In-Field Calibration of Antennas or Antenna Arrays Using Wavefield Modeling

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***Abstract*—Direction-of-arrival (DoA) estimation of radio signals requires accurate knowledge of the employed antenna array characteristics, which is usually obtained by calibration in a measurement chamber. Contrarily we propose an in-field calibration method, which does not require knowledge of the propagation channel or synchronization between transmitter and receiver. In contrast to the literature, the proposed method can be applied to multiport antenna systems of arbitrary type and interpolation of the antenna response between the discrete calibration DoAs is inherently performed using wavefield modeling. Simulations show that in-field calibration at sufficiently high signal-to-noise ratio (SNR) enables precise DoA estimation.**

I. Introduction

Multiport antenna systems like phased arrays [1], colocated antennas [2], [3] or multi-mode antennas [4], [5] enable direction-of-arrival (DoA) estimation of radio signals or transmit beamforming. This requires a precise knowledge of the antenna characteristics. To obtain that, the antenna system is usually calibrated in a measurement chamber. In many cases antennas are to be mounted on large metallic structures like a vehicle or plane, which influences the electric field and changes the antenna characteristics. Calibration has thus to be carried out in a large measurement facility, which is costly. An alternative is calibrating the antenna directly in-field using transmitters in known directions. In contrast to a calibrated measurement chamber, the propagation channel is unknown in this case.

In-field calibration is closely related to auto-calibration, also termed self-calibration, which aims at estimating both antenna and wavefield parameters (DoA) at the same time [6]. The unknown antenna parameters are either considered as deterministic [7] or stochastic with a known prior distribution [8], [9]. Both approaches suffer from the identifiability issue [10], that the antenna or array parameters can in general not be determined together with the wavefield, i.e. DoA. To circumvent this, strong assumptions have to be made. For example, assuming the different antenna ports share the same gain pattern, their phase patterns can be determined [11]. Restricting the geometry to uniform linear array (ULA), gain and phase patterns can be estimated [10].

These assumptions limit the applicability of auto-calibration to antenna arrays, but exclude colocated antennas and multi-mode antennas. Another drawback is that for practical multiport antennas, these assumptions often do not hold, resulting in a model mismatch and impaired performance. Instead we propose an in-field calibration method using wavefield modeling and manifold separation [12], [13], which allows to generalize the approach to arbitrary multiport antenna systems. To ensure identifiability, calibration has to be carried out with a precise DoA reference.

II. Signal model

The M ports of the multiport antenna system are described by their respective gain pattern $g_m(\theta)$ and phase pattern $\Phi_m(\theta)$ for DoA θ [14], forming the antenna response for port m,

$$a_m(\theta) = \sqrt{g_m(\theta)} e^{\mathrm{j}\Phi_m(\theta)}. \quad (1)$$

Assuming the antenna is connected to a multichannel receiver, the sampled baseband signal $\boldsymbol{r}(n) = [r_1(n), ..., r_M(n)]^T$ with sample index n received at the M ports of the multiport antenna is given by

$$\boldsymbol{r}(n) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(n) + \boldsymbol{w}(n), \quad (2)$$

with the antenna response matrix

$$\boldsymbol{A}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{a}(\theta_1) & ... & \boldsymbol{a}(\theta_P) \end{bmatrix} \quad (3)$$

composed of antenna response vectors

$$\boldsymbol{a}(\theta) = \begin{bmatrix} a_1(\theta) & ... & a_M(\theta) \end{bmatrix}^T \quad (4)$$

and $\boldsymbol{s}(n) = [s_1(n), ..., s_P(n)]^T$ are the P arriving signals from DoAs $\theta_1, ..., \theta_P$, assuming their bandwidth is small compared to the carrier frequency [14], [15]. The system is assumed to be internally noise limited and the noise term $\boldsymbol{w}(n) \sim \mathcal{CN}(0, \sigma_w^2 \mathbb{I}_M)$, with the identity matrix $\mathbb{I}_M$ of size M, is i.i.d. white circular symmetric Gaussian noise. Using wavefield modeling and manifold separation [12], [13], the antenna response vector can be decomposed

$$\boldsymbol{a}(\theta) = \boldsymbol{G}\,\boldsymbol{b}(\theta) \quad (5)$$

into a product of the sampling matrix $\boldsymbol{G} \in \mathbb{C}^{M \times U}$, which is wavefield or DoA independent, and the basis vector $\boldsymbol{b}(\theta) \in \mathbb{C}^U$, which is antenna independent [12].

The antenna response must be square integrable and the U basis functions orthonormal on the manifold $\theta \in [-\pi, \pi)$. An extension to 3D is possible [12], [13], but out of scope for this paper. A suitable basis for 2D is given by the Fourier functions

$$\boldsymbol{b}(\theta) = \frac{1}{\sqrt{2\pi}} e^{\mathrm{j}\theta u_\theta}, u_\theta = \left\lfloor -\frac{U-1}{2} \right\rfloor, ..., 0, ..., \left\lfloor \frac{U-1}{2} \right\rfloor. \quad (6)$$

III. IN-FIELD CALIBRATION

A. Problem formulation

As a preliminary step, the estimated antenna response $\hat{\boldsymbol{e}}_q$ has to be determined for $q = 1, ..., Q$ different known directions θ_q. This can either be done by eigenvalue decomposition of the spatial covariance matrix of the received signals [15] or by more advanced blind source separation (BSS) algorithms [16], [17]. For all approaches the estimated antenna responses

$$\hat{\boldsymbol{e}}_q = \boldsymbol{a}(\theta_q) c_q + \boldsymbol{v}_q = \boldsymbol{G}\, \boldsymbol{b}(\theta_q) c_q + \boldsymbol{v}_q \quad (7)$$

suffer from gain and phase ambiguity, i.e. they are scaled by unknown complex coefficients c_q and corrupted by noise $\boldsymbol{v}_q$. For large N, $\boldsymbol{v}_q$ is approximately circularly symmetric Gaussian distributed, $\boldsymbol{v}_q \sim \mathcal{CN}(0, \sigma_v^2 \mathbb{I}_M)$. The goal is now to estimate $\boldsymbol{G}$ in order to obtain an analytical closed form antenna response $\boldsymbol{a}(\theta) = \boldsymbol{G}\, \boldsymbol{b}(\theta)$ which is valid on the whole manifold $\theta \in [-\pi, \pi)$.

First we rewrite the problem in matrix form with the estimated antenna responses $\hat{\boldsymbol{E}} = \begin{bmatrix} \hat{\boldsymbol{e}}_1 & ... & \hat{\boldsymbol{e}}_Q \end{bmatrix}$, the true antenna responses $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}(\theta_1) & ... & \boldsymbol{a}(\theta_Q) \end{bmatrix}$, the basis functions evaluated at the DoAs $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}(\theta_1) & ... & \boldsymbol{b}(\theta_Q) \end{bmatrix}$, the diagonal matrix with unknown complex coefficients $\boldsymbol{C} = \mathrm{diag}\left\{ \begin{bmatrix} c_1 & ... & c_Q \end{bmatrix} \right\}$, and the noise $\boldsymbol{V} = \begin{bmatrix} \boldsymbol{v}_1 & ... & \boldsymbol{v}_Q \end{bmatrix}$, yielding the model

$$\hat{\boldsymbol{E}} = \boldsymbol{A}\boldsymbol{C} + \boldsymbol{V} = \boldsymbol{G}\boldsymbol{B}\boldsymbol{C} + \boldsymbol{V}. \quad (8)$$

In a calibrated measurement chamber, i.e. $C = \mathbb{I}_Q$, $\hat{\boldsymbol{G}}$ can be obtained by least squares

$$\hat{\boldsymbol{G}} = \hat{\boldsymbol{E}}\boldsymbol{B}^H (\boldsymbol{B}\boldsymbol{B}^H)^{-1}. \quad (9)$$

For non-coherent in-field calibration, the unknown complex coefficients due to unknown propagation channel, unknown transmit power and lack of synchronization between transmitter and receiver have to be taken into account. Due to these complex coefficients, the estimated antenna response may exhibit jumps and its spatial bandwidth is in general larger compared to the true antenna response. While the jumps could theoretically be avoided by a straight forward normalization w.r.t. the first antenna port, this does not solve the issue of increased spatial bandwidth, causing a violation of wavefield modeling (WM) assumptions [18]. Furthermore it cannot be applied to all types of multi-port antennas. For multi-mode antennas (MMAs), where the antenna response may exhibit nulls in certain directions, the normalization would lead to a division by zero.

In the array processing literature, different methods can be found to determine unknown array parameters together with unknown DoAs based on maximum likelihood (ML) or maximum a posteriori (MAP) [7]–[9]. In a similar fashion, the sampling matrix $\boldsymbol{G}$ can be estimated if the DoAs are known, otherwise the problem is ill-conditioned since wavefield and array parameters are not simultaneously identifiable [10]. For both ML and MAP a nonlinear, nonconvex function with a large number of unknowns has to be solved, which is unfavorable in practice.

B. Convex optimization

Instead we rewrite (8) to

$$\hat{\boldsymbol{E}}\boldsymbol{C}^{-1} = \boldsymbol{A} + \boldsymbol{W}\boldsymbol{C}^{-1} = \boldsymbol{G}\boldsymbol{B} + \boldsymbol{V}' \quad (10)$$

with $\boldsymbol{V}' = \begin{bmatrix} \boldsymbol{v}_1' & ... & \boldsymbol{v}_Q' \end{bmatrix}$ and $\boldsymbol{v}_q' \sim \mathcal{CN}(0, \boldsymbol{\Sigma}_q)$, $\boldsymbol{\Sigma}_q = \mathrm{diag}\left\{ \begin{bmatrix} ... & \frac{\sigma_v^2}{|c_q|^2} & ... \end{bmatrix} \right\}$. With Gaussian $\boldsymbol{V}'$, the maximum likelihood estimator is given by

$$\{\hat{\boldsymbol{G}}, \hat{\boldsymbol{C}}^{\boldsymbol{-1}}\} = \arg \min_{\boldsymbol{G}, \boldsymbol{C}^{-1}} ||\hat{\boldsymbol{E}}\boldsymbol{C}^{-1} - \boldsymbol{G}\boldsymbol{B}||_F^2. \quad (11)$$

Assuming $|c_q|^2$ are of similar magnitude, we obtain the LS solution for $\boldsymbol{G}$,

$$\hat{\boldsymbol{G}} \approx \hat{\boldsymbol{E}}\boldsymbol{C}^{-1}\boldsymbol{B}^H (\boldsymbol{B}\boldsymbol{B}^H)^{-1} \quad (12)$$

This is an approximation, optimal would be weighted least squares, but this would lead to a non-convex optimization problem. By plugging (12) into (11) we obtain

$$\hat{\boldsymbol{C}}^{\boldsymbol{-1}} = \arg \min_{\boldsymbol{C}^{-1}} ||\hat{\boldsymbol{E}}\boldsymbol{C}^{-1}(\mathbb{I}_Q - \boldsymbol{B}^\dagger \boldsymbol{B})||_F^2 \quad (13)$$

where $\boldsymbol{B}^\dagger \boldsymbol{B}$ is a projector onto the column space of $\boldsymbol{B}$ and $\mathbb{I}_Q - \boldsymbol{B}^\dagger \boldsymbol{B}$ is a projector onto the nullspace of $\boldsymbol{B}^H$. The complex coefficients $\hat{\boldsymbol{C}}^{\boldsymbol{-1}}$ are thus chosen such that the corrected antenna response observations $\hat{\boldsymbol{E}}\boldsymbol{C}^{-1}$ can be optimally represented by the given basis $\boldsymbol{B}$.

The term inside the norm is a complex affine expression w.r.t. to the unknown variable $\boldsymbol{C}^{-1}$ and the Frobenius norm itself is also a convex function, i.e. the optimization problem

$$\begin{aligned} \underset{\boldsymbol{C}^{-1}}{\text{minimize}} \quad & ||\hat{\boldsymbol{G}}\boldsymbol{B} - \hat{\boldsymbol{E}}\boldsymbol{C}^{-1}||_F \\ \text{subject to} \quad & \hat{\boldsymbol{G}} = \hat{\boldsymbol{E}}\boldsymbol{C}^{-1}\boldsymbol{B}^\dagger \\ & \boldsymbol{C}^{-1} \text{ is diagonal} \end{aligned} \quad (14)$$

is convex. The trivial solution $\boldsymbol{C}^{-1} = \mathrm{diag}\{\mathbf{0}\}$ can be avoided by adding the constraint

$$[\boldsymbol{C}^{-1}]_{q,q} = MQ \left(\sum_{m=1}^{M} \sum_{q=1}^{Q} |[\hat{\boldsymbol{E}}]_{m,q}| \right)^{-1} \quad (15)$$

where q is chosen such that θ_q lies within the main beam of the antenna A different convex constraint that prevents the trivial solution can be chosen as well. The final convex optimization problem to be solved is then given by

$$\begin{aligned} \underset{\boldsymbol{C}^{-1}}{\text{minimize}} \quad & ||\hat{\boldsymbol{G}}\boldsymbol{B} - \hat{\boldsymbol{E}}\boldsymbol{C}^{-1}||_F \\ \text{subject to} \quad & \hat{\boldsymbol{G}} = \hat{\boldsymbol{E}}\boldsymbol{C}^{-1}\boldsymbol{B}^\dagger \\ & [\boldsymbol{C}^{-1}]_{1,1} = \frac{MQ}{2} \left(\sum\nolimits_{m=1}^{M} \sum\nolimits_{q=1}^{Q} \left| [\hat{\boldsymbol{E}}]_{m,q} \right| \right)^{-1} \\ & \boldsymbol{C}^{-1} \text{ is diagonal.} \end{aligned} \quad (16)$$

By solving we obtain $\hat{\boldsymbol{G}}$ and thus a continuous expression for the equivalent antenna response

$$\hat{\boldsymbol{a}}(\theta) = \hat{\boldsymbol{G}}\boldsymbol{b}(\theta) \overset{\triangle}{\approx} \boldsymbol{a}(\theta), \tag{17}$$

where $\overset{\triangle}{\approx}$ means approximately equivalent under the transformation

$$\boldsymbol{a}(\theta) \approx \hat{\boldsymbol{a}}(\theta)c_{\mathrm{t}}(\theta) \tag{18}$$

with $c_{\mathrm{t}}(\theta) = \hat{\boldsymbol{a}}(\theta)^{\dagger}(\theta)\boldsymbol{a}(\theta)$ for arbitrary θ.

IV. DoA estimation

For DoA estimation we use the deterministic ML estimator [19] given by

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \mathrm{Re}\{\mathrm{tr}\{\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp}\hat{\boldsymbol{R}}_{\boldsymbol{r}}\}\}, \tag{19}$$

with sample covariance matrix of the received signal (2),

$$\hat{\boldsymbol{R}}_{\boldsymbol{r}} = \frac{1}{N}\sum_{n=1}^{N}\boldsymbol{r}(n)\boldsymbol{r}^{H}(n), \tag{20}$$

the projector onto the noise subspace

$$\boldsymbol{\Pi}_{\boldsymbol{A}}^{\perp} = \mathbb{I}_M - \boldsymbol{\Pi}_{\boldsymbol{A}} \tag{21}$$

and the projector onto the signal subspace

$$\boldsymbol{\Pi}_{\boldsymbol{A}} = \boldsymbol{A}(\boldsymbol{\theta})\left(\boldsymbol{A}^{H}(\boldsymbol{\theta})\boldsymbol{A}(\boldsymbol{\theta})\right)^{-1}\boldsymbol{A}^{H}(\boldsymbol{\theta}), \tag{22}$$

which requires any subset of P antenna response vectors $\boldsymbol{a}(\theta_p)$, see (3), to be linearly independent. This is equivalent to $\mathrm{rank}\{\boldsymbol{A}(\boldsymbol{\theta})\} = P$. By in-field calibration, instead of $\boldsymbol{A}(\boldsymbol{\theta})$ we obtain $\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{C}(\boldsymbol{\theta})$ with $\boldsymbol{C}(\boldsymbol{\theta}) = \mathrm{diag}\left\{\begin{bmatrix} c(\theta_1) & \dots & c(\theta_P)\end{bmatrix}\right\}$ composed of arbitrary complex coefficients $c(\theta_p)$. The signal subspace projector then becomes

$$\begin{aligned}\boldsymbol{\Pi}_{\boldsymbol{AC}} =& \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{C}(\boldsymbol{\theta})\left(\boldsymbol{C}^{H}(\boldsymbol{\theta})\boldsymbol{A}^{H}(\boldsymbol{\theta})\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{C}(\boldsymbol{\theta})\right)^{-1} \\ &\cdot\boldsymbol{C}^{H}(\boldsymbol{\theta})\boldsymbol{A}^{H}(\boldsymbol{\theta}) \\ =& \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{C}^{-1}(\boldsymbol{\theta})\left(\boldsymbol{A}^{H}(\boldsymbol{\theta})\boldsymbol{A}(\boldsymbol{\theta})\right)^{-1} \\ &\cdot(\boldsymbol{C}^{H})^{-1}(\boldsymbol{\theta})\boldsymbol{C}^{H}(\boldsymbol{\theta})\,\boldsymbol{A}^{H}(\boldsymbol{\theta}) \\ =& \boldsymbol{A}(\boldsymbol{\theta})\left(\boldsymbol{A}^{H}(\boldsymbol{\theta})\boldsymbol{A}(\boldsymbol{\theta})\right)^{-1}\boldsymbol{A}^{H}(\boldsymbol{\theta}),\end{aligned} \tag{23}$$

which is identical to (22). The estimator (19) is thus valid for true (4) or equivalent (17) antenna response.

V. Simulation results

The described in-field calibration algorithm is demonstrated for a multi-mode antenna. Its true antenna response $\boldsymbol{a}(\theta)$ is considered to be known and defined by (5) and (6) with $U = 9$ coefficients. The received signals are generated based on (2) for $Q = 12$ different directions, which are uniformly distributed over the manifold. The received signal for each direction is generated independently with 20 dB signal-to-noise ratio (SNR), corresponding to a time-division multiple access (TDMA) system. In a preliminary step, the discrete estimates of the equivalent antenna responses $\hat{\boldsymbol{e}}_q$ are obtained for all calibration directions θ_q with $q = 1,..,Q$ based on eigenvalue decomposition of the spatial covariance matrix.

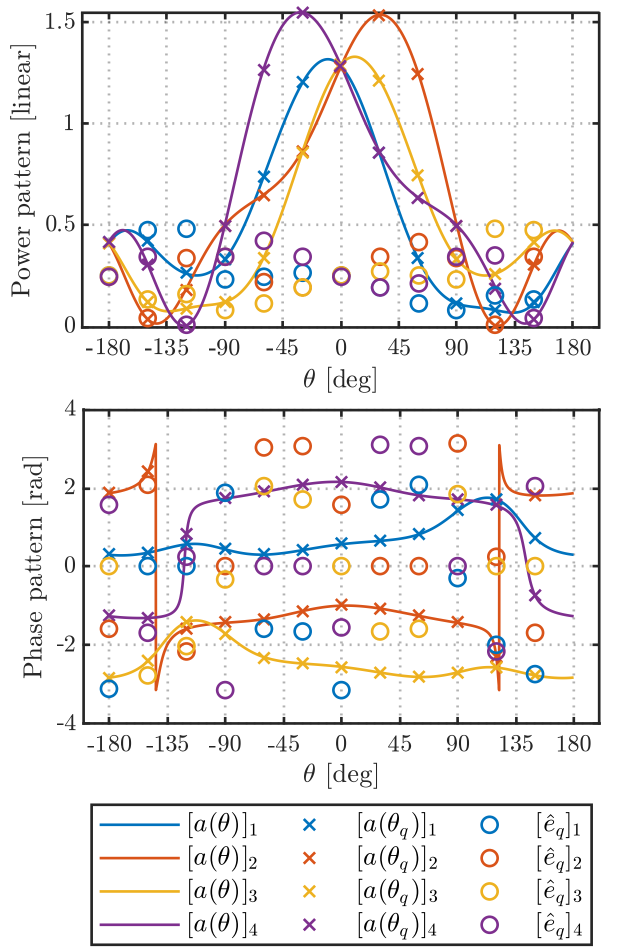

Fig. 1. True antenna response $\boldsymbol{a}(\theta)$, true antenna response sampled at calibration DoAs $\boldsymbol{a}(\theta_q)$ and the respective estimates $\hat{\boldsymbol{e}}_q$.

Figure 1 shows the true antenna response $\boldsymbol{a}(\theta)$, the true antenna response sampled at calibration DoAs $\boldsymbol{a}(\theta_q)$ and the discrete estimates of the equivalent antenna responses at the calibration DoAs $\hat{\boldsymbol{e}}_q$. The gain and phase ambiguity of $\hat{\boldsymbol{e}}_q$ is apparent.

As main step of the algorithm, the convex optimization problem (16) is solved using the CVX toolbox [20]. The outcome is the continuous equivalent antenna response $\hat{\boldsymbol{a}}(\theta)$, see Figure 2. Since the true antenna response is known in this example, the transformation (18) can be performed to compare the equivalent to the true antenna response. Figure 2 shows that the transformed equivalent antenna response $\hat{\boldsymbol{a}}(\theta)c_{\mathrm{t}}(\theta)$ matches perfectly the calibration points of the true antenna response $\boldsymbol{a}(\theta_q)$.

Finally the DoA estimation performance using true and equivalent antenna response is compared. DoA estimation using wavefield modeling [4] is performed with the true sampling matrix $\boldsymbol{G}$ and the estimated sampling matrix $\hat{\boldsymbol{G}}$ respectively. Figure 3 shows the DoA estimation RMSE over θ and 100 Monte Carlo runs. Only the estimator using the true sampling matrix asymptotically approaches the Cramér-Rao

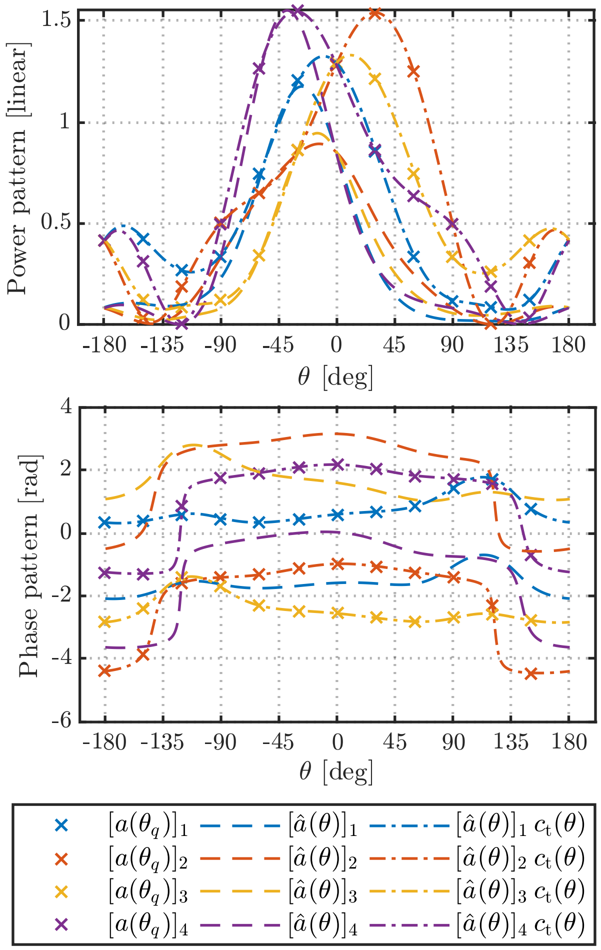

Fig. 2. True antenna response sampled at calibration DoAs $\boldsymbol{a}(\theta_q)$, estimated equivalent antenna response $\hat{\boldsymbol{a}}(\theta)$ and transformed equivalent antenna response $\hat{\boldsymbol{a}}(\theta)c_{\mathrm{t}}(\theta)$.

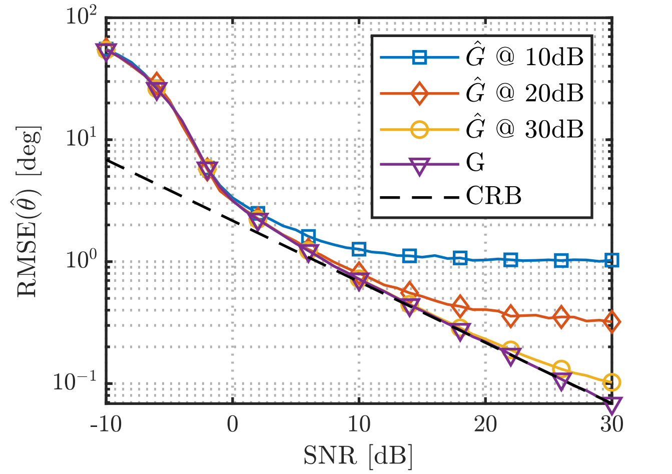

Fig. 3. DoA estimation RMSE depending on SNR using the true sampling matrix $\boldsymbol{G}$, estimated sampling matrices $\hat{\boldsymbol{G}}$ obtained by in-field calibration at different calibration SNRs and the CRB.

bound (CRB) for high SNR. Using the estimated sampling matrices, the achievable RMSE is limited by the calibration SNR. Nevertheless sub-degree accuracy is achievable for this antenna with a calibration SNR above 10 dB.

VI. Conclusion

In conclusion, in-field calibration is a viable method to obtain a continuous equivalent antenna response, which accurately reflects the true antenna characteristics. Depending on the desired DoA estimation accuracy, in-field calibration should be performed with sufficiently high SNR.

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