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# Controller Reduction Using Accuracy-Enhancing Methods

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**Summary.** The efficient solution of several classes of controller approximation problems by using frequency-weighted balancing related model reduction approaches is considered. For certain categories of performance and stability enforcing frequency-weights, the computation of the frequency-weighted controllability and observability Gramians can be achieved by solving reduced order Lyapunov equations. All discussed approaches can be used in conjunction with square-root and balancing-free accuracy enhancing techniques. For a selected class of methods robust numerical software is available.

## 9.1 Introduction

The design of low order controllers for high order plants is a challenging problem both theoretically as well as from a computational point of view. The advanced controller design methods like the LQG/LTR loop-shaping, $H_\infty$-synthesis, $\mu$ and linear matrix inequalities based synthesis methods produce typically controllers with orders comparable with the order of the plant. Therefore, the orders of these controllers tend often to be too high for practical use, where simple controllers are preferred over complex ones. To allow the practical applicability of advanced controller design methods for high order systems, the model reduction methods capable to address controller reduction problems are of primary importance. Comprehensive presentations of controller reduction methods and the reasons behind different approaches can be found in the textbook [ZDG96] and in the monograph [OA00].

The goal of controller reduction is to determine a low order controller starting from a high order one to ensure that the closed loop system formed from the original (high order) plant and low order controller behaves like the original plant with the original high order controller. Thus a basic requirement for controller reduction is preserving the closed-loop stability and many controller reduction approaches have been derived to fulfil just this goal [AL89, LAL90]. However, to be useful, the low order controller resulting in this way must provide an acceptable performance degradation of the closed loop behavior. This led to methods which try to enforce also the preservation of closed-loop performance [AL89, GG98, Gu95, WSL01, EJL01].

In our presentation we focus on controller reduction methods related to balancing techniques. The *balanced truncation* (BT) based approach proposed in [Moo81] is a general method to reduce the order of stable systems. Bounds on the additive approximation errors have been derived in [Enn84, Glo84] and they theoretically establish the remarkable approximation properties of this approach. In a series of papers [LHPW87, TP87, SC89, Var91b] the underlying numerical algorithms for this method have been progressively improved and accompanying robust numerical software is freely available [Var01a]. The main computations in the so-called *square-root* and *balancing-free* accuracy enhancing method of [Var91b] is the high-accuracy computation of the controllability/observability Gramians (using square-root techniques) and employing well-conditioned truncation matrices (via a balancing-free approach). Note that the BT method is able to handle the reduction of unstable systems either via modal decomposition or coprime factorization techniques [Wal90, Var93]. A closely related approach is the *singular perturbation approximation* (SPA) [LA89] which later has been turned into a reliable computational technique in [Var91a].

Controller reduction problems are often formulated as frequency-weighted model reduction problems [AL89]. An extension of balancing techniques to address *frequency-weighted model reduction* (FWMR) problems has been proposed in [Enn84] by defining so-called frequency-weighted controllability and observability Gramians. The main difficulty with this method, is the lack of guarantee of stability of the reduced models in the case of two-sided weighting. To overcome this weakness, several improvements of the basic method of [Enn84] have been suggested in [LC92, WSL99, VA03], by proposing alternative choices of the frequency-weighted controllability and observability Gramians and/or employing the SPA approach instead of BT. Although still no *a priory* approximation error bounds for this method exist, the *frequency-weighted balanced truncation* (FWBT) or *frequency-weighted singular perturbation approximation* (FWSPA) approaches with the proposed enhancements are well-suited to solve many controller reduction problems. In contrast, *Hankel-norm approximation* (HNA) related approaches [Glo84, LA85] appear to be less suited for this class of problems due to special requirements to be fulfilled by the weights (e.g., anti-stable and anti-minimum-phase).

The recent developments in computational algorithms for controller reduction focus on fully exploiting the structural features of the *frequency-weighted controller reduction* (FWCR) problems [VA03, Var03b, Var03a]. In these papers it is shown that for several categories of performance and stability enforcing frequency-weights, the computation of the frequency-weighted controllability and observability Gramians can be done by solving reduced order

Lyapunov equations. Moreover, all discussed approaches can be used in conjunction with *square-root* and *balancing-free* accuracy enhancing techniques. For a selected class of methods robust numerical software is available.

The paper is organized as follows. In Section 9.2 we describe shortly the basic approaches to controller reduction. A general computational framework using balancing-related frequency-weighted methods is introduced in Section 9.3 and the related main aspects are addressed like the definition of frequency-weighted Gramians, using accuracy enhancing techniques, and algorithmic performance issues. The general framework is specialized to several controller reduction problems in Section 9.4, by addressing the reduction of both general as well as state feedback and observer-based controllers, in conjunction with various stability and performance preserving problem formulations. In each case, we discuss the applicability of square-root techniques and show the achievable computational effort saving by exploiting the problem structure. In Section 9.5 we present an overview of existing software. In Section 9.6 we present an example illustrating the typical controller reduction problematic.

**Notation.** Throughout the paper, the following notational convention is used. The bold letter notation $\mathbf{G}$ is used to denote a state-space system $\mathbf{G} := (A, B, C, D)$ with the *transfer-function matrix* (TFM)

$$G(\lambda) = C(\lambda I - A)^{-1}B + D := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Depending on the system type, $\lambda$ is either the complex variable $s$ appearing in the Laplace transform in the case of a continuous-time system or the variable $z$ appearing in the $Z$-transform in the case of a discrete-time system. Throughout the paper we denote $G(\lambda)$ simply as $G$, when the system type is not relevant. The bold-notation is used consistently to denote system realizations corresponding to particular TFMs: $\mathbf{G}_1\mathbf{G}_2$ denotes the series coupling of two systems having the TFM $G_1(\lambda)G_2(\lambda)$, $\mathbf{G}_1 + \mathbf{G}_2$ represents the (additive) parallel coupling of two systems with TFM $G_1(\lambda) + G_2(\lambda)$, $\mathbf{G^{-1}}$ represents the inverse systems with TFM $G^{-1}$, $[\,\mathbf{G}_1\ \mathbf{G}_2\,]$ represents the realization of the compound TFM $[\,G_1\ G_2\,]$, etc.

## 9.2 Controller Reduction Approaches

Let $\mathbf{K} = (A_c, B_c, C_c, D_c)$ be a stabilizing controller of order $n_c$ for an $n$-th order plant $\mathbf{G} = (A, B, C, D)$. We want to find $\mathbf{K}_r$, an $r_c$-th order approximation of $\mathbf{K}$ such that the reduced controller $\mathbf{K}_r$ is stabilizing and essentially preserves the closed-loop system performances of the original controller. To guarantee closed-loop stability, sometimes we would like to additionally preserve the same number of unstable poles in $\mathbf{K}_r$ as in $\mathbf{K}$.

To solve controller reduction problems, virtually any model reduction method in conjunction with the modal separation approach (to preserve the unstable poles) can be employed. However, when employing general purpose model reduction methods to perform controller order reduction, the closed-loop stability and performance aspects are completely ignored and the resulting controllers are usually unsatisfactory.

To address stability and performance preserving issues, controller reduction problems are frequently formulated as FWMR problems with special weights [AL89]. This amounts to find $\mathbf{K}_r$, the $r_c$-th order approximation of $\mathbf{K}$ (having possibly the same number of unstable poles as $\mathbf{K}$), such that a weighted error of the form

$$\|W_o(K - K_r)W_i\|_\infty, \tag{9.1}$$

is minimized, where $W_o$ and $W_i$ are suitably chosen weighting TFMs.

Commonly used frequency-weights (see Section 9.3 and [AL89]) have minimal state-space realizations of orders as large as $n + n_c$ and thus employing general FWMR techniques could be expensive for high order plants/controllers, because they involve the computation of Gramians for systems of order $n + 2n_c$. A possible approach to alleviate the situation is to reduce first the weights using any of the standard methods (e.g., BT, SPA or HNA) and then apply the general FWBT or FWSPA approach with the enhancements proposed in [VA03]. Although apparently never discussed in the literature, this approach could be effective in some cases.

The idea to apply frequency-weighted balancing techniques to reduce the stable coprime factors of the controller has been discussed in several papers [AL89, LAL90, ZC95]. For example, given a *right coprime factorization* (RCF) $K = UV^{-1}$ of the controller, we would like to find a reduced controller in the RCF form $K_r = U_r V_r^{-1}$ such that

$$\left\| W_o \begin{bmatrix} U - U_r \\ V - V_r \end{bmatrix} W_i \right\|_\infty = \min . \tag{9.2}$$

Similarly, given a *left coprime factorization* (LCF) $K = V^{-1}U$ of the controller, we would like to find a reduced controller in the LCF form $K_r = V_r^{-1}U_r$ such that

$$\left\| \widetilde{W}_o [\, U - U_r \;\; V - V_r \,] \widetilde{W}_i \right\|_\infty = \min . \tag{9.3}$$

In (9.2) and (9.3) the weights have usually special forms to enforce either closed-loop stability [AL89, LAL90] or to preserve the closed-loop performance bounds for $\mathcal{H}_\infty$ controllers [GG98, Gu95, WSL01, EJL01]. The main appeal of coprime factorization based techniques is that in many cases (e.g., feedback controllers resulting from LQG, $\mathcal{H}_2$ or $\mathcal{H}_\infty$ designs) fractional representations of the controller can be obtained practically without any computation from the underlying synthesis approach. For example, this is the case for state feedback and observer-based controllers as well as for $\mathcal{H}_\infty$ controllers.

Interestingly, many stability/performance preserving controller reduction problems have very special structure which can be exploited when developing efficient numerical algorithms for controller reduction. For example, it has been shown in [VA02] that for the frequency-weighted balancing related approaches applied to several controller reduction problems with the special stability/performance enforcing weights proposed in [AL89], the computation of Gramians can be done by solving reduced order Lyapunov equations. Similarly, it was recently shown in [Var03b] that this is also true for a class of frequency-weighted coprime factor controller reduction methods.

The approach which we pursue in this paper is the specialization of the FWMR methods to derive FWCR approaches which exploit all particular features of the underlying frequency-weighted problem. The main benefit of such a specialization in the case of arbitrary controllers is the cheaper computation of frequency-weighted Gramians by solving reduced order Lyapunov equations (typically of order $n+n_c$ instead the expected order $n+2n_c$). A further simplification arises when considering reduction of controllers resulting from LQG, $\mathcal{H}_2$ or $\mathcal{H}_\infty$ designs. For such controllers, the Gramians can be computed by solving Lyapunov equations only of order $n_c$. In what follows, we present an overview of recent enhancements obtained for different categories of problems. More details on each problem can be found in several recent works of the author [VA02, VA03, Var03b, Var03a].

## 9.3 Frequency-Weighted Balancing Framework

In this section we describe the general computational framework to perform FWCR using balancing-related approaches. The following procedure to solve the frequency-weighted approximation problem (9.1), with a possible unstable controller $\mathbf{K}$, is applicable (with obvious replacements) to solve the coprime factor approximation problems (9.2) and (9.3) as well, where obvious simplifications arise because the factors are stable systems.

**FWCR Procedure.**

1. Compute the additive stable-unstable spectral decomposition
$$\mathbf{K} = \mathbf{K}_s + \mathbf{K}_u,$$
where $\mathbf{K}_s$, of order $n_{cs}$, contains the stable poles of $\mathbf{K}$ and $\mathbf{K}_u$, of order $n_c - n_{cs}$, contains the unstable poles of $\mathbf{K}$.
2. Compute the controllability Gramian of $\mathbf{K}_s\mathbf{W}_i$ and the observability Gramian of $\mathbf{W}_o\mathbf{K}_s$ and define, according to [Enn84], [WSL99] or [VA03], appropriate $n_{cs}$ order frequency-weighted controllability and observability Gramians $P_w$ and $Q_w$, respectively.
3. Using $P_w$ and $Q_w$ in place of standard Gramians of $\mathbf{K}_s$, determine a reduced order approximation $\mathbf{K}_{sr}$ by applying the BT or SPA methods.
4. Form $\mathbf{K}_r = \mathbf{K}_{sr} + \mathbf{K}_u$.

This procedure originates from the works of Enns [Enn84] and automatically ensures that the resulting reduced order controller $\mathbf{K}_r$ has exactly the same unstable poles as the original one, provided the approximation $\mathbf{K}_{sr}$ of the stable part $\mathbf{K}_s$ is stable. To guarantee the stability of $\mathbf{K}_{sr}$, specific choices of frequency-weighted Gramians have been proposed in [VA03] to enhance the original method proposed by Enns. In the following subsection, we present shortly the possible choices of the frequency-weighted controllability and observability Gramians to be employed in the **FWCR Procedure** and indicate the related computational aspects when employed in conjunction with square-root techniques.

### 9.3.1 Frequency-Weighted Gramians

To simplify the discussions we temporarily assume that the controller $\mathbf{K} = (A_c, B_c, C_c, D_c)$ is stable and the two weights $W_o$ and $W_i$ are also stable TFMs having minimal realizations of orders $n_o$ and $n_i$, respectively. In the case of an unstable controller, the discussion applies to the stable part $\mathbf{K}_s$ of the controller.

Consider the minimal realizations of the frequency weights

$$\mathbf{W_o} = (A_o, B_o, C_o, D_o), \quad \mathbf{W_i} = (A_i, B_i, C_i, D_i)$$

and construct the realizations of $\mathbf{KW}_i$ and $\mathbf{W_oK}$ as

$$\mathbf{KW_i} = \left[\begin{array}{c|c} \overline{A}_i & \overline{B}_i \\ \hline \overline{C}_i & \overline{D}_i \end{array}\right] =: \left[\begin{array}{cc|c} A_c & B_cC_i & B_cD_i \\ 0 & A_i & B_i \\ \hline C_c & D_cC_i & D_cD_i \end{array}\right], \tag{9.4}$$

$$\mathbf{W_oK} = \left[\begin{array}{c|c} \overline{A}_o & \overline{B}_o \\ \hline \overline{C}_o & \overline{D}_o \end{array}\right] =: \left[\begin{array}{cc|c} A_o & B_oC_c & B_oD_c \\ 0 & A_c & B_c \\ \hline C_o & D_oC_c & D_oD_c \end{array}\right]. \tag{9.5}$$

Let $\overline{P}_i$ and $\overline{Q}_o$ be the controllability Gramian of $\mathbf{KW_i}$ and the observability Gramian of $\mathbf{W_oK}$, respectively. Depending on the system type, continuous-time (c) or discrete-time (d), $\overline{P}_i$ and $\overline{Q}_o$ satisfy the corresponding Lyapunov equations

$$(c) \begin{cases} \overline{A}_i\overline{P}_i + \overline{P}_i\overline{A}_i^T + \overline{B}_i\overline{B}_i^T = 0 \\ \overline{A}_o^T\overline{Q}_o + \overline{Q}_o\overline{A}_o + \overline{C}_o^T\overline{C}_o = 0 \end{cases}, \quad (d) \begin{cases} \overline{A}_i\overline{P}_i\overline{A}_i^T + \overline{B}_i\overline{B}_i^T = \overline{P}_i \\ \overline{A}_o^T\overline{Q}_o\overline{A}_o + \overline{C}_o^T\overline{C}_o = \overline{Q}_o \end{cases}. \tag{9.6}$$

Partition $\overline{P}_i$ and $\overline{Q}_o$ in accordance with the structure of the matrices $\overline{A}_i$ and $\overline{A}_o$, respectively, i.e.

$$\overline{P}_i = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad \overline{Q}_o = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \tag{9.7}$$

where $P_E := P_{11}$ and $Q_E := Q_{22}$ are $n_c \times n_c$ matrices. The approach proposed by Enns [Enn84] defines

$$P_w = P_E, \quad Q_w = Q_E \tag{9.8}$$

as the frequency-weighted controllability and observability Gramians, respectively. Although successfully employed in many applications, the stability of the reduced controller is not guaranteed in the case of two-sided weighting, unless either $W_o = I$ or $W_i = I$. Occasionally, quite poor approximations result even for one-sided weighting.

In the context of FWMR, alternative choices of frequency-weighted Gramians guaranteeing stability have been proposed in [LC92] and [WSL99] (only for continuous-time systems). The choice proposed in [LC92] assumes that no pole-zero cancellations occur when forming $\mathbf{KW_i}$ and $\mathbf{W_oK}$, a condition which generally is not fulfilled by the special weights used in controller reduction problems. The alternative choice of [WSL99] has been improved in [VA03] by reducing the gap to Enns' choice and also extended to discrete-time systems.

The Gramians $P_w$ and $Q_w$ in the modified method of Enns proposed in [VA03] are determined as

$$P_w = P_V, \quad Q_w = Q_V, \tag{9.9}$$

where $P_V$ and $Q_V$ are the solutions of the appropriate pair of Lyapunov equations

$$(c) \left\{ \begin{array}{l} A_c P_V + P_V A_c^T + \widetilde{B}_c \widetilde{B}_c^T = 0 \\ Q_V A_c + A_c^T Q_V + \widetilde{C}_c^T \widetilde{C}_c = 0 \end{array} \right. , \quad (d) \left\{ \begin{array}{l} A_c P_V A_c^T + \widetilde{B}_c \widetilde{B}_c^T = P_V \\ A_c^T Q_V A_c + \widetilde{C}_c^T \widetilde{C}_c = Q_V \end{array} \right. . \tag{9.10}$$

Here, $\widetilde{B}_c$ and $\widetilde{C}_c$ are fictitious input and output matrices determined from the orthogonal eigendecompositions of the symmetric matrices $X$ and $Y$ defined as

$$(c) \left\{ \begin{array}{l} X = -A_c P_E - P_E A_c^T \\ Y = -A_c^T Q_E - Q_E A_c \end{array} \right. , \quad (d) \left\{ \begin{array}{l} X = -A_c P_E A_c^T + P_E \\ Y = -A_c^T Q_E A_c + Q_E \end{array} \right. . \tag{9.11}$$

The eigendecompositions of $X$ and $Y$ are given by

$$X = U \Theta U^T, \quad Y = V \Gamma V^T, \tag{9.12}$$

where $\Theta$ and $\Gamma$ are real diagonal matrices. Assume that $\Theta = \operatorname{diag}(\Theta_1, \Theta_2)$ and $\Gamma = \operatorname{diag}(\Gamma_1, \Gamma_2)$ are determined such that $\Theta_1 > 0$ and $\Theta_2 \leq 0$, $\Gamma_1 > 0$ and $\Gamma_2 \leq 0$. Partition $U = [\, U_1 \; U_2 \,]$ and $V = [\, V_1 \; V_2 \,]$ in accordance with the partitioning of $\Theta$ and $\Gamma$, respectively. Then $\widetilde{B}$ and $\widetilde{C}$ are defined in [VA03] as

$$\widetilde{B}_c = U_1 \Theta_1^{\frac{1}{2}}, \quad \widetilde{C}_c = \Gamma_1^{\frac{1}{2}} V_1^T. \tag{9.13}$$

It is easy to see that with this choice of Gramians we have $P_V - P_E \geq 0$ and $Q_V - Q_E \geq 0$, thus, the triple $(A_c, \widetilde{B}_c, \widetilde{C}_c)$ is minimal provided the original triple $(A_c, B_c, C_c)$ is minimal. Note that any combination of Gramians $(P_E, Q_V)$, $(P_V, Q_E)$, or $(P_V, Q_V)$ guarantees the stability of approximations for two-sided weighting.

### 9.3.2 Accuracy Enhancing Techniques

There are two main techniques to enhance the accuracy of computations in model and controller reduction. One of them is the *square-root* technique introduced in [TP87] and relies on computing exclusively with better conditioned "square-root" quantities, namely, with the Cholesky factors of Gramians, instead of the Gramians themselves. In the context of unweighted additive error model reduction (e.g., employing BT, SPA or HNA methods), this involves to solve the Lyapunov equations satisfied by the Gramians directly for their Cholesky factors by using the well-know algorithms proposed by Hammarling [Ham82]. This is not generally possible in the case of FWMR/FWCR since the frequency-weighted Gramians $P_w$ and $Q_w$ are "derived" quantities defined, for example, via (9.8) or (9.9). In this subsection we show how square-root formulas can be employed to compute the frequency-weighted Gramians for the specific choices described in the previous subsection.

Assume $\overline{S}_i$ and $\overline{R}_o$ are the Cholesky factors of $\overline{P}_i$ and $\overline{Q}_o$ in (9.7), respectively, satisfying $\overline{P}_i = \overline{S}_i\overline{S}_i^T$ and $\overline{Q}_o = \overline{R}_o^T\overline{R}_o$. These factors are upper triangular and can be computed using the method of Hammarling [Ham82] to solve the Lyapunov equations (9.6) directly for the Cholesky factors. The solution of these Lyapunov equations involves the reduction of each of the matrices $\overline{A}_i$ and $\overline{A}_o$ to a *real Schur form* (RSF). For efficiency reasons the reduction of $A$, $A_i$ and $A_o$ to RSF is preferably done independently and only once. This ensures that $\overline{A}_i$ and $\overline{A}_o$ in the realizations (9.4) of $\mathbf{KW_i}$ and (9.5) of $\mathbf{W_oK}$ are automatically in RSF.

If we partition $\overline{S}_i$ and $\overline{R}_o$ in accordance with the partitioning of $\overline{P}_i$ and $\overline{Q}_o$ in (9.7) as

$$\overline{S}_i = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad \overline{R}_o = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$

we have immediately that the Cholesky factors of $P_E = S_E S_E^T$ and $Q_E = R_E^T R_E$ corresponding to Enns' choice satisfy

$$S_E S_E^T = S_{11}S_{11}^T + S_{12}S_{12}^T = [\, S_{11}\ S_{12}\,][\, S_{11}\ S_{12}\,]^T, \tag{9.14}$$

$$R_E^T R_E = R_{12}^T R_{12} + R_{22}^T R_{22} = \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}^T \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}. \tag{9.15}$$

Thus, to obtain $S_E$ the RQ-factorization of the matrix $[\, S_{11}\ S_{12}\,]$ must be additionally performed, while for obtaining $R_E$ the QR-factorization of $[\, R_{12}^T\ R_{22}^T\,]^T$ must be performed. Both these factorizations can be computed using well established factorization updating techniques [GGMS74] which fully exploit the upper triangular shapes of $S_{11}$ and $R_{22}$.

For the choice (9.9) of Gramians, the Cholesky factors of $P_V = S_V S_V^T$ and $Q_V = R_V^T R_V$ result by solving (9.10) directly for these factors using the algorithm of Hammarling [Ham82]. Note that for computing $X$ and $Y$, we can use the Cholesky factors $S_E$ and $R_E$ determined above for Enns' choice.

Assume that $P_w = S_w S_w^T$ and $Q_w = R_w^T R_w$ are the Cholesky factorizations of the frequency weighted Gramians corresponding to one of the above choices of the Gramians (9.8) or (9.9). To determine the reduced order controller we determine two *truncation matrices* $L$ and $T$ such that the reduced controller is given by

$$(A_{cr}, B_{cr}, C_{cr}, D_{cr}) = (LA_cT,\ LB_c,\ C_cT,\ D_c).$$

The computation of $L$ and $T$ can be done from the singular value decomposition (SVD)

$$R_w S_w = \begin{bmatrix} U_1 \ U_2 \end{bmatrix} \operatorname{diag}(\Sigma_1, \Sigma_2) \begin{bmatrix} V_1 \ V_2 \end{bmatrix}^T, \qquad (9.16)$$

where

$$\Sigma_1 = \operatorname{diag}(\sigma_1, \ldots, \sigma_{r_c}), \quad \Sigma_2 = \operatorname{diag}(\sigma_{r_c+1}, \ldots, \sigma_{n_c}),$$

and $\sigma_1 \geq \ldots \geq \sigma_{r_c} > \sigma_{r_c+1} \geq \ldots \geq \sigma_{n_c} \geq 0$. To compute the SVD in (9.16), instead of using standard algorithms as those described in [GV89], special numerically stable algorithms for matrix products can be employed to avoid the forming of the product $R_w S_w$ [GSV00].

The so-called *square-root* (**SR**) methods determine $L$ and $T$ as [TP87]

$$L = \Sigma_1^{-1/2} U_1^T R_w, \qquad T = S_w V_1 \Sigma_1^{-1/2}. \qquad (9.17)$$

A potential disadvantage of this choice is that accuracy losses can be induced in the reduced controller if either of the truncation matrices $L$ or $T$ is ill-conditioned (i.e., nearly rank deficient). Note that in the case of BT based model reduction, the above choice leads, in the continuous-time, to *balanced* reduced models (i.e., the corresponding Gramians are equal and diagonal).

The second technique to enhance accuracy is the computation of well-conditioned truncation matrices $L$ and $T$, by avoiding completely any kind of balancing implied by using the (**SR**) formulas (9.17). This leads to a *balancing-free* (**BF**) approach (originally proposed in [SC89]) in which $L$ and $T$ are always well-conditioned. A *balancing-free square-root* (**BFSR**) algorithm which combines the advantages of the **BF** and **SR** approaches has been introduced in [Var91b]. $L$ and $T$ are determined as

$$L = (Y^T X)^{-1} Y^T, \qquad T = X,$$

where $X$ and $Y$ are $n_c \times r_c$ matrices with orthogonal columns computed from two QR decompositions

$$S_w V_1 = XW, \quad R_w^T U_1 = YZ$$

with $W$ and $Z$ non-singular and upper-triangular. The reduced controller obtained in this way is related to that one obtained by the **SR** approach by a non-orthogonal state coordinate transformation. Since the accuracy of the **BFSR** algorithm is usually better than either of **SR** or **BF** techniques, this approach is the default option in high performance controller reduction software (see Section 9.5).

Assume now that the singular value decomposition of $R_w S_w$ is

$$R_w S_w = \left[\, U_1 \; U_2 \; U_3 \,\right] \operatorname{diag}(\Sigma_1, \Sigma_2, 0) \left[\, V_1 \; V_2 \; V_3 \,\right]^T,$$

where

$$\Sigma_1 = \operatorname{diag}(\sigma_1, \ldots, \sigma_{r_c}), \quad \Sigma_2 = \operatorname{diag}(\sigma_{r_c+1}, \ldots, \sigma_{\overline{n}_c}),$$

and $\sigma_1 \geq \ldots \geq \sigma_{r_c} > \sigma_{r_c+1} \geq \ldots \geq \sigma_{\overline{n}_c} > 0$. Assume we employ the **SR** formulas to compute a minimal realization of the controller of order $\overline{n}_c$ as

$$\left[\begin{array}{c|c} LA_cT & LB_c \\ \hline C_cT & D_c \end{array}\right] = \left[\begin{array}{cc|c} A_{c,11} & A_{c,12} & B_{c,1} \\ A_{c,21} & A_{c,22} & B_{c,2} \\ \hline C_{c,1} & C_{c,2} & D_c \end{array}\right],$$

where the system matrices are compatibly partitioned with $A_{c,11} \in R^{r_c \times r_c}$. The SPA method (see [LA89]) determines the reduced controller matrices as

$$\left[\begin{array}{c|c} A_{cr} & B_{cr} \\ \hline C_{cr} & D_{cr} \end{array}\right] = \left[\begin{array}{c|c} A_{c,11} - A_{c,12}A_{c,22}^{-1}A_{c,21} & B_{c,1} - A_{c,12}A_{c,22}^{-1}B_{c,2} \\ \hline C_{c,1} - C_{c,2}A_{c,22}^{-1}A_{c,21} & D_c - C_{c,2}A_{c,22}^{-1}B_{c,2} \end{array}\right].$$

This approach has been termed the **SR** SPA method. Note that the resulting reduced controller is in a balanced state-space coordinate form both in continuous- as well as in discrete-time cases.

A **SRBF** version of the SPA method has been proposed in [Var91a] to combine the advantages of the **BF** and **SR** approaches. The truncation matrices $L$ and $T$ are determined as

$$L = \begin{bmatrix} (Y_1^T X_1)^{-1} Y_1^T \\ (Y_2^T X_2)^{-1} Y_2^T \end{bmatrix}, \qquad T = [\, X_1 \; X_2 \,],$$

where $X_1$ and $Y_1$ are $\overline{n}_c \times r_c$ matrices, and $X_2$ and $Y_2$ are $\overline{n}_c \times (\overline{n}_c - r_c)$ matrices. All these matrices with orthogonal columns are computed from the QR decompositions

$$S_w V_i = X_i W_i, \quad R_w^T U_i = Y_i Z_i, \quad i = 1, 2$$

with $W_i$ and $Z_i$ non-singular and upper-triangular.

### 9.3.3 Algorithmic Efficiency Issues

The two main computational problems of controller reduction by using the frequency weighted BT or SPA approaches are the determination of frequency-weighted Gramians and the computation of the corresponding truncation matrices. All computation ingredients for these computations are available as robust numerical implementations either in the LAPACK [ABB99] or SLICOT [BMSV99] libraries. To compare the effectiveness of different methods, we roughly evaluate in what follows the required computational effort for the main computations in terms of required *floating-point operations* (*flops*). Note that 1 *flop* corresponds to 1 addition/subtraction or 1 multiplication/division performed on the floating point processor. In our evaluations we tacitly assume that the number of system inputs $m$ and system outputs $p$ satisfy $m, p \ll n_c$, thus many computations involving the input and output matrices (e.g., products) are negligible.

The main computational ingredient for computing Gramians is the solution of Lyapunov equations as those in (9.6). This involves the reduction of the matrices $\overline{A}_i$ and $\overline{A}_o$ to the *real Schur form* (RSF) using the Francis' QR-algorithm [GV89]. By exploiting the block upper triangular structure of these matrices, this reduction can be performed by reducing independently $A_i$, $A_c$ and $A_o$, which amounts to about $25n_i^3$, $25n_c^3$ and $25n_o^3$ *flops*, respectively. The Cholesky factors $\overline{S}_i$ and $\overline{R}_o$ of Gramians $\overline{P}_i$ and $\overline{Q}_o$ in (9.6) can be computed using the method of Hammarling [Ham82] and this requires about $8(n_i+n_c)^3$ and $8(n_o+n_c)^3$ *flops*, respectively. The computation of the Cholesky factors $S_E$ and $R_E$ using the algorithm of [GGMS74] for the updating formulas (9.14) and (9.15) requires additionally about $2n_i n_c^2$ and $2n_o n_c^2$ *flops*, respectively. Thus, the computation of the pair $(S_E, R_E)$ requires

$$N_E = 25(n_i^3 + n_c^3 + n_o^3) + 8(n_i + n_c)^3 + 8(n_o + n_c)^3 + 2(n_i + n_o)n_c^2 \quad (9.18)$$

*flops*. Note that $N_E$ represents the cost of evaluating Gramians when applying the FWBT or FWSPA approaches to solve the controller reduction problem as a general FWMR problem, without any structure exploitation. In certain problems with two-sided weights, the input and output weights share the same state matrix. In this case $n_i = n_o$ and $N_E$ reduces with $25n_i^3$ *flops*.

The computation of one of the factors $S_V$ (or $R_V$) corresponding to the modified Lyapunov equations (9.10) requires up to $19.5n_c^3$ *flops*, of which about $9n_c^3$ *flops* account for the eigendecomposition of $X$ in (9.12) to form the constant term of the Lyapunov equation satisfied by $P_V$ and $8n_c^3$ *flops* account to solve the Lyapunov equation (9.10) for the factor $S_V$. Note that the reduction of $A_c$ to a RSF is performed only once, when computing the factors $S_E$ and $R_E$. The additional number of operations required by different choices of the frequency-weighted Gramians is

$$N_V = \begin{cases} 0, & (S_w, R_w) = (S_E, R_E) \\ 19.5n_c^3, & (S_w, R_w) = (S_V, R_E) \text{ or } (S_w, R_w) = (S_E, R_V) \\ 39n_c^3, & (S_w, R_w) = (S_V, R_V) \end{cases} .$$

The determination of the truncation matrices $L$ and $T$ involves the computation of the singular value decomposition of the $n_c \times n_c$ matrix $R_w S_w$, which requires at least $N_T = 22n_c^3$ *flops*. The rest of computations is negligible if $r_c \ll n_c$.

From the above analysis it follows that for $n_i$ and $n_o$ of comparable sizes with $n_c$, the term $N_E$, which accounts for the computations of the Cholesky factors for Enns' choice of the frequency weighted Gramians, has the largest contribution to $N_{tot} = N_E + N_V + N_T$, the total number of operations. Note that $N_V + N_T$ depends only on the controller order $n_c$ and the choice of Gramian modification scheme, thus this part of $N_{tot}$ appears as "constant" in all evaluations of the computational efforts. It is interesting to see the relative values of $N_E$ and $N_{tot}$ for some typical cases. For an unweighted controller reduction problem $N_E = 41n_c^3$ and $N_{tot} = 63n_c^3$, thus $N_E/N_{tot} = 0.65$. These values of $N_E$ and $N_{tot}$ can be seen as lower limits for all controller reduction problems using balancing related approaches. In the case when $n_i, n_o \ll n_c$, $N_E \approx 41n_c^3$ and $63n_c^3 \leq N_{tot} \leq 102n_c^3$, thus in this case $0.40 \leq N_E/N_{tot} \leq 0.65$. At the other extreme, assuming the typical values of $n_c = n$, $n_i = n_o = 2n$ for a state feedback and observer-based controller, we have $N_E = 865n^3$ and $887n_c^3 \leq N_{tot} \leq 926n_c^3$, and thus the ratio of $N_E/N_{tot}$ satisfies $0.93 \leq N_E/N_{tot} \leq 0.98$. These figures show that solving FWCR problems can be tremendously expensive when employing general purpose model reduction algorithms. In the following sections we show that for several classes of controller reduction problems, structure exploitation can lead to significant computation savings expressed by much smaller values of $N_E$.

## 9.4 Efficient Solution of Controller Reduction Problems

To develop efficient numerical methods for controller reduction, the general framework for controller reduction described in the previous section needs to be specialized to particular classes of problems by fully exploiting the underlying problem structures. When deriving efficient specialized versions of the **FWCR Algorithm**, the main computational saving arises in determining the frequency-weighted Gramians for each particular case via the corresponding Cholesky factors. In what follows we consider several controller reduction problems with particular weights and give the main results concerning the computation of Gramians. We focus only on Enns' choice, since it enters also in all other alternative choices discussed in the previous section.

### 9.4.1 Frequency-Weighted Controller Reduction

We consider the solution of the FWCR problem (9.1) for the specific stability and performance preserving weights discussed in [AL89]. To enforce closed-loop stability, one-sided weights of the form

$$\text{SW1:} \qquad W_o = (I + GK)^{-1}G, \quad W_i = I, \tag{9.19}$$

or

$$\text{SW2:} \qquad W_o = I, \quad W_i = G(I + KG)^{-1}, \tag{9.20}$$

can be used, while performance-preserving considerations lead to two-sided weights

$$\text{PW:} \qquad W_o = (I + GK)^{-1}G, \quad W_i = (I + GK)^{-1}\,, \tag{9.21}$$

The unweighted reduction corresponds to the weights

$$\text{UW:} \qquad W_o = I, \quad W_i = I. \tag{9.22}$$

It can be shown (see [ZDG96]), that for the weights (9.19) and (9.20) the stability of the closed-loop system is guaranteed if $\|W_o(K - K_r)W_i\|_\infty < 1$, provided $K$ and $K_r$ have the same number of unstable poles. Similarly, minimizing $\|W_o(K - K_r)W_i\|_\infty$ for the weights in (9.21) ensures the best matching of the closed-loop TFM for a given order of $K_r$.

To solve the FWCR problems corresponding to the above weights, we consider both the case of a general stabilizing controller as well as the case of state feedback and observer-based controllers. In each case we show how to compute efficiently the Cholesky factors of frequency-weighted Gramians in order to apply the **SR** and **SRBF** accuracy enhancing techniques. Finally, we give estimates of the necessary computational efforts and discuss the achieved saving by using structure exploitation.

### General Controller

Since the controller can be generally unstable, only the stable part of the controller is reduced and a copy of the unstable part is kept in the reduced controller. Therefore, we assume a state-space representation of the controller with $A_c$ already reduced to a block-diagonal form

$$\mathbf{K} = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{cc|c} A_{c1} & 0 & B_{c1} \\ 0 & A_{c2} & B_{c2} \\ \hline C_{c1} & C_{c2} & D_c \end{array}\right], \tag{9.23}$$

where $\Lambda(A_{c1}) \subset \mathrm{C}^+$ and $\Lambda(A_{c2}) \subset \mathrm{C}^-$. Here $\mathrm{C}^-$ denotes the open left half complex plane of C in a continuous-time setting or the interior of the unit circle in a discrete-time setting, while $\mathrm{C}^+$ denotes the complement of $\mathrm{C}^-$ in C. The above form corresponds to an additive decomposition of the controller TFM as $K = K_u + K_s$, where $\mathbf{K}_u = (A_{c1}, B_{c1}, C_{c1}, 0)$ contains the unstable poles of $\mathbf{K}$ and $\mathbf{K}_s = (A_{c2}, B_{c2}, C_{c2}, D_c)$, of order $n_{cs}$, contains the stable poles of $\mathbf{K}$.

For our developments, we build the state matrix of the realizations of the weights in (9.19), (9.20), or (9.21) in the form

$$A_w = \begin{bmatrix} A - BD_cR^{-1}C & B\widetilde{R}^{-1}C_c \\ -B_cR^{-1}C & A_c - B_cR^{-1}DC_c \end{bmatrix},$$

where $R = I + DD_c$ and $\widetilde{R} = I + D_cD$. Since the controller is stabilizing, $A_w$ has all its eigenvalues in $\mathrm{C}^-$.

The following theorem, proved in [VA02], extends the results of [LAL90, SM96] to the case of an arbitrary stabilizing controller:

**Theorem 9.4.1** *For a given $n$-th order system $\mathbf{G} = (A, B, C, D)$ assume that $\mathbf{K} = (A_c, B_c, C_c, D_c)$ is an $n_c$-th order stabilizing controller with $I + DD_c$ nonsingular. Then the frequency-weighted Gramians for Enns' method [Enn84] applied to the frequency-weighted controller reduction problems with weights defined in (9.19), (9.20), or (9.21) can be computed by solving the corresponding Lyapunov equations of order at most $n + n_c$ as follows:*

1. *For $W_o = (I + GK)^{-1}G$ and $W_i = I$, $P_E$ satisfies*

$$(c)\ A_{c2}P_E + P_E A_{c2}^T + B_{c2}B_{c2}^T = 0, \quad (d)\ A_{c2}P_E A_{c2}^T + B_{c2}B_{c2}^T = P_E \tag{9.24}$$

   *and $Q_E$ is the $n_{cs} \times n_{cs}$ trailing block of $Q_o$ satisfying*

$$(c)\ A_w^T Q_o + Q_o A_w + C_o^T C_o = 0, \quad (d)\ A_w^T Q_o A_w + C_o^T C_o = Q_o \tag{9.25}$$

   *with $C_o = \begin{bmatrix} -R^{-1}C & -R^{-1}DC_c \end{bmatrix}$.*
2. *For $W_o = I$ and $W_i = G(I + GK)^{-1}$, $P_E$ is the $n_{cs} \times n_{cs}$ trailing block of $P_i$ satisfying*

$$(c)\ A_w P_i + P_i A_w^T + B_i B_i^T = 0, \quad (d)\ A_w P_i A_w^T + B_i B_i^T = P_i \tag{9.26}$$

   *with $B_i = \begin{bmatrix} -B\widetilde{R}^{-1} \\ B_c D\widetilde{R}^{-1} \end{bmatrix}$ and $Q_E$ satisfies*

$$(c)\ A_{c2}^T Q_E + Q_E A_{c2} + C_{c2}^T C_{c2} = 0, \quad (d)\ A_{c2}^T Q_E A_{c2} + C_{c2}^T C_{c2} = Q_E \tag{9.27}$$

3. *For $W_o = (I+GK)^{-1}G$ and $W_i = (I+GK)^{-1}$, $P_E$ is the $n_{cs} \times n_{cs}$ trailing block of $P_i$ satisfying (9.26) with $B_i = \begin{bmatrix} BD_cR^{-1} \\ B_cR^{-1} \end{bmatrix}$ and $Q_E$ is the $n_{cs} \times n_{cs}$ trailing block of $Q_o$ satisfying (9.25).*

### State Feedback and Observer-Based Controller

Simplifications arise also in the case of a state feedback and full order observer-based controller of the form

$$\mathbf{K} = \left[\begin{array}{c|c} A + BF + LC + LDF & -L \\ \hline F & 0 \end{array}\right]. \tag{9.28}$$

The following result extends *Lemma 1* of [LAL90] to the case of possibly unstable controllers.

**Corollary 9.4.2** *For a given $n$-th order system $\mathbf{G} = (A, B, C, D)$ suppose that $F$ is a state feedback gain and $L$ is a state estimator gain, such that $A + BF$ and $A + LC$ are stable. Then the frequency-weighted Gramians for Enns' method [Enn84] applied to the frequency-weighted controller reduction problems with weights defined in (9.19), (9.20), or (9.21) can be computed by solving Lyapunov equations of order at most $2n$.*

In the case of state feedback and observer-based controllers important computational effort saving results if we further exploit the problem structure. In this case

$$A_w = \begin{bmatrix} A & BF \\ -LC & A+BF+LC \end{bmatrix}$$

and this matrix can be put in an upper block diagonal form using the transformation matrix

$$T = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}.$$

We obtain the transformed matrices $\widetilde{A}_w := T^{-1}A_wT$, $\widetilde{B}_i := T^{-1}B_i$, and $\widetilde{C}_o := C_oT$, where

$$\widetilde{A}_w = \begin{bmatrix} A+BF & BF \\ 0 & A+LC \end{bmatrix}.$$

If $\widetilde{P}_i$ and $\widetilde{Q}_o$ satisfy

$$(c) \begin{cases} \widetilde{A}_w\widetilde{P}_i + \widetilde{P}_i\widetilde{A}_w^T + \widetilde{B}_i\widetilde{B}_i^T = 0 \\ \widetilde{A}_w^T\widetilde{Q}_o + \widetilde{Q}_o\widetilde{A}_w + \widetilde{C}_o^T\widetilde{C}_o = 0 \end{cases}, \quad (d) \begin{cases} \widetilde{A}_w\widetilde{P}_i\widetilde{A}_w^T + \widetilde{B}_i\widetilde{B}_i^T = \widetilde{P}_i \\ \widetilde{A}_w^T\widetilde{Q}_o\widetilde{A}_w + \widetilde{C}_o^T\widetilde{C}_o = \widetilde{Q}_o \end{cases}, \quad (9.29)$$

then $P_i$ in (9.26) and $Q_o$ in (9.25) are given by $P_i = T\widetilde{P}_iT^T$ and $Q_o = T^{-T}\widetilde{Q}_oT^{-1}$, respectively. The computational saving arises from the need to reduce $A_w$ to a RSF when solving the Lyapunov equations (9.25) and (9.26). Instead of reducing the $2n \times 2n$ matrix $A_w$, we can reduce two $n \times n$ matrices $A + BF$ and $A + LC$ to obtain $\widetilde{A}_w$ in a RSF. This means a 4 times speedup of computations for this step.

### Square-Root Techniques

We can employ the method of [Ham82] to solve (9.26) and (9.25) directly for the Cholesky factors $S_i$ of $P_i = S_iS_i^T$ and $R_o$ of $Q_o = R_o^TR_o$, respectively. In the case of an unstable controller, we assume a state-space realization of $\mathbf{K}$ as in (9.23) with the $n_{cs} \times n_{cs}$ matrix $A_{c2}$ containing the stable eigenvalues of $A_c$. If we partition $S_i$ and $R_o$ in the form

$$S_i = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad R_o = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix},$$

where both $S_{22}$ and $R_{22}$ are $n_{cs} \times n_{cs}$, then the Cholesky factor of the trailing block of $P_i$ in (9.26) corresponding to the stable part of $\mathbf{K}$ is simply $S_E = S_{22}$, while the Cholesky factor $R_E$ of the trailing block of $Q_o$ in (9.25) satisfies $R_E^TR_E = R_{22}^TR_{22} + R_{12}^TR_{12}$. Thus the computation of $R_E$ involves an additional QR-decomposition of $[\,R_{22}^T\ R_{12}^T\,]^T$ and can be computed using standard updating techniques [GGMS74]. Updating can be avoided in the case of the one-sided weight $W_o = (I + GK)^{-1}G$, by using alternative state-space realizations of $\mathbf{W_o}$ and $\mathbf{K}$. For details, see [VA02]. Still in the case of two-sided weighting with $W_o = (I + GK)^{-1}G$ and $W_i = (I + GK)^{-1}$ we prefer the approach of the Theorem 9.4.1 with $\mathbf{W_i}$ and $\mathbf{W_o}$ sharing the same state matrix $A_w$, because the computation of both Gramians can be done with a single reduction of this $(n + n_c) \times (n + n_c)$ matrix to the RSF. In this case the cost to compute the two Gramians is only slightly larger than for one Gramian.

For a state feedback and full order observer-based controller, let $\widetilde{S}_i$ be the Cholesky factor of $\widetilde{P}_i$ in (9.29) partitioned as

$$\widetilde{S}_i = \begin{bmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} \\ 0 & \widetilde{S}_{22} \end{bmatrix}.$$

The $n_{cs} \times n_{cs}$ Cholesky factor $S_E$ corresponding to the trailing $n_{cs} \times n_{cs}$ part of $P_i$ is the trailing $n_{cs} \times n_{cs}$ block of an upper triangular matrix $\widehat{S}_{22}$ which satisfies

$$\widehat{S}_{22}\widehat{S}_{22}^T = \widetilde{S}_{11}\widetilde{S}_{11}^T + (\widetilde{S}_{12} + \widetilde{S}_{22})(\widetilde{S}_{12} + \widetilde{S}_{22})^T.$$

$\widehat{S}_{22}$ can be computed easily from the RQ-decomposition of $\begin{bmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} + \widetilde{S}_{22} \end{bmatrix}$ using standard factorization updating formulas [GGMS74]. No difference appears in the computation of the Cholesky factor $R_E$.

### Efficiency Issues

In Table 9.1 we give for the different weights (assuming $n_{cs} = n_c$) the number of operations $\widetilde{N}_E$ necessary to determine the Cholesky factors of the frequency-weighted Gramians and the achieved operation savings $\Delta_E = N_E - \widetilde{N}_E$, (see also (9.18) for $N_E$) with respect to using standard FWMR techniques to reduce a general controller:

**Table 9.1.** Operation counts: general controller

| Weight | $\widetilde{N}_E$ | $\Delta_E$ |
|---|---|---|
| SW1/SW2 | $33(n + n_c)^3 + 33n_c^3$ | $24n^2n_c + 74nn_c^2 + 58n_c^3$ |
| PW | $41(n + n_c)^3 + 2nn_c^2$ | $48n^2n_c + 146nn_c^2 + 141n_c^3$ |

In the case of a state feedback and observer-based controller ($n_c = n$), the corresponding values are shown in Table 9.2:
Observe the large computational effort savings obtained in all cases through structure exploitation for both general as well as state feedback controllers. For example, for the SW1/SW2 and PW problems with a state feedback controller the effort to compute the Gramians is about 2.7 times less than without structure exploitation.

**Table 9.2.** Operation counts: observer-based controller

| Weight | $\widetilde{N}_E$ | $\Delta_E$ |
|---|---|---|
| SW1/SW2 | $122n^3$ | $331n^3$ |
| PW | $181n^3$ | $484n^3$ |

### 9.4.2 Stability Preserving Coprime Factor Reduction

In this subsection, we discuss the efficient solution of frequency-weighted balancing-related coprime factor controller reduction problems for the special stability preserving frequency-weights proposed in [LAL90]. We show that for both general controllers as well as for state feedback and observer-based controllers, the computation of frequency-weighted Gramians for the coprime factor controller reduction can be done efficiently by solving lower order Lyapunov equations. Further, we show that these factors can be directly obtained in Cholesky factored forms allowing the application of the **SRBF** accuracy enhancing technique.

The following stability enforcing one-sided weights are used: for the right coprime factor reduction problem the weights are

$$\text{SRCF:} \qquad W_o = V^{-1}(I+GK)^{-1}[\,G\;I\,], \quad W_i = I, \tag{9.30}$$

while for the left coprime factor reduction the weights are

$$\text{SLCF:} \qquad \widetilde{W}_o = I, \quad \widetilde{W}_i = \begin{bmatrix} G \\ I \end{bmatrix}(I+KG)^{-1}\widetilde{V}^{-1}, \tag{9.31}$$

All above weights are stable TFMs with realizations of order $n+n_c$. It can be shown (see for example [ZDG96]) that with the above weights, the stability of the closed-loop system is guaranteed if $\left\| W_o \begin{bmatrix} U-U_r \\ V-V_r \end{bmatrix} \right\|_\infty < 1$ or $\|[\,\widetilde{U} - \widetilde{U}_r\;\widetilde{V} - \widetilde{V}_r\,]\widetilde{W}_i\|_\infty < 1$. These results justify the frequency-weighted coprime factor controller reduction methods introduced in [LAL90] for the reduction of state feedback and observer-based controllers. The case of arbitrary stabilizing controllers has been considered in [ZDG96]. Both cases are addressed in what follows. Note that in contrast to the approach of the previous subsection, the reduction of coprime factors can be performed even for completely unstable controllers.

#### RCF of a General Controller

We consider the efficient computation of the frequency-weighted controllability Gramian for the weights defined in (9.30). Let $F_c$ be any matrix such that $A_c + B_cF_c$ is stable (i.e., the eigenvalues of $A_c + B_cF_c$ lie in the open left half plane for a continuous-time system or in the interior of the unit circle for a discrete-time system). Then, a RCF of $K = UV^{-1}$ is given by

$$\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} = \left[ \begin{array}{c|c} A_c + B_c F_c & B_c \\ \hline C_c + D_c F_c & D_c \\ F_c & I \end{array} \right] .$$

The output weighting $\mathbf{W_o}$ is a stable TFM having a state-space realization $\mathbf{W_o} = (A_o, *, C_o, *)$ of order $n + n_c$ [ZDG96, p.503], where

$$\begin{aligned} A_o &= \begin{bmatrix} A_c - B_c R^{-1} D C_c & -B_c R^{-1} C \\ B \widetilde{R}^{-1} C_c & A - B D_c R^{-1} C \end{bmatrix} , \\ C_o &= [\, R^{-1} D C_c - F_c \quad - R^{-1} C \,] . \end{aligned}$$

The solution of the controller reduction problem for the special weights defined in (9.30) involves the solution of a Lyapunov equation of order $n_c$ to compute the controllability Gramian $P_E$ and the solution of a Lyapunov equation of order $n + 2n_c$ to determine the frequency-weighted observability Gramian $Q_E$. The following theorem [Var03b] shows that it is always possible to solve a Lyapunov equation of order $n + n_c$ to compute the frequency-weighted observability Gramian for the special weights in (9.30).

**Theorem 9.4.3** *For a given $n$-th order system $\mathbf{G} = (A, B, C, D)$ assume that $\mathbf{K} = (A_c, B_c, C_c, D_c)$ is an $n_c$-th order stabilizing controller with $I + DD_c$ nonsingular. Then the frequency-weighted Gramians for Enns' method [Enn84] applied to the frequency-weighted right coprime factorization based controller reduction problem with weights defined in (9.30) can be computed by solving the corresponding Lyapunov equations of order at most $n + n_c$ as follows: $P_E$ satisfies*

$$\begin{array}{l} (c)\ (A_c + B_c F_c) P_E + P_E (A_c + B_c F_c)^T + B_c B_c^T = 0 \\ (d)\ (A_c + B_c F_c) P_E (A_c + B_c F_c)^T + B_c B_c^T = P_E \end{array} ,$$

*while $Q_E$ is the leading $n_c \times n_c$ diagonal block of $Q_o$ satisfying*

$$(c)\ A_o^T Q_o + Q_o A_o + C_o^T C_o = 0 , \quad (d)\ A_o^T Q_o A_o + C_o^T C_o = Q_o . \qquad (9.32)$$

### RCF of a State Feedback and Observer-Based Controller

In the case of a state feedback and full order observer-based controller (9.28), we obtain a significant reduction of computational costs. In this case, with $F_c = -(C + DF)$ we get (see [ZDG96])

$$\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} = \left[ \begin{array}{c|c} A + BF & -L \\ \hline F & 0 \\ C + DF & I \end{array} \right]$$

and the output weighting $\mathbf{W_o}$ has the following state-space realization of order $n$ [ZDG96, p.503]

$$\mathbf{W_o} = \left[\begin{array}{c|cc} A + LC & -B - LD & L \\ \hline C & -D & I \end{array}\right]. \tag{9.33}$$

The following is a dual result to *Lemma 2* of [LAL90] to the case of nonzero feedthrough matrix $D$ and covers also the discrete-time case.

**Corollary 9.4.4** *For a given $n$-th order system $\mathbf{G} = (A, B, C, D)$ and the observer-based controller $\mathbf{K}$ (9.28), suppose $F$ is a state feedback gain and $L$ is a state estimator gain, such that $A + BF$ and $A + LC$ are stable. Then the frequency-weighted Gramians for Enns' method [Enn84] applied to frequency-weighted right coprime factorization based controller reduction problem with weights defined in (9.30) can be computed by solving the corresponding Lyapunov equations of order $n$, as follows:*

$$(c)\ \begin{array}{l} (A + BF)P_E + P_E(A + BF)^T + LL^T \ = 0 \\ (A + LC)^T Q_E + Q_E(A + LC) + C^T C = 0 \end{array},$$

$$(d)\ \begin{array}{l} (A + BF)P_E(A + BF)^T + LL^T \ = P_E \\ (A + LC)^T Q_E(A + LC) + C^T C = Q_E \end{array}.$$

### LCF of a General Controller

Let $L_c$ be any matrix such that $A_c + L_c C_c$ is stable. Then, a LCF of $K = \widetilde{V}^{-1}\widetilde{U}$ is given by

$$[\,\widetilde{\mathbf{U}}\ \widetilde{\mathbf{V}}\,] = \left[\begin{array}{c|cc} A_c + L_c C_c & B_c + L_c D_c & L_c \\ \hline C_c & D_c & I \end{array}\right].$$

The input weight $\widetilde{W}_i$ is a stable TFM having a state-space realization $\widetilde{\mathbf{W}}_\mathbf{i} :=$ $(A_i, B_i, *, *)$ of order $n + n_c$ [ZDG96, see p.503], where

$$A_i = \begin{bmatrix} A - B\widetilde{R}^{-1}D_c C & B\widetilde{R}^{-1}C_c \\ -B_c R^{-1} C & A_c - B_c D\widetilde{R}^{-1}C_c \end{bmatrix},\ B_i = \begin{bmatrix} -B\widetilde{R}^{-1} \\ B_c D\widetilde{R}^{-1} - L_c \end{bmatrix}, \tag{9.34}$$

with $R := I + DD_c$ and $\widetilde{R} = I + D_c D$.

We have a result similar to Theorem 9.4.3 showing that $P_E$ can be efficiently determined by solving only a reduced order Lyapunov equation.

**Theorem 9.4.5** *For a given $n$-th order system $\mathbf{G} = (A, B, C, D)$ assume that $\mathbf{K} = (A_c, B_c, C_c, D_c)$ is an $n_c$-th order stabilizing controller with $I + DD_c$ nonsingular. Then the frequency-weighted Gramians for Enns' method [Enn84] applied to the frequency-weighted left coprime factorization based controller reduction problem with weights defined in (9.31) can be computed by solving the corresponding Lyapunov equations of order at most $n + n_c$ as follows: $P_E$ is the trailing $n_c \times n_c$ block of $P_i$ satisfying*

$$(c)\ A_i P_i + P_i A_i^T + B_i B_i^T = 0, \quad (d)\ A_i P_i A_i^T + B_i B_i^T = P_i\,, \tag{9.35}$$

*while* $Q_E$ *satisfies*

$$
\begin{aligned}
&(c)\ (A_c + L_cC_c)^TQ_E + Q_E(A_c + L_cC_c) + C_c^TC_c = 0\ ,\\
&(d)\ (A_c + L_cC_c)^TQ_E(A_c + L_cC_c) + C_c^TC_c \qquad = Q_E\ .
\end{aligned}
$$

### LCF of a State Feedback and Observer-Based Controller

Significant simplifications arise in the case of a state feedback and full order observer-based controller (9.28), where it is assumed that $A+BF$ and $A+LC$ are both stable. In this case (see [ZDG96]), with $L_c = -(B + LD)$ we get

$$
[\,\widetilde{\mathbf{U}}\ \widetilde{\mathbf{V}}\,] = \left[\begin{array}{c|cc} A + LC & -L & -(B + LD) \\ \hline F & 0 & I \end{array}\right]
$$

and the input weighting $\widetilde{\mathbf{W}}_\mathbf{i}$ has the following state-space realization of order $n$ [ZDG96, p.503]

$$
\widetilde{\mathbf{W}}_\mathbf{i} = \left[\begin{array}{c|c} A + BF & B \\ \hline C + DF & D \\ F & I \end{array}\right] .
$$

The following result is an extension of *Lemma 2* of [LAL90] to the case of a nonzero feedthrough matrix $D$ and covers both the continuous- as well as the discrete-time case.

**Corollary 9.4.6** *For a given n-th order system* $\mathbf{G} = (A, B, C, D)$ *and the observer-based controller* $\mathbf{K}$ *(9.28), suppose* $F$ *is a state feedback gain and* $L$ *is a state estimator gain, such that* $A + BF$ *and* $A + LC$ *are stable. Then the frequency-weighted Gramians for Enns' method [Enn84] applied to the frequency-weighted left coprime factorization based controller reduction problem with weights defined in (9.31) can be computed by solving the corresponding Lyapunov equations of order* $n$ *as follows:*

$$
(c)\ \begin{array}{l} (A + BF)P_E + P_E(A + BF)^T + BB^T = 0 \\ (A + LC)^TQ_E + Q_E(A + LC) + F^TF = 0 \end{array}
$$

$$
(d)\ \begin{array}{l} (A + BF)P_E(A + BF)^T + BB^T = P_E \\ (A + LC)^TQ_E(A + LC) + F^TF\ = Q_E \end{array}
$$

### Square-Root Techniques

In the case of general right coprime factorized controllers, the method of Hammarling [Ham82] can be employed to solve (9.32) directly for the $(n + n_c) \times (n + n_c)$ Cholesky factor $R_o$ of $Q_o = R_o^TR_o$. By partitioning $R_o$ in the form

$$
R_o = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix},
$$

with $R_{11}$ an $n_c \times n_c$ matrix, the Cholesky factor $R_E$ of the leading block of $Q_o$ is $R_E = R_{11}$.

Similarly, in the case of general left coprime factorized controllers, (9.35) can be solved directly for the $(n + n_c) \times (n + n_c)$ Cholesky factor $S_i$ of $P_i = S_i S_i^T$. By partitioning $S_i$ in the form

$$S_i = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix},$$

with $S_{22}$ an $n_c \times n_c$ matrix, the Cholesky factor of the trailing block of $P_i$ is $S_E = S_{22}$.

The Cholesky factors of Gramians for the remaining cases are directly obtained by solving the appropriate Lyapunov equations using Hammarling's algorithm [Ham82].

### Efficiency Issues

In Table 9.3 we give for the RCF and LCF based approaches the number of operations $\widetilde{N}_E$ necessary to determine the Cholesky factors of the frequency-weighted Gramians and the achieved operation savings $\Delta_E = N_E - \widetilde{N}_E$, (see (9.18) for $N_E$) with respect to using standard FWMR techniques to reduce the coprime factors of the controller:

**Table 9.3.** Operation counts: general coprime factorized controller

| Weight | $\widetilde{N}_E$ | $\Delta_E$ |
|---|---|---|
| SRCF/SLCF | $33(n + n_c)^3 + 33n_c^3$ | $24n^2 n_c + 74nn_c^2 + 58n_c^3$ |

To these figures we have to add the computational effort involved to compute a stabilizing state feedback (output injection) gain to determine the RCF (LCF) of the controller. When employing the Schur method of [Var81], it is possible to arrange the computations such that the resulting closed-loop state matrix $A_c + B_c F_c$ $(A_c + L_c C_c)$ is in a RSF. In this way it is possible to avoid the reduction of this matrix to determine the unweighted Gramian $P_E$ $(Q_E)$ when solving the corresponding Lyapunov equation.

In the case of a state feedback and observer-based controller $(n_c = n)$, the corresponding values are shown in Table 9.4.

**Table 9.4.** Operation counts: observer-based coprime factorized controller

| Weight | $\widetilde{N}_E$ | $\Delta_E$ |
|---|---|---|
| SRCF/SLCF | $66n^3$ | $58n^3$ |

Observe the substantial computational effort savings obtained through structure exploitation for both general as well as state feedback controllers.

### 9.4.3 Performance Preserving Coprime Factors Reduction

In this subsection we consider the efficient computation of low order controllers by using the coprime factors reduction procedures to solve the frequency-weighted coprime factorization based $\mathcal{H}_\infty$ controller reduction problems formulated in [GG98]. Let

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{9.36}$$

be the TFM used to parameterize all admissible $\gamma$-suboptimal controllers [ZDG96] in the form

$$K = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21},$$

where $Q$ is a stable and proper rational matrix satisfying $\|Q\|_\infty < \gamma$. Since for standard $\mathcal{H}_\infty$ problems both $M_{12}$ and $M_{21}$ are invertible and minimum-phase [ZDG96], a "natural" RCF of the central controller ($Q = 0$) as $K_0 = UV^{-1}$ can be obtained with

$$U = M_{11}M_{21}^{-1}, \quad V = M_{21}^{-1},$$

while a "natural" LCF of the central controller as $K_0 = \widetilde{V}^{-1}\widetilde{U}$ can be obtained with

$$\widetilde{U} = M_{12}^{-1}M_{11}, \quad \widetilde{V} = M_{12}^{-1}.$$

These factorizations can be used to perform unweighted coprime factor controller reduction using accuracy-enhanced model reduction algorithms [Var92].

A frequency-weighted right coprime factor reduction can be formulated with the one sided weights [ZDG96, GG98]

$$\text{PRCF:} \qquad W_o = \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1}, \quad W_i = I, \tag{9.37}$$

where

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} := \begin{bmatrix} M_{12} - M_{11}M_{21}^{-1}M_{22} & M_{11}M_{21}^{-1} \\ -M_{21}^{-1}M_{22} & M_{21}^{-1} \end{bmatrix}.$$

With the help of the submatrices of $\Theta$ it is possible to express $K$ also as

$$K = (\Theta_{12} + \Theta_{11}Q)(\Theta_{22} + \Theta_{21}Q)^{-1}$$

and thus the central controller is factorized as $K_0 = \Theta_{12}\Theta_{22}^{-1}$.

Similarly, a frequency-weighted left coprime factor reduction formulated in [GG98] is one sided with

$$\text{PLCF:} \qquad \widetilde{W}_o = I, \quad \widetilde{W}_i = \widetilde{\Theta}^{-1} \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix}, \tag{9.38}$$

where

$$\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} \\ \widetilde{\Theta}_{21} & \widetilde{\Theta}_{22} \end{bmatrix} := \begin{bmatrix} M_{21} - M_{22}M_{12}^{-1}M_{11} & -M_{22}M_{12}^{-1} \\ M_{12}^{-1}M_{11} & M_{12}^{-1} \end{bmatrix}.$$

This time we have the alternative representation of $K$ as

$$K = (\widetilde{\Theta}_{22} + Q\widetilde{\Theta}_{12})^{-1}(\widetilde{\Theta}_{21} + Q\widetilde{\Theta}_{11})$$

and the central controller is factorized as $K_0 = \widetilde{\Theta}_{22}^{-1}\widetilde{\Theta}_{21}$. Note that both $\Theta$ and $\widetilde{\Theta}$ are stable, invertible and minimum-phase.

The importance of the above frequency-weighted coprime factor reduction can be seen from the results of [GG98]. If $K_0$ is a stabilizing continuous-time $\gamma$-suboptimal $\mathcal{H}_\infty$ central controller, and $K_r$ is an approximation of $K_0$ computed by applying the coprime factors reduction approach with the weight defined above, then $K_r$ stabilizes the closed-loop system and preserves the $\gamma$-suboptimal performance, provided the weighted approximation error (9.2) or (9.3) is less than $1/\sqrt{2}$. We conjecture that this result holds also in the discrete-time case, and can be proved along the lines of the proof provided in [ZDG96].

### RCF Controller Reduction

We consider the efficient computation of the frequency-weighted controllability Gramian for the weights defined in (9.37). Let us consider a realization of the parameterization TFM $M$ (9.36) in the form

$$\mathbf{M} = \left[\begin{array}{c|cc} \widehat{A} & \widehat{B}_1 & \widehat{B}_2 \\ \hline \widehat{C}_1 & \widehat{D}_{11} & \widehat{D}_{12} \\ \widehat{C}_2 & \widehat{D}_{21} & \widehat{D}_{22} \end{array}\right].$$

Note that for the central controller we have $(A_c, B_c, C_c, D_c) = (\widehat{A}, \widehat{B}_1, \widehat{C}_1, \widehat{D}_{11})$. Since $M_{12}$ and $M_{21}$ are stable, minimum-phase and invertible TFMs, it follows that $\widehat{D}_{12}$ and $\widehat{D}_{21}$ are invertible, $\widehat{A}$, $\widehat{A} - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{C}_1$ and $\widehat{A} - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{C}_2$ are all stable matrices, i.e., have eigenvalues in the open left half plane for a continuous-time controller and in the interior of the unit circle for a discrete-time controller.

The realizations of $\mathbf{\Theta}$ and $\mathbf{\Theta}^{-1}$ can be computed as [ZDG96]

$$\mathbf{\Theta} = \left[\begin{array}{c|c} A_\Theta & B_\Theta \\ \hline C_\Theta & D_\Theta \end{array}\right] = \left[\begin{array}{c|cc} \widehat{A} - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{C}_2 & \widehat{B}_2 - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{D}_{22} & \widehat{B}_1\widehat{D}_{21}^{-1} \\ \hline \widehat{C}_1 - \widehat{D}_{11}\widehat{D}_{21}^{-1}\widehat{C}_2 & \widehat{D}_{12} - \widehat{D}_{11}\widehat{D}_{21}^{-1}\widehat{D}_{22} & \widehat{D}_{11}\widehat{D}_{21}^{-1} \\ -\widehat{D}_{21}^{-1}\widehat{C}_2 & -\widehat{D}_{21}^{-1}\widehat{D}_{22} & \widehat{D}_{21}^{-1} \end{array}\right],$$

$$\mathbf{\Theta}^{-1} = \left[\begin{array}{c|c} A_{\Theta^{-1}} & B_{\Theta^{-1}} \\ \hline C_{\Theta^{-1}} & D_{\Theta^{-1}} \end{array}\right] = \left[\begin{array}{c|cc} \widehat{A} - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{B}_2\widehat{D}_{12}^{-1} & \widehat{B}_1 - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{D}_{11} \\ \hline -\widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{D}_{12}^{-1} & -\widehat{D}_{12}^{-1}\widehat{D}_{11} \\ \widehat{C}_2 - \widehat{D}_{22}\widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{D}_{22}\widehat{D}_{12}^{-1} & \widehat{D}_{21} - \widehat{D}_{22}\widehat{D}_{12}^{-1}\widehat{D}_{11} \end{array}\right].$$

Since the realization of $\mathbf{W_o}\begin{bmatrix}\mathbf{U}\\ \mathbf{V}\end{bmatrix}$ has apparently order $2n_c$, it follows that the solution of the controller reduction problem for the special weights defined in (9.37) involves the solution of a Lyapunov equation of order $n_c$ to determine the frequency-weighted controllability Gramian $P_E$ and a Lyapunov equation of order $2n_c$ to compute the observability Gramian $Q_E$. The following result [Var03a] shows that it is always possible to solve two Lyapunov equations of order $n_c$ to compute the frequency-weighted Gramians for the special weights in (9.37).

**Theorem 9.4.7** *The controllability Gramian $P_E$ and the frequency-weighted observability Gramian $Q_E$ according to Enns' choice [Enn84] for the frequency-weighted RCF controller reduction problem with weights (9.37) satisfy, according to the system type, the corresponding Lyapunov equations*

$$(c)\begin{cases} A_\Theta P_E + P_E A_\Theta^T + \widetilde{B}_\Theta \widetilde{B}_\Theta^T &= 0 \\ A_{\Theta^{-1}}^T Q_E + Q_E A_{\Theta^{-1}} + \widetilde{C}_{\Theta^{-1}}^T \widetilde{C}_{\Theta^{-1}} &= 0 \end{cases},$$

$$(d)\begin{cases} A_\Theta P_E A_\Theta^T + \widetilde{B}_\Theta \widetilde{B}_\Theta^T &= P_E \\ A_{\Theta^{-1}}^T Q_E A_{\Theta^{-1}} + \widetilde{C}_{\Theta^{-1}}^T \widetilde{C}_{\Theta^{-1}} &= Q_E \end{cases},$$

*where* $\widetilde{B}_\Theta = B_\Theta \begin{bmatrix}0\\ I\end{bmatrix} = \widehat{B}_1\widehat{D}_{21}^{-1}$ *and* $C_{\Theta^{-1}} = \operatorname{diag}(\gamma^{-1}I, I)C_{\Theta^{-1}}$.

### LCF Controller Reduction

We consider now the efficient computation of the frequency-weighted controllability Gramian for the weights defined in (9.38). The realizations of $\widetilde{\mathbf{\Theta}}$ and $\widetilde{\mathbf{\Theta}}^{-1}$ can be computed as [ZDG96]

$$\widetilde{\mathbf{\Theta}} = \left[\begin{array}{c|c} A_{\widetilde{\Theta}} & B_{\widetilde{\Theta}} \\ \hline C_{\widetilde{\Theta}} & D_{\widetilde{\Theta}} \end{array}\right] = \left[\begin{array}{c|cc} \widehat{A} - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{B}_1 - \widehat{B}_2\widehat{D}_{12}^{-1}\widehat{D}_{11} & -\widehat{B}_2\widehat{D}_{12}^{-1} \\ \hline \widehat{C}_2 - \widehat{D}_{22}\widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{D}_{21} - \widehat{D}_{22}\widehat{D}_{12}^{-1}\widehat{D}_{11} & -\widehat{D}_{22}\widehat{D}_{12}^{-1} \\ \widehat{D}_{12}^{-1}\widehat{C}_1 & \widehat{D}_{12}^{-1}\widehat{D}_{11} & \widehat{D}_{12}^{-1} \end{array}\right],$$

$$\widetilde{\mathbf{\Theta}}^{-1} = \left[\begin{array}{c|c} A_{\widetilde{\Theta}^{-1}} & B_{\widetilde{\Theta}^{-1}} \\ \hline C_{\widetilde{\Theta}^{-1}} & D_{\widetilde{\Theta}^{-1}} \end{array}\right] = \left[\begin{array}{c|cc} \widehat{A} - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{C}_2 & -\widehat{B}_1\widehat{D}_{21}^{-1} & \widehat{B}_2 - \widehat{B}_1\widehat{D}_{21}^{-1}\widehat{D}_{22} \\ \hline \widehat{D}_{21}^{-1}\widehat{C}_2 & \widehat{D}_{21}^{-1} & \widehat{D}_{21}^{-1}\widehat{D}_{22} \\ \widehat{C}_1 - \widehat{D}_{11}\widehat{D}_{21}^{-1}\widehat{C}_2 & -\widehat{D}_{11}\widehat{D}_{21}^{-1} & \widehat{D}_{12} - \widehat{D}_{11}\widehat{D}_{21}^{-1}\widehat{D}_{22} \end{array}\right].$$

Since the realization of $[\,\widetilde{\mathbf{U}}\ \ \widetilde{\mathbf{V}}\,]\widetilde{\mathbf{W_i}}$ has apparently order $2n_c$, it follows that the solution of the controller reduction problem for the special weights defined in (9.38) involves the solution of a Lyapunov equation of order $2n_c$

to determine the frequency-weighted controllability Gramian $P_E$ and a Lyapunov equation of order $n_c$ to compute the observability Gramian $Q_E$. The following result [Var03a] shows that it is always possible to solve two Lyapunov equations of order $n_c$ to compute the frequency-weighted Gramians for the special weights in (9.38).

**Theorem 9.4.8** *The frequency-weighted controllability Gramian $P_E$ and observability Gramian $Q_E$ according to Enns' choice [Enn84] for the frequency-weighted LCF controller reduction problem with weights (9.38) satisfy the corresponding Lyapunov equations*

$$(c)\begin{cases} A_{\widetilde{\Theta}^{-1}} P_E + P_E A^T_{\widetilde{\Theta}^{-1}} + \widetilde{B}_{\widetilde{\Theta}^{-1}} \widetilde{B}^T_{\widetilde{\Theta}^{-1}} & = 0 \\ A^T_{\widetilde{\Theta}} Q_E + Q_E A_{\widetilde{\Theta}} + \widetilde{C}^T_{\widetilde{\Theta}} \widetilde{C}_{\widetilde{\Theta}} & = 0 \end{cases},$$

$$(d)\begin{cases} A_{\widetilde{\Theta}^{-1}} P_E A_{\widetilde{\Theta}^{-1}} + \widetilde{B}_{\widetilde{\Theta}^{-1}} \widetilde{B}^T_{\widetilde{\Theta}^{-1}} & = P_E \\ A^T_{\widetilde{\Theta}} Q_E A_{\widetilde{\Theta}} + \widetilde{C}^T_{\widetilde{\Theta}} \widetilde{C}_{\widetilde{\Theta}} & = Q_E \end{cases},$$

*where* $\widetilde{B}_{\widetilde{\Theta}^{-1}} = B_{\widetilde{\Theta}^{-1}} \mathrm{diag}\,(\gamma^{-1} I, I)$ *and* $\widetilde{C}_{\widetilde{\Theta}} = \widehat{D}_{12}^{-1} \widehat{C}_1$.

### Efficiency Issues

In Table 9.5 we give for the RCF and LCF based approaches the number of operations $\widetilde{N}_E$ necessary to determine the Cholesky factors of the frequency-weighted Gramians and the achieved operation savings $\Delta_E = N_E - \widetilde{N}_E$, (see (9.18) for $N_E$) with respect to using standard FWMR techniques to reduce the coprime factors of the controller.

**Table 9.5.** Operation counts: coprime factorized $\mathcal{H}_\infty$-controller

| Weight | $\widetilde{N}_E$ | $\Delta_E$ |
|---|---|---|
| PRCF/PLCF | $66n_c^3$ | $58n_c^3$ |

Observe the substantial (47%) computational effort savings obtained through structure exploitation.

## 9.4.4 Relative Error Coprime Factors Reduction

An alternative approach to $\mathcal{H}_\infty$ controller reduction uses the relative error method as suggested in [Zho95]. Using this approach in conjunction with the RCF reduction we can define the weights as

$$W_o = I, \quad W_i = \begin{bmatrix} U \\ V \end{bmatrix}^+, \tag{9.39}$$

where $\begin{bmatrix} U \\ V \end{bmatrix}^+$ denotes a stable left inverse of $\begin{bmatrix} U \\ V \end{bmatrix}$. A variant of this approach (see [ZDG96]) is to perform a relative error coprime factor reduction on an invertible augmented minimum-phase system $\begin{bmatrix} U_a \\ V_a \end{bmatrix}$ instead of $\begin{bmatrix} U \\ V \end{bmatrix}$. In our case, $\Theta$ can be taken as the augmented system. Thus this method essentially consists of determining an approximation $\Theta_r$ of $\Theta$ by solving the relative error minimization problems

$$\|(\Theta - \Theta_r)\Theta^{-1}\|_\infty = \min \tag{9.40}$$

or

$$\|\Theta^{-1}(\Theta - \Theta_r)\|_\infty = \min\,. \tag{9.41}$$

These are a frequency-weighted problems with the corresponding weights

$$\text{RCFR1:} \qquad W_o = I, \quad W_i = \Theta^{-1} \tag{9.42}$$

and respectively

$$\text{RCFR2:} \qquad W_o = \Theta^{-1}, \quad W_i = I\,. \tag{9.43}$$

The reduced controller is recovered from the sub-blocks (1,2) and (2,2) of $\Theta_r$ as $K_r = \Theta_{r,12}\Theta_{r,22}^{-1}$. This method has been also considered in [EJL01] for the case of normalized coprime factor $\mathcal{H}_\infty$ controller reduction.

In the same way, a relative error LCF reduction can be formulated with the weights

$$\widetilde{W}_o = [\,\widetilde{U}\ \widetilde{V}\,]^+, \quad \widetilde{W}_i = I \tag{9.44}$$

where $[\,\widetilde{U}\ \ \widetilde{V}\,]^+$ denotes a stable right inverse of $[\,\widetilde{U}\ \ \widetilde{V}\,]$. Alternatively, an augmented relative error problem can be solved by approximating $\widetilde{\Theta}$ by a reduced order system $\widetilde{\Theta}_r$ by solving the relative error norm minimization problems

$$\|\widetilde{\Theta}^{-1}(\widetilde{\Theta} - \widetilde{\Theta}_r)\|_\infty \tag{9.45}$$

or

$$\|(\widetilde{\Theta} - \widetilde{\Theta}_r)\widetilde{\Theta}^{-1}\|_\infty\,. \tag{9.46}$$

These are frequency-weighted problems with weights

$$\text{LCFR1:} \qquad \widetilde{W}_o = \widetilde{\Theta}^{-1}, \quad \widetilde{W}_i = I \tag{9.47}$$

and respectively

$$\text{LCFR2:} \qquad \widetilde{W}_o = I, \quad \widetilde{W}_i = \widetilde{\Theta}^{-1}\,. \tag{9.48}$$

The reduced controller is recovered from the sub-blocks (2,1) and (2,2) of $\widetilde{\Theta}_r$ as $K_r = \widetilde{\Theta}_{r,22}^{-1}\widetilde{\Theta}_{r,21}$.

**Relative Error RCF Reduction**

For the solution of the relative error approximation problems (9.40) and (9.41) we have the following straightforward results [ZDG96, Theorem 7.5]:

**Theorem 9.4.9** *The frequency-weighted controllability Gramian $P_E$ and observability Gramian $Q_E$ for Enns' method [Enn84] applied to the frequency-weighted approximation problems (9.40) and (9.41) satisfy, depending on the system type, the corresponding Lyapunov equations, as follows:*

1. *For the problem (9.40)*

$$(c)\begin{cases} A_{\Theta^{-1}}P_E + P_E A_{\Theta^{-1}}^T + B_{\Theta^{-1}}B_{\Theta^{-1}}^T &= 0 \\ A_\Theta^T Q_E + Q_E A_\Theta + C_\Theta^T C_\Theta &= 0 \end{cases},$$

$$(d)\begin{cases} A_{\Theta^{-1}}P_E A_{\Theta^{-1}} + B_{\Theta^{-1}}B_{\Theta^{-1}}^T &= P_E \\ A_\Theta^T Q_E A_\Theta + C_\Theta^T C_\Theta &= Q_E \end{cases}.$$

2. *For the problem (9.41)*

$$(c)\begin{cases} A_\Theta P_E + P_E A_\Theta^T + B_\Theta B_\Theta^T &= 0 \\ A_{\Theta^{-1}}^T Q_E + Q_E A_{\Theta^{-1}} + C_{\Theta^{-1}}^T C_{\Theta^{-1}} &= 0 \end{cases},$$

$$(d)\begin{cases} A_\Theta P_E A_\Theta^T + B_\Theta B_\Theta^T &= P_E \\ A_{\Theta^{-1}}^T Q_E A_{\Theta^{-1}} + C_{\Theta^{-1}}^T C_{\Theta^{-1}} &= Q_E \end{cases}.$$

**Relative Error LCF Reduction**

For the solution of the relative error approximation problems (9.45) and (9.46) we have the following straightforward results [ZDG96, Theorem 7.5]:

**Theorem 9.4.10** *The frequency-weighted controllability Gramian $P_E$ and observability Gramian $Q_E$ for Enns' method [Enn84] applied to the frequency-weighted approximation problem (9.45) and (9.46) satisfy, according to the system type, the corresponding Lyapunov equations, as follows:*

1. *For the problem (9.45)*

$$(c)\begin{cases} A_{\widetilde{\Theta}}P_E + P_E A_{\widetilde{\Theta}}^T + B_{\widetilde{\Theta}}B_{\widetilde{\Theta}}^T &= 0 \\ A_{\widetilde{\Theta}^{-1}}^T Q_E + Q_E A_{\widetilde{\Theta}^{-1}} + C_{\widetilde{\Theta}^{-1}}^T C_{\widetilde{\Theta}^{-1}} &= 0 \end{cases},$$

$$(d)\begin{cases} A_{\widetilde{\Theta}}P_E A_{\widetilde{\Theta}} + B_{\widetilde{\Theta}}B_{\widetilde{\Theta}}^T &= P_E \\ A_{\widetilde{\Theta}^{-1}}^T Q_E A_{\widetilde{\Theta}^{-1}} + C_{\widetilde{\Theta}^{-1}}^T C_{\widetilde{\Theta}^{-1}} &= Q_E \end{cases}.$$

2. *For the problem (9.46)*

$$(c)\begin{cases} A_{\widetilde{\Theta}^{-1}}P_E + P_E A_{\widetilde{\Theta}^{-1}}^T + B_{\widetilde{\Theta}^{-1}}B_{\widetilde{\Theta}^{-1}}^T &= 0 \\ A_{\widetilde{\Theta}}^T Q_E + Q_E A_{\widetilde{\Theta}} + C_{\widetilde{\Theta}}^T C_{\widetilde{\Theta}} &= 0 \end{cases},$$

$$(d)\begin{cases} A_{\widetilde{\Theta}^{-1}}P_E A_{\widetilde{\Theta}^{-1}}^T + B_{\widetilde{\Theta}^{-1}}B_{\widetilde{\Theta}^{-1}}^T &= P_E \\ A_{\widetilde{\Theta}}^T Q_E A_{\widetilde{\Theta}} + C_{\widetilde{\Theta}}^T C_{\widetilde{\Theta}} &= Q_E \end{cases}.$$

**Efficiency Issues**

In Table 9.6 we give for the RCF and LCF based approaches the number of operations $\widetilde{N}_E$ necessary to determine the Cholesky factors of the frequency-weighted Gramians and the achieved operation savings $\Delta_E = N_E - \widetilde{N}_E$, with respect to using standard FWMR techniques to reduce the coprime factors of the controller.

**Table 9.6.** Operation counts: coprime factorized $\mathcal{H}_\infty$-controller parametrizations

| Weight | $\widetilde{N}_E$ | $\Delta_E$ |
|---|---|---|
| RCFR1/RCFR2 | $66n_c^3$ | $58n_c^3$ |
| LCFR1/LCFR2 | $66n_c^3$ | $58n_c^3$ |

Observe the substantial (47%) computational effort savings obtained through structure exploitation.

## 9.5 Software for Controller Reduction

In this section we present an overview of available software tools to support controller reduction. We focus on tools developed within the NICONET project. For details about other tools see Chapter 7 of [Var01a].

### 9.5.1 Tools for Controller Reduction in SLICOT

A powerful collection of Fortran 77 subroutines for model and controller reduction has been implemented within the NICONET project [Var01a, Var02b] as part of the SLICOT library. The model and controller reduction software in SLICOT implements the latest algorithmic developments for the following approaches:

- absolute error model reduction using the balanced truncation [Moo81], singular perturbation approximation [LA89], and Hankel-norm approximation [Glo84] methods;
- relative error model reduction using the balanced stochastic truncation approach [DP84, SC88, VF93];
- frequency-weighted balancing related model reduction methods [Enn84, LC92, WSL99, VA01, VA03] and frequency-weighted Hankel-norm approximation methods [LA85, HG86, Var01b];
- controller reduction methods using frequency-weighted balancing related methods [LAL90, VA02, VA03] and unweighed and frequency-weighted coprime factorization based techniques [LAL90].

The model and controller reduction routines in SLICOT are among the most powerful and numerically most reliable software tools available for model and controller reduction. All routines can be employed to reduce both stable and unstable, continuous- or discrete-time models or controllers. The underlying numerical algorithms rely on *square-root* (**SR**) [TP87] and *balancing-free square-root* (**BFSR**) [Var91b] accuracy enhancing techniques. The Table 9.7 contains the list of the user callable subroutines available for controller reduction in SLICOT.

**Table 9.7.** User callable SLICOT controller reduction routines

| Name | Function |
|---|---|
| SB16AD | FWBT/FWSPA-based controller reduction for closed-loop stability and performance preserving weights |
| SB16BD | state feedback/observer-based controller reduction using coprime factorization in conjunction with FWBT and FWSPA techniques |
| SB16CD | state feedback/observer-based controller reduction using frequency-weighted coprime factorization in conjunction with FWBT technique |

In implementing these routines, a special attention has been paid to ensure their numerical robustness. All implemented routines rely on the **SR** and **BFSR** accuracy enhancing techniques [TP87, Var91b, Var91a]. Both techniques substantially contribute to improve the numerical reliability of computations. Furthermore, all routines optionally perform the scaling of the original system. When calling each routine, the order of the reduced controller can be selected by the user or can be determined automatically on the basis of computed quantities which can be assimilated to the usual Hankel singular values. Each of routines can handle both continuous- and discrete-time controllers. In what follows we shortly discuss some particular functionality provided by these user callable routines.

The FWCR routine SB16AD is a specialization of a general purpose FWMR routine, for the special one-sided weights (9.19) and (9.20) used to enforce closed-loop stability as well as two-sided weights (9.21) for performance preservation. This routine works on a general stabilizing controller. Unstable controllers are handled by separating their stable and unstable parts and applying the controller reduction only to the stable parts. This routine has a large flexibility in combining different choices of the Gramians (see subsection 9.3.1) and can handle the unweighted case as well.

The coprime factorization based controller reduction routines SB16BD and SB16CD are specially adapted to reduce state feedback and observer-based controllers. The routine SB16BD allows arbitrary combinations of BT and SPA methods with "natural" left and right coprime factorizations of the controller. The routine SB16CD, implementing the frequency-weighted coprime factorization based stability preserving approach, can be employed only in conjunction with the BT technique. This routine allows to work with both left and right coprime factorization based approaches.

In implementing the new controller reduction software, a special emphasis has been put on an appropriate modularization of the routines by isolating some basic computational tasks and implementing them in supporting computational routines. For example, the balancing related approach (implemented in SB16AD) and the frequency-weighted coprime factorization based controller reduction method (implemented in SB16CD), share a common two step computational scheme: (1) compute two non-negative definite matrices called generically "frequency-weighted Gramians"; (2) determine suitable truncation matrices and apply them to obtain the matrices of the reduced model/controller using the BT or SPA methods. For the first step, separate routines have been implemented to compute appropriate Gramians according to the specifics of each method. To employ the accuracy enhancing **SR** or **BFSR** techniques, these routines compute in fact, instead of Gramians, their Cholesky factors. For the second step, a unique routine has been implemented, which is called by both above routines. For a detailed description of the controller reduction related software available in SLICOT see [Var02a].

### 9.5.2 SLICOT Based User-Friendly Tools

One of the main objectives of the NICONET project was to provide, additionally to standardized Fortran codes, high quality software embedded into user-friendly environments for *computer aided control system design*. The popular computational environment MATLAB[1] allows to easily add external functions implemented in general purpose programming languages like C or Fortran. The external functions are called *mex*-functions and have to be programmed according to precise programming standards. Two *mex*-functions have been implemented as main MATLAB interfaces to the controller reduction routines available in SLICOT. To provide a convenient interface to work with control objects defined in the MATLAB Control Toolbox, easy-to-use higher level controller reduction *m*-functions have been additionally implemented. The list of available *mex*- and *m*-functions is given in Table 9.8.

**Table 9.8.** *mex*- and *m*-functions for controller reduction

| Name | Function |
|---|---|
| *mex*: `conred`<br>*m*: `fwbconred` | frequency-weighted balancing related controller reduction (based on SB16AD) |
| *mex*: `sfored`<br>*m*: `sfconred` | coprime factorization based reduction of state feedback controllers (based on SB16BD and SB16CD) |

[1] MATLAB is a registered trademark of The MathWorks, Inc.

All these functions are able to reduce both continuous- and discrete-time, stable as well as unstable controllers. The functions can be used for unweighted reduction as well, without any significant computational overhead.

In the implementation of the *mex*- and *m*-functions, one main goal was to allow the access to the complete functionality provided by the underlying Fortran routines. To manage the multitude of possible user options, a so-called SYSRED structure has been defined. The controller reduction relevant fields which can be set in the SYSRED structure are shown below:

```
       BalredMethod: [ {bta} | spa ]
 AccuracyEnhancing: [ {bfsr} | sr ]
            Tolred: [ positive scalar {0} ]
        TolMinreal: [ positive scalar {0} ]
             Order: [ integer {-1} ]
   FWEContrGramian: [ {standard} | enhanced ]
  FWEObservGramian: [ {standard} | enhanced ]
     CoprimeFactor: [ left | {right} ]
      OutputWeight: [ {stab} | perf | none]
       InputWeight: [ {stab} | none]
    CFConredMethod: [ {fwe}  | nofwe ]
   FWEConredMethod: [ none | outputstab | inputstab | {performance} ]
```

This structure is created and managed via special functions. For more details on this structure see [Var02a].

Functionally equivalent user-friendly tools can be also implemented in the MATLAB-like environment `Scilab` [Gom99]. In `Scilab`, external functions can be similarly implemented as in MATLAB and only several minor modifications are necessary to the MATLAB *mex*-functions to adapt them to `Scilab`.

## 9.6 Controller Reduction Example

We consider the standard $\mathcal{H}_\infty$ optimization setup for the four-disk control system [ZDG96] described by

$$\begin{aligned} \dot{x} &= Ax + b_1 w + b_2 u \\ z &= \begin{bmatrix} 10^{-3}h \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= c_2 x + [0\ 1]w \end{aligned}$$

where $u$ and $w$ are the control and disturbance inputs, respectively, $z$ and $y$ are the performance and measurement outputs, respectively, and $x \in \mathbb{R}^7$ is the state vector. For completeness, we give the matrices of the model

$$A = \begin{bmatrix} -0.161 & -6.004 & -0.58215 & -9.9835 & -0.40727 & -3.982 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$b_1 = \begin{bmatrix} b_2 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.55 & 11 & 1.32 & 18 \end{bmatrix}$$

$$c_2 = \begin{bmatrix} 0 & 0 & 0.00064432 & 0.0023196 & 0.071252 & 1.0002 & 0.10455 & 0.99551 \end{bmatrix}$$

Using the `hinf` function of the Robust Control Toolbox [CS02], we computed the $\mathcal{H}_\infty$ controller $K(s)$ and the controller parameterization $M(s)$ using the loop-shifting formulae of [SLC89]. The optimal $\mathcal{H}_\infty$-norm of the TFM $T_{zw}$ from the disturbance input $w$ to performance output $z$ is $\gamma_{opt} = 1.1272$. We employed the same value $\gamma = 1.2$ as in [ZDG96] to determine an 8th order $\gamma$-suboptimal controller and the corresponding parameterization. The resulting controller is itself stable and has been reduced to orders between 0 and 7 using the methods presented in this paper. Provided the corresponding closed-loop system was stable, we computed for each reduced order controller the value of the $\mathcal{H}_\infty$-norm of the TFM $T_{zw}$. The results are presented in the Table 9.9, where U signifies that the closed-loop system with the resulting reduced order controller is unstable.

For each controller order, the bolded numbers indicate the best achieved approximation of the closed-loop TFM $T_{zw}$ in terms of the corresponding $\mathcal{H}_\infty$-norms. Observe that the FWSPA approach is occasionally superior for this example to the FWBT method. Several methods were able to obtain very good approximations until orders as low as 4. Even the best second order approximation appears to be still satisfactory. Interestingly, this controller provides a better approximation of the closed-loop TFM than the best third order controller. None of the employed methods was able to produce a stabilizing first order controller, although such a controller apparently exists (see results reported for the frequency-weighted HNA in [ZDG96]). As a curiosity, the standard unweighted SPA provided a stabilizing constant output feedback gain controller albeit this exhibits a very poor closed-loop performance.

## 9.7 Conclusions

We discussed recent enhancements of several frequency-weighted balancing related controller reduction methods. These enhancements are in three main directions: (1) enhancing the capabilities of underlying approximation methods by employing new choices of Gramians guaranteeing stability for two-sided weights or by employing alternatively the SPA approach instead of traditionally employed BT method; (2) improving the accuracy of computations by

**Table 9.9.** $\mathcal{H}_\infty$-norm of the closed-loop TFM $T_{zw}$

| Order of $K_r$ | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|
| UW (BT) | U | 1.318 | U | U | U | U | U | U |
| UW (SPA) | 1.200 | 1.200 | U | U | U | U | U | **6490.9** |
| RCF (BT) | 1.198 | **1.196** | 1.198 | **1.196** | 385.99 | 494.1 | U | U |
| RCF (SPA) | 1.196 | **1.196** | U | **1.196** | U | 34.99 | U | **6490.9** |
| LCF (BT) | 2.061 | 1.260 | 33.810 | 5.197 | U | U | U | U |
| LCF (SPA) | 1.196 | **1.196** | 1.588 | 2.045 | U | U | U | **6490.9** |
| SW1 (BT) | 1.321 | 1.199 | 2.287 | 1.591 | 23.381 | U | U | U |
| SW1 (SPA) | 1.196 | **1.196** | **1.196** | 1.484 | 3.218 | U | U | **6490.9** |
| SRCF (BT) | 1.232 | 1.197 | 1.254 | 1.202 | 13.514 | **1.413** | U | U |
| SRCF (SPA) | 1.196 | **1.196** | 16.274 | **1.196** | U | U | U | **6490.9** |
| SLCF (BT) | 1.418 | 1.216 | 37.647 | 3.062 | U | U | U | U |
| SLCF (SPA) | 1.196 | **1.196** | 1.197 | 1.799 | 15.151 | U | U | **6490.9** |
| PRCF (BT) | 1.199 | **1.196** | 1.207 | **1.196** | 2.760 | 1.734 | U | U |
| PRCF (SPA) | 1.196 | **1.196** | 1.542 | **1.196** | U | U | U | **6490.9** |
| PLCF (BT) | 1.196 | **1.196** | U | 1.197 | U | U | U | U |
| PLCF (SPA) | 1.196 | **1.196** | **1.196** | **1.196** | 7.609 | U | U | **6490.9** |
| PW (BT) | 1.334 | 1.198 | U | 1.212 | U | U | U | U |
| PW (SPA) | 1.196 | **1.196** | **1.196** | **1.196** | 3.465 | U | U | **6490.9** |
| RCFR1 (BT) | U | 1.197 | U | **4.1233** | U | U | U | U |
| RCFR1 (SPA) | **1.195** | **1.196** | U | U | U | U | U | **6490.9** |
| LCFR1 (BT) | U | 1.197 | U | **4.1233** | U | U | U | U |
| LCFR1 (SPA) | **1.195** | **1.196** | U | U | U | U | U | **6490.9** |
| RCFR2 (BT) | **1.195** | **1.196** | 1.199 | **1.196** | **2.758** | 1.6811 | U | U |
| RCFR2 (SPA) | 1.196 | **1.196** | U | **1.196** | U | U | U | **6490.9** |
| LCFR2 (BT) | U | 1.197 | U | **4.1233** | U | U | U | U |
| LCFR2 (SPA) | **1.195** | **1.196** | U | U | U | U | U | **6490.9** |

extending the **SR** and **BFSR** accuracy enhancing techniques to frequency-weighted balancing; and (3) improving the computational efficiency of several balancing related controller reduction approaches by fully exploiting the underlying problem structure when computing frequency-weighted Gramians. To ease the implementation of these approaches, we provide complete directly implementable formulas for frequency-weighted Gramian computations.

As can be seen clearly from Table 9.9, none of existing methods seems to be universally applicable and their performances are very hard to predict. However, having several alternative approaches at our disposal certainly increases the chance of obtaining acceptable low order controller approximations. For several approaches, ready to use controller reduction software is freely available in the Fortran 77 library SLICOT, together with user friendly interfaces to the computational environments MATLAB and `Scilab`. For the rest of methods described in this paper, similar software can be easily implemented using standard computational tools provided in SLICOT.

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