

The main contribution of this paper is the design of a dynamic state feedback, which allows smooth transitions between a high-gain behavior in a neighborhood of the working point and a low-gain behavior when the system is far from it. This is possible by using state-dependent gains rather then a more classic constant value. The nominal constant gain is multiplied by an adjustable windowing function, which adapts the value of the gain in an expanding or contracting neighborhood of the equilibrium point. As the windowing effect is realized by means of a not-normalized Gaussian function with an adaptive variance, the control algorithm is referred to as *Adaptive Variance Algorithm* (AVA).

Varying gains is at the basic of gain-scheduling approaches surveyed in Leith and Leithead (2000). The design principle in gain scheduling is typically to use different gains to decompose the nonlinear design task in a number of linear sub-tasks. Adaptation is the key feature of adaptive control schemes like those in Sastry and Bodson (1994). Compared to those, the goal here is not to guarantee the control specifications despite varying or initially uncertain parameters, but rather to adapt the behavior of the control variable depending on a measure of the distance between the system state and the final goal. Unlike all previously mentioned works on saturated PD schemes, the proposed control law does not aim at guaranteeing that the hardware limitations are not exceeded, but rather to automatically adjust the gains depending on the error. This will reduce, as byproduct, the likelihood of saturation. On the other hand, unlike those schemes which have predefined regions where the saturation acts, the adaptive nature of AVA allows the controller to automatically change its behavior. To visualize this phenomenon, imagine having a saturated proportional action in a mechanical system. The local stiffness for the saturated scheme would always be (nearly) zero far from the equilibrium, while with AVA the stiffness is allowed to increase if it is within the limitations of the system. Related to the schemes with bounded output is also Model Predictive Control, thanks to its ability to satisfy a set of constraints, see Allgöwer et al. (1999). Nevertheless, the drawback is the necessity of repeatedly optimizing a possibly complex problem and stability cannot be always easily guaranteed.

The paper is organized as follows. Section 2 presents the main idea of the paper in the case of two very simple systems. In Section 3, the control objective is described and the results are formalized in a theorem, which represents the main result of the paper. The behavior of the control laws is shown with simulations in Section 4. Additionally, practical considerations and a discussion about the role of the parameters are considered in that section. Finally, Section 5 summarizes the work and points to possible extensions of the proposed method.

1.1 Preliminaries

The usual Euclidean norm will be denoted by $|\cdot|$.

A function $\gamma : [0, \infty) \to [0, \infty)$ is a class $\mathcal{K}$ function if it is continuous, strictly increasing and $\gamma(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if in addition [1] $\gamma(r) \to \infty$ as $r \to \infty$. A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a class $\mathcal{KL}$ function if given $\beta(r, t)$ then for each fixed $t \geq 0$, $\beta(\cdot, t)$ is a class $\mathcal{K}_\infty$ function and for each fixed $r \geq 0$, $\beta(r, \cdot)$ is decreasing to zero as $t \to \infty$.

Local and global asymptotic stability will be denoted in short as LAS and GAS, respectively. By 0-GAS it is meant that the system $\dot{x} = f(x, 0)$ is GAS. This is the unforced system associated to the system with inputs $\dot{x} = f(x, u)$. Finally, ISS denotes input-to-state stability. When not specifically stated, these properties are always meant to hold for the origin of the state space.

2. SIMPLE CANONICAL EXAMPLES

In this section, the main idea of the control law is derived using a single and double integrator as canonical examples of first and second-order control systems, respectively. In the remainder, it will be considered (without loss of generality) the regulation to the origin of the state space.

2.1 Single integrator

Consider the single input integrator with state $\xi_1 \in \mathbb{R}$

$$\dot{\xi}_1 = v \ , \tag{1}$$

where $v \in \mathbb{R}$ is the control input to be chosen such that $\xi_1 \to 0$ as $t \to \infty$. Clearly, $v = -k_1\xi_1$, with $k_1 > 0$ achieves the goal and renders the system GAS. The following theorem states that the same can be guaranteed also with a dynamic state feedback, which modifies the gain by an adaptive windowing function.

[1] The class $\mathcal{K}_\infty$ notation is often used also for functions that are defined only on bounded intervals $[0, r]$. In this case, the function is bounded, but it can always be extended to a $\mathcal{K}_\infty$ function on $[0, \infty)$.

Theorem 1. For the integrator system (1), *the dynamic state feedback with internal state* $\sigma \in \mathbb{R}$

$$v = -k_1 e^{-V(\xi_1,\sigma)}\xi_1 \tag{2a}$$

$$\dot{\sigma} = \big(k_3 - k_2 e^{-V(\xi_1,\sigma)}\big)\sigma \ , \tag{2b}$$

where, given the constant $\bar{\sigma} > 0$, $V : \mathbb{R}^2 \to \mathbb{R}$ *is defined as*

$$V(\xi_1, \sigma) = \frac{\xi_1^2}{2(\bar{\sigma}^2 + \sigma^2)} \tag{3}$$

leads to a GAS closed-loop system, provided that the gains k_i, *with* $i \in \{1, 2, 3\}$, *satisfy the inequalities* $k_1 > 0$ *and* $k_2 > k_3 > 0$.

Proof. See Appendix B.

Given the expression of $V(\xi_1, \sigma)$, the term $e^{-V(\xi_1,\sigma)}$ is a not-normalized unimodal Gaussian function with zero mean and variance $\bar{\sigma}^2 + \sigma^2$. The idea behind (2) is to use a smooth windowing function to reduce the control effort far from the equilibrium point, i.e., for large control errors. Additionally, the internal state of the controller evolves to automatically adapt the windowing size and still yield a GAS system. The condition $k_2 > k_3 > 0$ intuitively guarantees that the windowing size starts shrinking back to its original value as the system approaches the goal. Although σ will eventually converge to zero, the dynamic in (2b) implies an exponential growth of σ for values that are far from $\xi_1 = 0$. Here by "far" it is understood values outside the region defined by the variance of the Gaussian. Given an initial condition $\xi_{1,0}$ far from the origin, then $k_1 e^{-V(\xi_1,\sigma)}$ will be initially small and, as σ increases, it will tend to the nominal value k_1. On the other hand, the controller will always behave almost as a static state feedback within a neighborhood of the origin. The amplitude of such neighborhood is adjustable via $\bar{\sigma}$.

Remark: The function $V(\xi_1, \sigma)$ could be used as a semidefinite Lyapunov function as in Iggidr et al. (1996), if additionally $k_1 \geq k_2$. Computing the derivative of $V(\xi_1, \sigma)$ along the flow of the closed-loop system leads to

$$\dot{V} = -2\big[\big(k_1 - k_2\alpha(\sigma)\big)e^{-V(\xi_1,\sigma)} + k_3\alpha(\sigma)\big]V(\xi_1, \sigma) \ , \tag{4}$$

where $\alpha(\sigma) = \frac{\sigma^2}{\bar{\sigma}^2+\sigma^2}$ assumes values within the interval $[0, 1)$, while $e^{-V(\xi_1,\sigma)}$ has value in $(0, 1]$. Choosing the constants such that $k_1 \geq k_2 > k_3 > 0$, guarantees that $\dot{V} = 0 \iff \xi_1 = 0$ and with $\xi_1 = 0$, (2b) reduces to $\dot{\sigma} = -(k_2 - k_3)\sigma$. Besides imposing stronger restrictions on the gains, this allows to show only LAS and not GAS.

2.2 Double integrator

As example of a second-order control system, the canonical double integrator is considered

$$\dot{\xi}_1 = \xi_2 \tag{5a}$$

$$\dot{\xi}_2 = v \ , \tag{5b}$$

which can be thought of as the simplest model of a mass in a one-dimensional space under the effect of a time-varying force input v. As before, the control input has to be chosen such that the closed-loop system is GAS. To this end, let s be the variable defined as

$$s = \xi_2 + k_1 e^{-V(\xi_1,\sigma)}\xi_1 \ , \tag{6}$$

with $V(\xi_1, \sigma)$ given in (3). The previously derived results are extended to the second-order system (5) by designing a second-order dynamic feedback with state $(\sigma, \rho) \in \mathbb{R}^2$. The controller is provided in the following theorem.

Theorem 2. *For the double integrator system* (5)*, the dynamic state feedback with internal state* $(\sigma, \rho) \in \mathbb{R}^2$

$$v = -k_1 e^{-V(\xi_1,\sigma)} \left(\xi_2 - \xi_1 \dot{V}(\xi_1,\sigma)\right) - h_1 e^{-V(s,\rho)} s \quad (7a)$$

$$\dot{\sigma} = \left(k_3 - k_2 e^{-V(\xi_1,\sigma)}\right)\sigma \quad (7b)$$

$$\dot{\rho} = \left(h_3 - h_2 e^{-V(s,\rho)}\right)\rho \ , \quad (7c)$$

where s *is defined in* (6) *and* $V(\cdot,\cdot)$ *is given in* (3)*, leads to a GAS closed-loop system provided that the control gains* k_i*,* h_i*, with* $i \in \{1,2,3\}$ *satisfy the inequalities* $k_1 > 0$*,* $k_2 > k_3 > 0$ *and* $h_1 > 0$*,* $h_2 > h_3 > 0$*.*

Proof. See Appendix B.

3. MAIN RESULT

Since the focus of the paper is to provide first results for global asymptotic stabilization with smooth high-gain/low-gain transitions, a class of systems with trivial zero dynamics will be considered.

Consider a system described by the nonlinear equations (e.g., a mechanical system)

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sum_{i=1}^{n} \boldsymbol{g}_i(\boldsymbol{x}) u_i \quad (8a)$$

$$\boldsymbol{y} = \boldsymbol{\lambda}(\boldsymbol{x}) \ , \quad (8b)$$

where $\boldsymbol{x} \in \mathbb{R}^{2n}$ is an available state vector, $\boldsymbol{u} \in \mathbb{R}^n$ is the control input and $\boldsymbol{y} \in \mathbb{R}^n$ is the output of interest. As in Isidori (1995), the vector fields $\boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{g}_i(\boldsymbol{x})$ and the functions $\lambda_i(\boldsymbol{x})$, $i \in \{1,\ldots,n\}$, are smooth and defined on an open set of $\mathbb{R}^{2n}$. Additionally, the matrix $\boldsymbol{g}(\boldsymbol{x})$ has full-rank in the domain of interest and the system has (vector) relative degree $\{r_1,\ldots,r_n\}$ with $r_i = 2$.

In the above conditions, the *State Space Exact Linearization Problem* is solvable, see Isidori (1995). The control goal is to design an asymptotically stabilizing control input for the controllable feedback linearized system satisfying the following requirements. The aim of the control law is to fulfill the main stabilizing objective and smoothly pass from a high-gain behavior in a neighborhood of the equilibrium point to a low-gain behavior for large control errors, in order to avoid large values of the control input.

In view of Theorem 2, a possible solution to the previously stated problem is provided by the following result. Therein, $L_f^k \lambda(x)$ denotes that λ is differentiated k times along f.

Theorem 3. *Given the system* (8)*, let* $\boldsymbol{u} = \boldsymbol{a}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x})\boldsymbol{v}$ *be the linearizing feedback leading to* n *controllable linear systems in the form*

$$\dot{\xi}_1^i = \xi_2^i \quad (9a)$$

$$\dot{\xi}_2^i = v_i \ , \quad (9b)$$

with linearizing coordinates $\boldsymbol{\xi}$ *and* $\xi_k^i = L_{\boldsymbol{f}}^{k-1}\lambda_i(\boldsymbol{x})$ *for* $1 \le k \le 2$*,* $1 \le i \le n$*. Then choosing* v^i *as in* (7)*, with variables* s^i *as in* (6) *and* $V^i(\xi_1^i, \sigma^i)$ *as in* (3)*, renders the origin* $\boldsymbol{\xi} = \boldsymbol{0}$ *GAS provided that the gains* k_1^i*,* k_2^i*,* k_3^i *and* h_1^i*,* h_2^i*,* h_3^i *satisfy the inequalities* $k_1^i > 0$*,* $k_2^i > k_3^i > 0$ *and* $h_1^i > 0$*,* $h_2^i > h_3^i > 0$*. Moreover,* $\bar{\sigma}^i$ *allows to influence the region for the transition from high to low gain.*

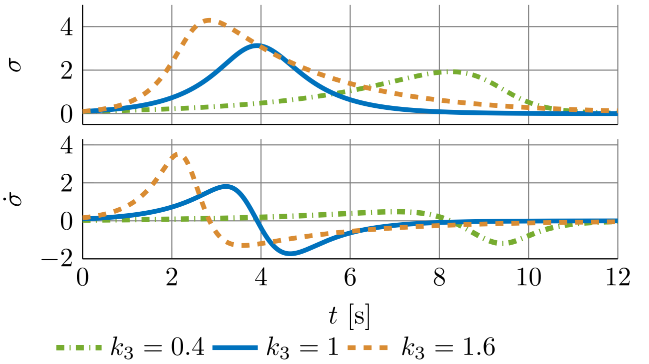

Fig. 1. Internal state σ and its derivative for the controller (2) of system (1).

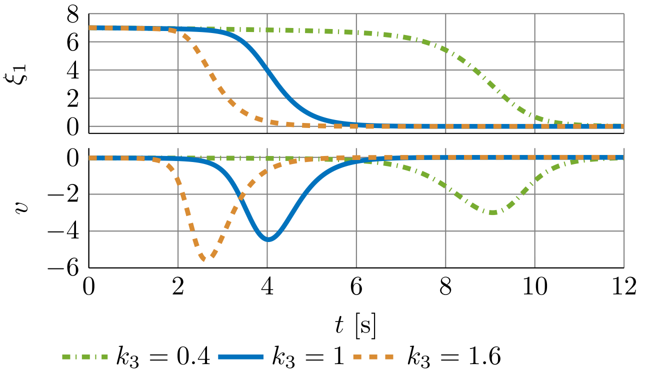

Fig. 2. State ξ_1 of the system (1) in feedback with (2), together with the control input v.

Proof. The proof consists in repeatedly applying Theorem 2 for each of the n double integrators.

4. SIMULATIONS AND DISCUSSION

In this section, the behavior of the control laws is shown with simulations, which provide also an overview on the role of the parameters.

The effects of the different gain values will be analyzed first. The simple integrator system will be considered for this, as the role of the parameters is the same also for (5). The way k_1 influences the system response is analogous to what happens in a more standard state feedback. Starting, instead, from (2b), one can recognize that k_3 and k_2 will influence how rapidly the windowing function expands and contracts, respectively. For the system (1), with $k_1 = 2$, the choice $k_3 = 0.5\,k_2$ leads to an almost symmetrical behavior. Fig. 1 shows the different evolution of σ and $\dot{\sigma}$, for gains $k_1 = 2$, $k_2 = 2$ and $k_3 \in \{0.4, 1, 1.6\}$.

As a result of a faster increase of σ, the state will more rapidly reach the goal as the gain will transition more quickly towards the high gain value. This effect is visible in Fig. 2, where the state ξ_1 and the control input v from the previous simulation are plotted, and also in Fig. 4.

Looking at the control input v in Fig. 2, it is possible to observe that the initial state $\xi_{1,0} = 7$, given $\bar{\sigma} = 2$, is so far from the origin that the control gain is close to zero. On the contrary, in Fig. 3 the initial state $\xi_{1,0} = 1$ is close enough (given $\bar{\sigma} = 2$) to have a system response analogous

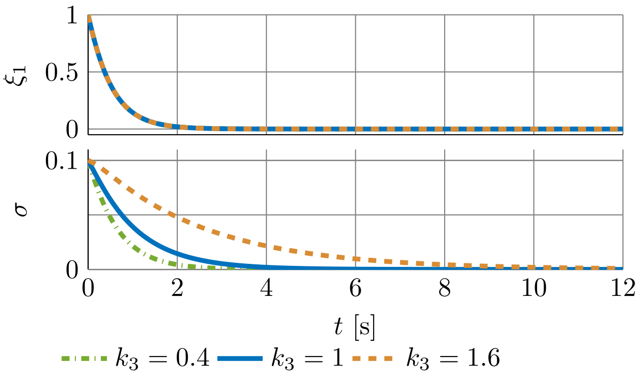

Fig. 3. Evolution of the state of the closed-loop system given by (1) and (2), when the initial condition is close to the goal.

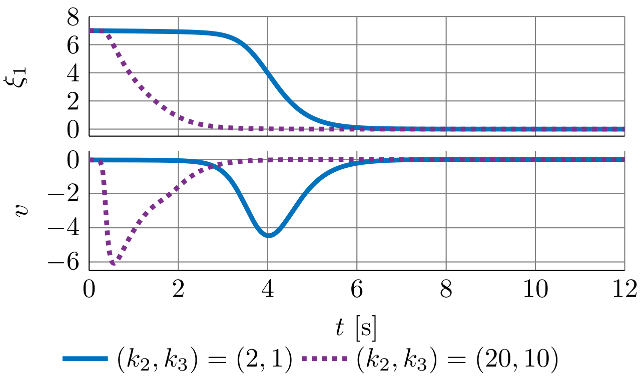

Fig. 4. State ξ_1 of the system (1) in feedback with (2) and the control input v, for slow and fast evolution of σ.

to a linear system. Accordingly, the different evolution of σ, due to the different gain values, has no influence on ξ_1.

A last consideration about the gains is that the condition $k_1 \geq k_2$ that would be required when using $V(\xi_1, \sigma)$ as semidefinite Lyapunov function is rather conservative. In Fig. 4, for example, both $k_2 = k_1$ and $k_2 = 10\, k_1$ have been used. The plots in blue in Fig. 2 and Fig. 4 are identical, as all the parameters are the same. The plots in purple, instead, were obtained with $k_2 = 20$ and $k_3 = 10$. As already noticed in Fig. 2, the higher k_3 will lead to a faster increase of σ and the state will reach the goal faster.

The next simulations show the different response of the system (5) when in feedback interconnection with the proposed approach (AVA), a classic proportional-derivative control (PD) and the combination of an interpolator and a PD controller (IP). The particular expressions of the PD controller and the interpolator are

$$v_{PD} = -2k_1(\xi_2 - \xi_{2,d}) - k_1^2(\xi_1 - \xi_{1,d}) \qquad (10)$$
$$\dddot{\xi}_{IP} = -3k_1\ddot{\xi}_{IP} - 3k_1^2\dot{\xi}_{IP} - k_1^3\xi_{IP}\ , \qquad (11)$$

where $\xi_{1,d} = \xi_{2,d} = 0$ for the PD scheme, while $\xi_{1,d} = \xi_{IP}$, $\xi_{2,d} = \dot{\xi}_{IP}$ for the scheme with inpterpolator and PD controller. In particular, to show a difference between the last two methods, for the latter a third-order filter is used. In this way, the scheme is effectively realizing a sort of dynamic extension, which is detailly explained in Zhan et al. (1991). Fig. 5 and Fig. 6 show the comparison of the three methods far from and close to the goal, respectively.

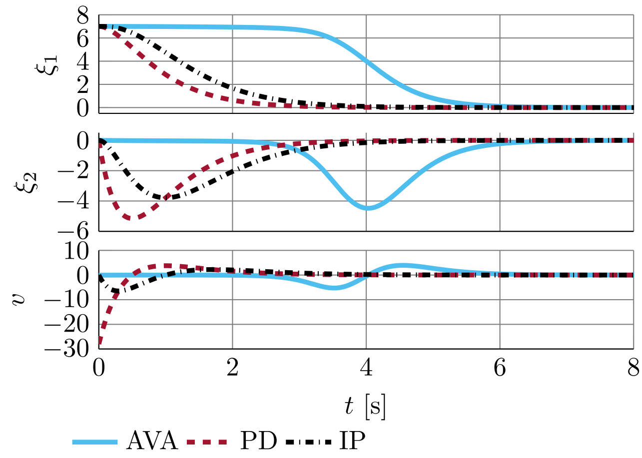

Fig. 5. Behavior of (5) in feedback with three different schemes, starting far from the goal.

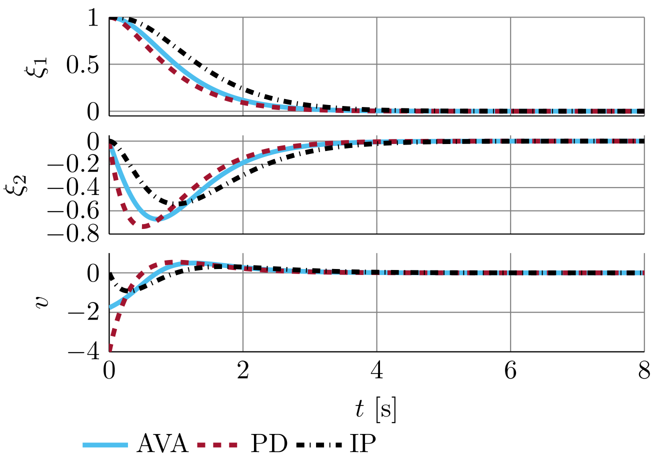

Fig. 6. Behavior of (5) in feedback with three different schemes, starting close to the goal.

The PD controller and the IP scheme are at the antipodes; the first always starts with a nonzero input (which in Fig. 5 is considerably high due to the large initial error), while the second always starts from zero (due to the fact that it behaves as a third-order system). AVA is a compromise between the two as it performs the gain transition that allows it to have a high gain near the equilibrium point and a low gain far from it. AVA can reproduce the behavior of the PD and IP via the limiting cases of $\bar{\sigma} \to \infty$ and $\bar{\sigma} \to 0$, respectively. Finally, notice that it has been chosen to limit the gain tuning of all the methods by using multiples and powers of k_1. While a different tuning of the parameters allows to change the responses (e.g., AVA could converge more rapidly to the goal as shown in Fig. 4), the overall behavior will remain unchanged.

A possible critical situation for the AVA scheme could occur when a change of the desired goal is commanded before σ has converged back to its desired value. In this case, if the new desired goal would have normally required a low gain, the controller cannot guarantee the correct transition of the gain, i.e., the control input might still result in a large value as a result of a high control error. A possible solution, if applicable, could be to reset the control state σ in case of a change of the desired goal before the latter has been reached and σ has converged.

5. CONCLUSION

In this paper, a dynamic state feedback with adaptive state-dependent gains has been presented. The main focus of the paper was on developing an automatic high-gain/low-gain transition depending on the distance between the current system state and the final goal. This is realized while still guaranteeing the global asymptotic stability of the equilibrium point in the origin. The basic *Adaptive Variance Algorithm* (AVA) in its first and second-order variants has been used in combination with feedback linearization for systems with trivial zero dynamics. The stability of the algorithm is analyzed and simulations are used to validate the theoretical results, as well as to highlight the role of the different parameters.

Future work will consider the possibility of finding conditions under which a stabilizing controller for a nonlinear system can be modified with an AVA scheme in order to guarantee in general the high-gain/low-gain transitions.

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Appendix A

Useful theorems for the derivation of the results presented in this paper are reported here for completeness. All the vector fields are assumed to be locally Lipschitz and to be zero when evaluated in zero.

The following results from Seibert and Suarez (1990) hold for the cascaded system

$$\dot{x} = f(x) \tag{A.1a}$$
$$\dot{y} = g(y, x) \ . \tag{A.1b}$$

Theorem 4. *If the systems* (A.1a) *and* (A.1b) *are GAS and* 0*-GAS, respectively and if every orbit of* (A.1) *is bounded in the future, then* (A.1) *is GAS.*

Theorem 5. *If the positive real-valued, differentiable function W, defined on B, is unbounded on any unbounded set and $\dot{W} \leq 0$ holds on the intersection of an end set with some set of the form $\{(x, y) \mid |(x, y)| \geq M > 0\}$, then all orbits starting in B are bounded for $t > 0$.*

It is reported a version of the main result in Jiang et al. (1996) in absence of inputs and for the case of GAS, which is sufficient for the purpose of this paper.

Theorem 6. *Given the interconnected systems*

$$\dot{x}_1 = f_1(x_1, x_2) \tag{A.2a}$$
$$\dot{x}_2 = f_2(x_1, x_2) \ , \tag{A.2b}$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$ and $f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_i}$ is locally Lipschitz. Assume that, for $i = 1, 2$, there exists an ISS-Lyapunov function U_i for the x_i-subsystem such that the following holds:

(1) there exist functions ψ_{i1}, $\psi_{i2} \in \mathcal{K}_\infty$ so that

$$\psi_{i1}(|x_i|) \leq U_i(x_i) \leq \psi_{i2}(|x_i|) \ , \ \ \forall x_i \in \mathbb{R}^{n_i} \tag{A.3}$$

(2) there exist functions $\alpha_i \in \mathcal{K}_\infty$, $\chi_i \in \mathcal{K}$ so that $U_1 \geq \chi_1(U_2(x_2))$ implies

$$\frac{\partial U_1}{\partial x_1} f_1(x_1, x_2) \leq -\alpha_1(U_1(x_1)) \tag{A.4}$$

and $U_2(x_2) \geq \chi_2(U_1(x_1))$ implies

$$\frac{\partial U_2}{\partial x_2} f_2(x_1, x_2) \leq -\alpha_2(U_2(x_2)) \tag{A.5}$$

and $\chi_1 \circ \chi_2(r) < r$ or equivalently $\chi_2 \circ \chi_1(r) < r$, $\forall r > 0$, then the zero solution of (A.2) *is GAS.*

Appendix B

This appendix contains the proofs of Theorems 1-2.

The closed-loop system given by (1) and (2) is

$$\dot{\xi}_1 = f_1(\xi_1, \sigma) = -k_1 e^{-V(\xi_1,\sigma)} \xi_1 \tag{B.1a}$$

$$\dot{\sigma} = f_2(\xi_1, \sigma) = \left(k_3 - k_2 e^{-V(\xi_1,\sigma)}\right)\sigma \ . \tag{B.1b}$$

The proof of Theorem 1 relies on Theorem 6.

Proof. [Theorem 1] Consider the following ISS-Lyapunov functions $U_1(\xi_1) = \frac{e^{V(\xi_1,0)}-1}{2k_1}$ and $U_2(\sigma) = \frac{\sigma^2}{2k_2}$.

For the system $\dot{\xi}_1 = f_1(\xi_1, \sigma)$, define the class $\mathcal{K}_\infty$ function $\alpha_1(r) = \ln(2k_1 r + 1)$. Then $V(\xi_1, 0) = \alpha_1(U_1(\xi_1))$ and

$$\frac{\partial U_1}{\partial \xi_1} f_1(\xi_1, \sigma) = -V(\xi_1, 0) e^{\frac{\xi_1^2 \sigma^2}{2\bar{\sigma}^2(\bar{\sigma}^2+\sigma^2)}} \leq -\alpha_1(U_1(\xi_1)) \tag{B.2}$$

holds for all $\sigma \in \mathbb{R}$. Therefore, it must hold also for $U_1(\xi_1) \geq \chi_1 U_2(\sigma)$ and any $\chi_1 > 0$.

For the system $\dot{\sigma} = f_2(\xi_1, \sigma)$,

$$\frac{\partial U_2}{\partial \sigma} f_2(\xi_1, \sigma) = \frac{\sigma^2}{k_2}\left(k_3 - k_2 e^{-V(\xi_1,\sigma)}\right) \leq -\alpha_2 U_2(\sigma) \ , \tag{B.3}$$

with $0 < \alpha_2 < 2(k_2 - k_3)$, holds whenever

$$\sigma^2 \geq \frac{\xi_1^2}{2\left|\ln \frac{2k_3+\alpha_2}{2k_2}\right|} - \bar{\sigma}^2 \ . \tag{B.4}$$

Remember that $k_2 > k_3$. Given (B.4), then (B.3) will hold also for $U_2(\sigma) \geq \chi_2(U_1(\xi_1))$, with

$$\chi_2(r) = \frac{\bar{\sigma}^2}{2k_2\left|\ln \frac{2k_3+\alpha_2}{2k_2}\right|}\alpha_1(r) \ . \tag{B.5}$$

Since $\ln(r + 1) \leq r$ for all $r \geq 0$, then $\alpha_1(r) \leq 2k_1 r$ and one can always choose $\chi_1 > 0$ such that $\chi_1 \chi_2(r) < r$, $\forall r > 0$. Therefore, (B.1) is GAS by virtue of Theorem 6.

The closed-loop system given by (5) and (7) is

$$\dot{s} = -h_1 e^{-V(s,\rho)} s \tag{B.6a}$$

$$\dot{\rho} = \left(h_3 - h_2 e^{-V(s,\rho)}\right)\rho \tag{B.6b}$$

$$\dot{\xi}_1 = -k_1 e^{-V(\xi_1,\sigma)} \xi_1 + s \tag{B.6c}$$

$$\dot{\sigma} = \left(k_3 - k_2 e^{-V(\xi_1,\sigma)}\right)\sigma \ . \tag{B.6d}$$

The proof of Theorem 2 will use the results formulated in Seibert and Suarez (1990) and reported in Theorems 4-5. Denoting by $S1$ the subsystem (B.6a) - (B.6b) and by $S2$ the subsystem (B.6c) - (B.6d), then (B.6) can be seen as the cascade of $S1$ and the system $S2$ with input s. Moreover, Fig. B.1 can be used to visualize the regions defined within the proof.

Proof. [Theorem 2] The subsystem $S1$ is GAS and $S2$ is 0-GAS by virtue of Theorem 1. Therefore, according to Theorem 4, it is enough to show that every orbit of (B.6) is bounded in the future to conclude that (B.6) is GAS. This will be shown using Theorem 5, consequently an end set and for the system (B.6) is needed. Remember that, since $S1$ is GAS, any set of the form

$$\left\{(s, \rho, \xi_1, \sigma) \in \mathbb{R}^4 \mid |(s, \rho)| \leq N, (\xi_1, \sigma) \in \mathbb{R}^2\right\} \tag{B.7}$$

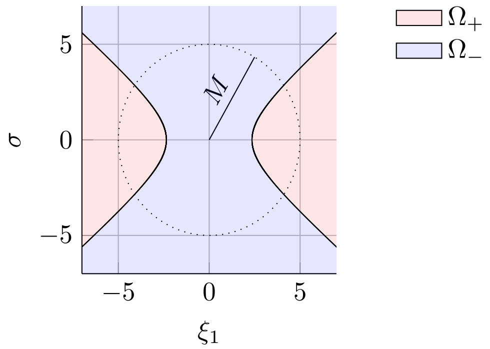

Fig. B.1. Regions of the state space of the system (B.1) for parameter values: $k_1 = 2$, $k_2 = 2$, $k_3 = 1$, $\bar{\sigma} = 2$. In absence of input, the trajectories of the system will always go from Ω_+ to Ω_-.

is an end set for the system (B.6). A tighter end set will be given next. Let $w(\xi_1, \sigma) = k_3 - k_2 e^{-V(\xi_1,\sigma)}$, then the set Ω_+ cannot be part of the end set for small enough s, where $\Omega_+ = \left\{(\xi_1, \sigma) \in \mathbb{R}^2 \mid w(\xi_1, \sigma) \geq 0\right\}$. In fact, assume $w(\xi_1, \sigma) = \bar{w} \geq 0$, then given Young's inequality

$$\begin{aligned}\dot{w} &= \frac{k_2 e^{-V(\xi_1,\sigma)}}{\bar{\sigma}^2+\sigma^2}\left(\xi_1 s - k_1 e^{-V(\xi_1,\sigma)}\xi_1^2 - \frac{\xi_1^2\sigma^2\bar{w}}{\bar{\sigma}^2+\sigma^2}\right) \\ &\leq \frac{k_3 - \bar{w}}{\bar{\sigma}^2+\sigma^2}\left[\frac{s^2}{2\epsilon_+} - \left(k_1\frac{k_3-\bar{w}}{k_2} - \frac{\epsilon_+}{2}\right)\xi_1^2\right] \ ,\end{aligned} \tag{B.8}$$

$\forall(\xi_1, \sigma) \in \Omega_+$. Since $0 \leq w(\xi_1, \sigma) < k_3$ and $\xi_1^2 \geq \bar{\xi}_1^2 > 0$ in Ω_+, then $\dot{w} < 0$ is satisfied with $\epsilon_+ = k_1\frac{k_3-\bar{w}}{k_2}$ and $s^2 \leq N^2$, $N^2 = \frac{\epsilon_+^2}{2}\bar{\xi}_1^2$. Therefore, $w(\xi_1, \sigma)$ decreases and $\dot{w}$ will in turn become even more negative, until the trajectories of the system leave from Ω_+.

The second step consists in finding a function $W(s, \rho, \xi_1, \sigma)$ satisfying the conditions of Theorem 5. Given the function

$$W_2(\xi_1, \sigma) = \frac{\xi_1^2}{2k_1} + U_2(\sigma) \ , \tag{B.9}$$

there exists $M > 0$ such that $\dot{W}_2 \leq 0$ for all (ξ_1, σ) in $\Omega_- \cap \Omega_M$ and with $s^2 \leq N^2$, where the set Ω_M is $\Omega_M = \left\{(\xi_1, \sigma) \in \mathbb{R}^2 \mid |(\xi_1, \sigma)| \geq M\right\}$. In fact, given the sets and using Young's inequality for products, with $\epsilon_- = \frac{k_1 k_3}{k_2}$, one gets

$$\begin{aligned}\dot{W}_2 &= -e^{-V(\xi_1,\sigma)}\xi_1^2 + \frac{\xi_1 s}{k_1} + w(\xi_1, \sigma)\frac{\sigma^2}{k_2} \\ &\leq -\left(\frac{k_3}{k_2} - \frac{\epsilon_-}{2k_1}\right)\xi_1^2 + \frac{N^2}{2\epsilon_- k_1} \\ &\leq \frac{k_3}{2k_2}\left(\sigma_0^2 - M^2\right) + \frac{N^2}{2\epsilon_- k_1} \ ,\end{aligned} \tag{B.10}$$

where σ_0 is the value of σ when the trajectory of the system first entered in Ω_- and $\sigma < \sigma_0$ since σ is strictly decreasing in Ω_-. Therefore, $\dot{W}_2 \leq 0$ in $\Omega_- \cap \Omega_M$ and with $s^2 \leq N^2$ holds for $M^2 = \frac{k_2 N^2}{k_1 k_3 \epsilon_-} + \sigma_0^2$.

Finally, since $S1$ is GAS, let $W_1(s, \rho)$ be a Lyapunov function for $S1$ obtained by a converse Lyapunov theorem. Then $W(s, \rho, \xi_1, \sigma) = W_1(s, \rho) + W_2(\xi_1, \sigma)$ satisfies the hypothesis of Theorem 5.