Numerical optimal control applied to an epidemiological model

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Abstract: A previously published SIR-ASI optimal control model of dengue fever is described and the optimal control problem is solved in this paper by an alternative solution approach, namely by a direct transcription method. In this method, the optimal control problem is substituted by a nonlinear programming problem. The nonlinear programming problem is solved by an interior point method. In the following an a posteriori check of the necessary optimality conditions of optimal control, which has not yet been done, is performed with the discretized states, adjoints, and controls. The check shows some accordance with the numerical results, but unfortunately in this setting a seldom discrepancy is observed, which seems to be connected with the modeling of the mechanical control.

Keywords: Dengue fever, mechanical control, insecticide, optimal control, direct transcription, a posteriori verification of numerical results

1. INTRODUCTION

Dengue fever is an infectious disease which occurs in tropical and subtropical areas. According to the World Health Organization (WHO (2017)) about 50% of the world population is nowadays at risk. Dengue fever is caused by viruses and transmitted to humans by the bite of an infected mosquito, mainly of the species Aedes aegypti. The symptoms of dengue fever include high fever, severe headache, muscle and joint pains, and nausea. Complications occur in seldom cases, but may lead to severe dengue, a potentially lethal course of the disease. A first vaccine against dengue fever has only recently been made available and several other vaccines are under development. Furthermore, there is no specific anti-viral treatment for dengue. For these reasons the best protection against dengue fever at present is avoiding the contact with infected mosquitoes, including the control of mosquitoes and their eggs as described in WHO (2009).

The course of the disease in a population can be simulated with mathematical models in order to make statements about future developments and to suggest possible strategies to attenuate the spread of the disease. The investigation of a model describing an epidemic outbreak of dengue fever in Cape Verde is the subject matter in Rodrigues et al. (2013b). In this paper three vector control operations are considered: larvicides which are distributed in the water reservoirs in order to combat the mosquitoes in the aquatic phase (eggs, larvae, and pupae); adulticides against the adult mosquitoes which can be sprayed in buildings or outdoor to achieve a fast and significant decrease in the mosquito population; and the mechanical control including the removal of standing waters and small water reservoirs to get rid of possible breeding grounds. The aim of using control measures is to keep the number of infected individuals as low as possible. This also leads to less medical treatment, hospitalizations, and absences from work due to illness and thus to less costs. On the other hand, the costs caused by the use of the controls should also be as low as possible, including the development and distribution of insecticides and search for breeding areas and their subsequent elimination. The goal is to optimize the tradeoff between costs and effectiveness.

Rodrigues et al. (2013a) consider an optimal control problem and solve it numerically, but no verification of the results is done. We apply an alternative numerical approach for the solution of this optimal control problem by direct transcription into a nonlinear programming problem, see e.g. Betts (2001). The message along the line of Vanderbei (2001); Dussault (2014) is that one has to be careful when applying the direct transcription approach to optimal problems which are non-convex in the controls which is the case here for one of the three controls. We use the solver IPOPT of Wächter and Biegler (2006) and compare our results to Rodrigues et al. (2013a) to investigate if our approach yields better outcomes. Furthermore, an a posteriori verification of the necessary optimality conditions of optimal control is performed. We present analytical control laws obtained from Pontryagin's minimum principle and use them to validate our numerical results.

In Section 2 the model is described. The numerical results and the verification are shown in Section 3.

2. DENGUE SIR-ASI-MODEL

We firstly describe the model from Rodrigues et al. (2013b). We consider a human population of N_h individuals and a mosquito population. Both populations are split into disjoint groups (compartments). The human population is divided into three groups, the susceptibles S_h, the infected and infectious I_h, and the recovered R_h. These variables denote total numbers. By birth, all humans are assumed to be susceptible which means they have not yet been infected by dengue fever. Once they have been bitten by an infected mosquito and become carriers of the virus, they are infected and able to transmit the virus to mosquitoes. After recovery the humans are healthy and cannot contract the disease again. The total human population N_h is assumed to be constant over time t, so $N_h = S_h(t)+I_h(t)+R_h(t)$. The mosquitoes are also divided into three groups. For this classification it is necessary to know the life cycle of the mosquitoes: Female mosquitoes bite humans and lay their eggs in small water containers, like empty cans or old car tyres. After a few days larvae hatch from the eggs and mature further to pupae. The adult mosquitoes emerge from the pupae. The first group of the mosquitoes, A_m, refers to the mosquitoes in the aquatic phase and includes the egg, larva, and pupa stages. The other two groups characterize the adult mosquitoes, the susceptibles S_m and the infected I_m. The mosquitoes get infected and become vectors of the disease by taking a blood meal from an infectious human. After that they are able to transmit the virus to humans by biting them. The mosquitoes live to briefly to become immune. Due to this six compartments the model is also called an SIR-ASI model. Individuals in one group are assumed to be homogeneous and they can change between compartments during the course of the disease. The dynamics of the flow between the compartments is described using differential equations. The mosquito population is affected by the use of vector control measures. The three control variables considered in this model include

- c_m, the proportion of adulticides, $(0 \leq c_m \leq 1)$,
- c_A, the proportion of larvicides, $(0 \leq c_A \leq 1)$, and
- α, the proportion of mechanical control $(0 < \alpha_{\min} \leq \alpha \leq 1)$.

Table 1. Parameters and values in the model according to Rodrigues et al. (2013b)

Parameter	Value	Description
N_h	$480000[-]$	total human population
B	$0.8\left[\frac{\text{bite}}{\text{day}}\right]$	average number of bites of a mosquito per day
$\frac{1}{\mu_h}$	$71 \cdot 365[\text{day}]$	average lifespan of humans in days
β_{mh}	$0.375\left[\frac{1}{\text{bite}}\right]$	probability of disease transmission from mosquitoes to humans per bite
β_{hm}	$0.375\left[\frac{1}{\text{bite}}\right]$	probability of disease transmission from humans to mosquitoes per bite
$\frac{1}{\eta_h}$	$3[\text{day}]$	duration of the infection in days
φ	$6\left[\frac{1}{\text{day}}\right]$	number of eggs at each deposit per mosquito per day
$\frac{1}{\mu_m}$	$10[\text{day}]$	average lifespan of adult mosquitoes in days
μ_A	$\frac{1}{4}\left[\frac{1}{\text{day}}\right]$	natural mortality rate of larvae per day
η_A	$0.08\left[\frac{1}{\text{day}}\right]$	maturation rate from larvae to adult per day
m	$3[-]$	number of female mosquitoes per human
k	$3[-]$	total number of larvae per human

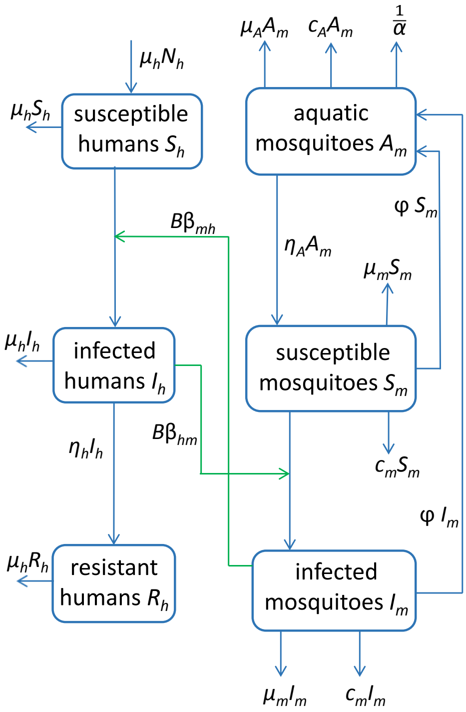

Fig. 1. SIR for humans – ASI for mosquitoes

The first control combats the adult mosquitoes in the compartments S_m and I_m, while the latter two are directed against the mosquitoes in the aquatic phase A_m. The maximal use of insecticides is attained at $c_m = 1$ and $c_A = 1$. Since $\alpha \neq 0$, because it appears in a denominator in the model, we introduce $\alpha_{\min}$ as a lower bound which corresponds to the maximal use of mechanical control. Here $\alpha_{\min} = 0.001$ is chosen. Note that the case $c_m = 0$, $c_A = 0$, and $\alpha = 1$ means no control at all. The parameters used in the model are summarized in Table 1 with the values of the Cape Verde data from Rodrigues et al. (2013b). The state variables are S_h, I_h, R_h, A_m, S_m, and I_m. Together with the control variables and the transition and contact rates between the groups, the model is described and a nonlinear system of differential equations is obtained. Fig. 1 shows a scheme of this model. For the analysis we use the normalized system. The following transformations are carried out: $s_h = \frac{S_h}{N_h}$, $i_h = \frac{I_h}{N_h}$, $r_h = \frac{R_h}{N_h}$, $a_m = \frac{A_m}{kN_h}$, $s_m = \frac{S_m}{mN_h}$, and $i_m = \frac{I_m}{mN_h}$. The model is thus given by:

$$
\begin{aligned}
\frac{ds_h}{dt} &= \mu_h - \left(B\beta_{mh} m i_m + \mu_h\right) s_h \\
\frac{di_h}{dt} &= B\beta_{mh} m i_m s_h - (\eta_h + \mu_h) i_h \\
\frac{dr_h}{dt} &= \eta_h i_h - \mu_h r_h \\
\frac{da_m}{dt} &= \varphi \frac{m}{k}\left(1 - \frac{a_m}{\alpha}\right)(s_m + i_m) - (\eta_A + \mu_A + c_A) a_m \\
\frac{ds_m}{dt} &= \eta_A \frac{k}{m} a_m - \left(B\beta_{hm} i_h + \mu_m + c_m\right) s_m \\
\frac{di_m}{dt} &= B\beta_{hm} i_h s_m - (\mu_m + c_m) i_m
\end{aligned}
\tag{1}
$$

3. OPTIMAL CONTROL

The costs and benefits of the controls should be weighed and both epidemiological and economic goals considered. This is taken into account in the objective function of the optimal control problem. This function, that should be minimized, is composed of the costs caused by the presence of infected people i_h and by the use of adulticides c_m, larvicides c_A, and mechanical control α. The positive constants γ_D, γ_S, γ_L, and γ_E serve as weights and enable different evaluations of these costs. They are chosen so that $\gamma_D+\gamma_S+\gamma_L+\gamma_E = 1$ is fulfilled. The constraints are given by the normalized system (1) and the optimization problem is completed through appropriate initial conditions for the states and bounds on the controls. The problem is given by:

$$\min \int_0^T \gamma_D i_h(t)^2 + \gamma_S c_m(t)^2 + \gamma_L c_A(t)^2 + \gamma_E(1-\alpha(t))^2 dt$$

s. t. equations (1),

$$s_h(0) = 0.9999, \quad i_h(0) = 0.0001, \quad r_h(0) = 0, \qquad (2)$$
$$a_m(0) = 1, \quad s_m(0) = 1, \quad i_m(0) = 0,$$
$$0 \le c_m \le 1, \quad 0 \le c_A \le 1, \quad 0 < \alpha_{\min} \le \alpha \le 1.$$

3.1 Numerical results

A *first-discretize-then-optimize* approach is chosen to simulate the optimal control problem. Using a direct approach we discretize both states and controls over a time grid with a constant discretization step size. A period of one year, or 365 days, respectively, is considered in all simulations. The simulations are accomplished with a step size of $\frac{365}{1000}$. The control functions are approximated over the time grid as piecewise constant functions. The computations are performed using the implicit Euler method and the cost functional of the optimal control problem (2) is discretized with the rectangular formula. This yields a nonlinear programming problem which is solved via the modelling language AMPL of Fourer et al. (2003) and the solver IPOPT developed by Wächter and Biegler (2006). The latter is a well-known interior point algorithm for solving large-scale nonlinear programming problems. One reason to use this combination is the ability of AMPL for automatic differentiation which provides IPOPT with exact derivatives. Furthermore an easy and intuitive implementation of optimal control problems is possible by using AMPL. The simulations are carried out with the data in Table 1. Solving problem (2) with the described approach, we obtain the optimal application of the three controls as shown in Fig. 2, over a period of one year with equal weights. According to these results, a low use of the control measures is optimal, while c_m is the most used control, specifically in the first 50 days. Fig. 3 shows a comparison between the proportion of infected people in a population without any controls and the corresponding proportion with an optimal use of the controls. In addition, the analogous figure for the proportion of recovered people is shown. The proportion of infected humans can be reduced, from over 0.16 in peak without control to about 0.09 in the case with controls. So even a little use of the controls

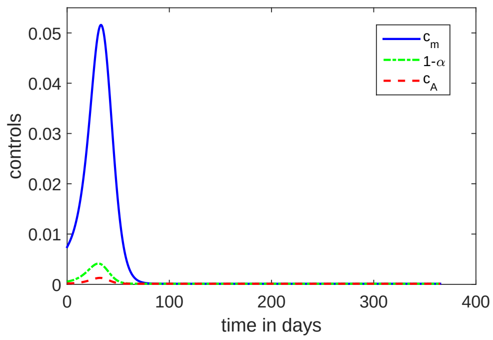

Fig. 2. Optimal control functions c_m, $1-\alpha$, and c_A with $\gamma_D = \gamma_S = \gamma_L = \gamma_E = 0.25$

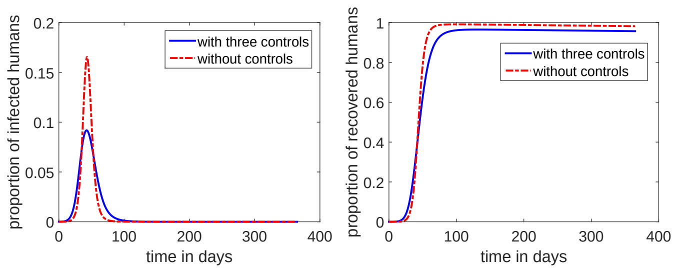

Fig. 3. Proportion of infected and recovered humans with and without control

shows a great effect. Compared to the numerical results in Rodrigues et al. (2013a) we obtain similar optimal control functions. But there are differences in the values of the objective function that should be minimized. For equal weights we receive 0.0592 whereas in Rodrigues et al. (2013a) a higher value of 0.0669 is reached.

3.2 Verification

After solving the nonlinear programming model with the described approach, an a posteriori verification of the numerical results is now carried out. In this step we try to check the necessary optimality conditions of optimal control in order to verify the candidate optimality of the continuous problem via the approximate discrete optimal solution. The optimality is checked by the use of Pontryagin's minimum principle and so the numerical results are compared to an analytical optimal control law. Therefore we firstly have to define the Hamiltonian function:

Definition 1. (The Hamiltonian). Let x be the state variable and u the control variable of an optimal control problem. The integrand of the corresponding cost functional is defined by $f_0(t, x, u)$ and $f(t, x, u) = \dot{x}$ represents the ODE constraints. The latter functions are assumed to be sufficiently differentiable. Then the function

$$\mathcal{H}(t, x, \lambda, u) = \lambda_0 f_0(t, x, u) + \lambda f(t, x, u) \qquad (3)$$

is called the Hamiltonian function, where λ is the adjoint variable. Here λ_0 is a scalar and λ a row vector of appropriate size.

In general $\lambda_0 = 1$ can be set. This definition follows the direct adjoining approach described in Section 4 of Hartl et al. (1995). For a proof of Pontryagin's minimum principle see Ioffe and Tihomirov (1979). We give here only the most important conclusion of this principle which is used in the following analyses. It states that for all $t \in [0,T]$ the optimal control u^* is the global minimizer of the Hamiltonian function, pointwisely evaluated along an optimal trajectory $(x^*(t), \lambda(t))$, where u lies in the admissible region U:

$$\mathcal{H}(t, x^*(t), \lambda(t), u^*(t)) = \min_{u \in U} \mathcal{H}(t, x^*(t), \lambda(t), u) \,. \tag{4}$$

This law is used to validate our (discretized) numerical results at least approximately. In the interior of the admissible region holds for all components u_j^* of the optimal control vector u^*

$$\frac{d\mathcal{H}}{du_j^*} = 0 \,. \tag{5}$$

Eq. (4) is a necessary optimality condition. So the (discretized) numerical solution of states, adjoints, and controls of the NLP from IPOPT should provide the global minimum of the Hamiltonian with respect to the admissible controls. The Hamiltonian function for our dengue model, with f_0 defined in the cost functional of (2) and f from the differential equations in (1), is given by:

$$\begin{aligned}\mathcal{H} = {} & \gamma_D i_h^2 + \gamma_S c_m^2 + \gamma_L c_A^2 + \gamma_E (1-\alpha)^2 \\ & + \lambda_1 (\mu_h - (B\beta_{mh} m i_m + \mu_h) s_h) \\ & + \lambda_2 (B\beta_{mh} m i_m s_h - (\eta_h + \mu_h) i_h) \\ & + \lambda_3 (\eta_h i_h - \mu_h r_h) \\ & + \lambda_4 \left(\varphi \frac{m}{k} \left(1 - \frac{a_m}{\alpha}\right)(s_m + i_m) - (\eta_A + \mu_A + c_A) a_m\right) \\ & + \lambda_5 \left(\eta_A \frac{k}{m} a_m - (B\beta_{hm} i_h + \mu_m + c_m) s_m\right) \\ & + \lambda_6 (B\beta_{hm} i_h s_m - (\mu_m + c_m) i_m)\end{aligned} \tag{6}$$

This function can be split into a sum of four functions $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$, as follows:

$$\mathcal{H}(c_m, c_A, \alpha) = \mathcal{H}_1(c_m) + \mathcal{H}_2(c_A) + \mathcal{H}_3(\alpha) + \mathcal{H}_4 \,. \tag{7}$$

The function $\mathcal{H}_1$ is composed of the terms of $\mathcal{H}$ which depend on the control c_m, the same applies to the functions $\mathcal{H}_2$ and $\mathcal{H}_3$ and the controls c_A and α, respectively. The function $\mathcal{H}_4$ summarizes all terms of $\mathcal{H}$ where none of the controls appear. According to the minimum principle the requested optimal controls minimize the Hamiltonian function $\mathcal{H}$ at any point of time. $\mathcal{H}$ is separated in c_m, c_A, and α, since no mixed terms of these controls occur. So the minimum of the Hamiltonian function (6) with respect to the controls can be expressed as the sum of the minima of the three single functions $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$:

$$\begin{aligned}\min_{(c_m, c_A, \alpha)} \mathcal{H}(c_m, c_A, \alpha) = {} & \min_{c_m} \mathcal{H}_1(c_m) + \min_{c_A} \mathcal{H}_2(c_A) \\ & + \min_{\alpha} \mathcal{H}_3(\alpha) + \mathcal{H}_4\end{aligned} \tag{8}$$

$\mathcal{H}_4$ is a constant in this minimization, so it can be neglected in the following. Now we compute the minima of the functions $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ separately and compare them to the corresponding solution of IPOPT. These functions are deduced from (6) and thus given by:

$$\begin{aligned}\mathcal{H}_1(c_m) &= \gamma_S c_m^2 - \lambda_5(t) s_m(t) c_m - \lambda_6(t) i_m(t) c_m \\ \mathcal{H}_2(c_A) &= \gamma_L c_A^2 - \lambda_4(t) a_m(t) c_A \\ \mathcal{H}_3(\alpha) &= \gamma_E (1-\alpha)^2 - \lambda_4(t) \varphi \frac{m}{k} \frac{a_m(t)}{\alpha} (s_m(t) + i_m(t))\end{aligned} \tag{9}$$

Note that $\mathcal{H}_1$ and $\mathcal{H}_2$ are strictly convex functions of c_m and c_A, respectively, but that $\mathcal{H}_3$ is a non-convex function of α. The states a_m, s_m, and i_m, as well as the adjoints λ_4, λ_5, and λ_6, which appear in these equations, are time-dependent. So there is a Hamiltonian function for each $t \in [0,T]$ which has to be minimized. At first we consider

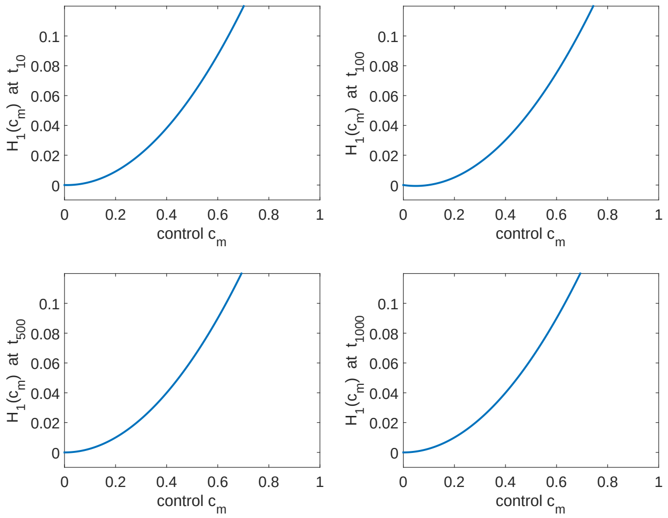

Fig. 4. $\mathcal{H}_1(c_m)$ (defined in (9)) with $\gamma_S = 0.25$ at different time steps $t_i = i \cdot \frac{365}{1000}$, $i \in \{10, 100, 500, 1000\}$

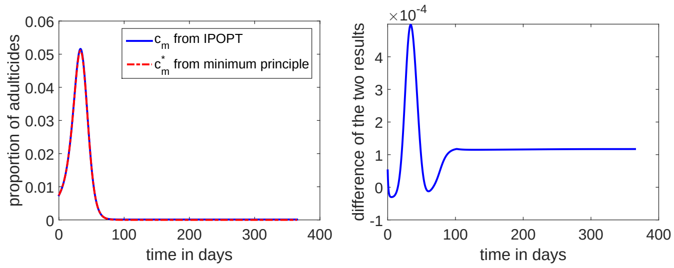

Fig. 5. Comparison of the two results for c_m: obtained from IPOPT and (11) from the minimum principle, with $\gamma_S = 0.25$

the optimal control c_m of adulticides and the minimum of $\mathcal{H}_1(c_m)$. IPOPT provides the values for the states and adjoints in each time step. Fig. 4 shows examples of the course of the function $\mathcal{H}_1(c_m)$ at four time steps. The function does not vary much for different times t and different weights γ_S, so just a selection of these four cases is shown. As can be seen, the (global) minimum of the function lies approximately between 0 and 0.2 for all time steps. For a detailed analysis the minimum of the function $\mathcal{H}_1$ is now computed by using (5). The (global) minimum of the optimization problem can either lie in the interior of the admissible set of control values or on the edge of this range. In case the (global) minimum is in the interior of the set, it is given by setting the derivative of $\mathcal{H}_1$ to zero and solving this equation for c_m:

$$\frac{d\mathcal{H}_1}{dc_m} \stackrel{!}{=} 0 \quad \Rightarrow \quad c_m^{int} = \frac{\lambda_5 s_m + \lambda_6 i_m}{2\gamma_S} \,. \tag{10}$$

If $c_m^{int} \notin (0,1)$ we have to clip its values to $[0,1]$, corresponding to the bounds given in (2). Thereby we obtain for the optimal control c_m^*:

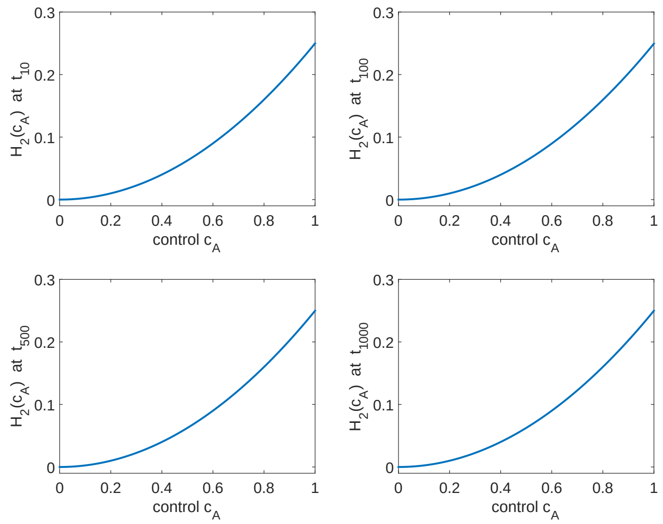

Fig. 6. $\mathcal{H}_2(c_A)$ (defined in (9)) with $\gamma_L = 0.25$ at different time steps $t_i = i \cdot \frac{365}{1000}$, $i \in \{10, 100, 500, 1000\}$

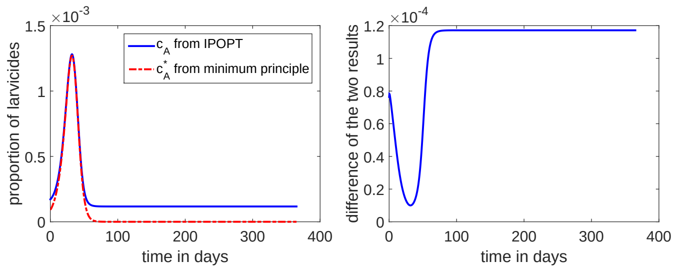

Fig. 7. Comparison of the two results for c_A: obtained from IPOPT and (13) from the minimum principle, with $\gamma_L = 0.25$

$$c_m^* = \min\left\{1, \max\left\{0, c_m^{int}\right\}\right\} . \tag{11}$$

Fig. 5 illustrates this optimal control derived from the minimum principle and the state and costate trajectory derived from IPOPT. This control is compared to the control solution found by IPOPT in the previous subsection. As shown in the figure, the two results coincide since the shapes of the functions are almost identical. The figure also shows the difference of these two results which varies in a very small range. The same procedure is applied to the optimal control of larvicides c_A. The corresponding part of the Hamiltonian function, $\mathcal{H}_2$ from (9), is shown in Fig. 6 at four time steps. Again, the shapes of the function show only very slight changes for different times and different weights γ_L. The minimum in the interior of the admissible set is calculated in the same way as before:

$$\frac{d\mathcal{H}_2}{dc_A} \overset{!}{=} 0 \quad \Rightarrow \quad c_A^{int} = \frac{\lambda_4 a_m}{2\gamma_L} . \tag{12}$$

Since the bounds for the use of larvicides are again given by 0 and 1, we obtain the optimal control c_A^* by

$$c_A^* = \min\left\{1, \max\left\{0, c_A^{int}\right\}\right\} . \tag{13}$$

This result achieved by the application of the minimum principle is depicted in Fig. 7 in comparison to the solution of IPOPT. As in the previous case with c_m, the two results are well matched. The differences of the two

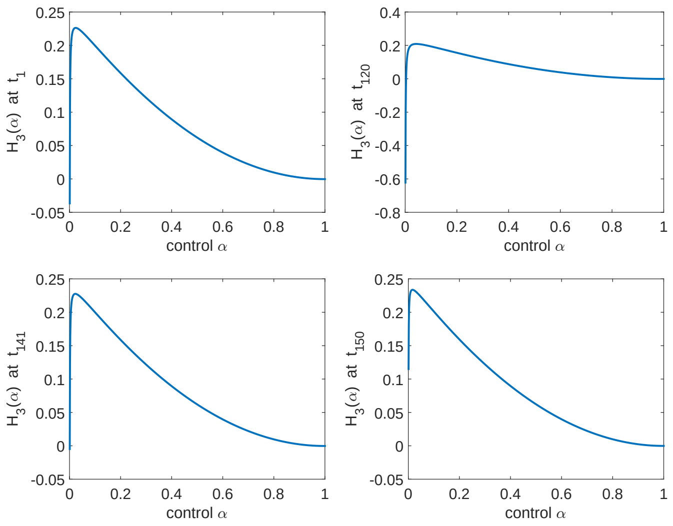

Fig. 8. $\mathcal{H}_3(\alpha)$ (defined in (9)) with $\gamma_E = 0.25$ at different time steps $t_i = i \cdot \frac{365}{1000}$, $i \in \{1, 120, 141, 150\}$

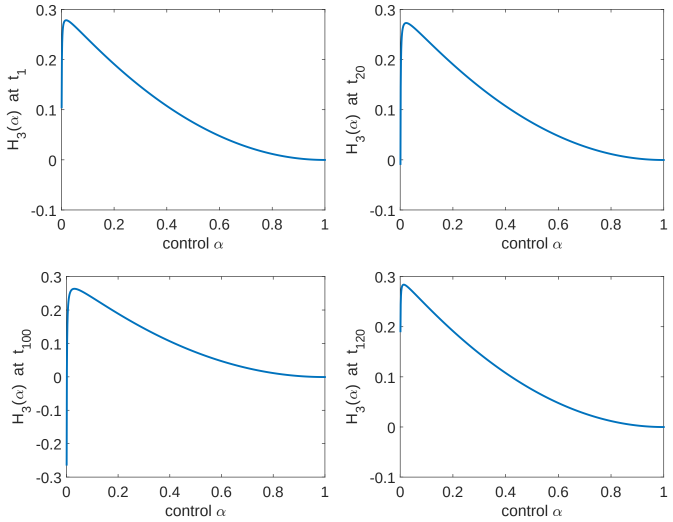

Fig. 9. $\mathcal{H}_3(\alpha)$ (defined in (9)) with $\gamma_E = 0.30$ at different time steps $t_i = i \cdot \frac{365}{1000}$, $i \in \{1, 20, 100, 120\}$

functions occur in a very small extent. In comparison to the adulticides c_m, much less of the larvicides c_A is used in an optimal case. The situation with the mechanical control α is different. Differentiating the corresponding part of the Hamiltonian function in (9) we obtain:

$$\frac{d\mathcal{H}_3}{d\alpha} = -2\gamma_E(1-\alpha) + \lambda_4 \varphi \frac{m}{k} a_m (s_m + i_m) \frac{1}{\alpha^2} \overset{!}{=} 0 . \tag{14}$$

Solving this equation for α yields three solutions. The shape of the Hamiltonian function $\mathcal{H}_3$ shows greater variations than in the previously considered cases for $\mathcal{H}_1$ and $\mathcal{H}_2$. This is shown in Fig. 8, which illustrates this function at four time steps. For small times the minimum of the function lies on the edge of the admissible set, at $\alpha_{\min} = 0.001$. Additionally there is a local minimum of the function close to one. Around the time step t_{141} the position of the global minimum changes and is from that moment until the end of the considered time interval near one. Nevertheless this observation cannot be generalized because there are also significant variations of the Hamiltonian function $\mathcal{H}_3$ for different values of the weight γ_E. Fig. 9 shows four cases of this function for $\gamma_E = 0.3$. In this case, the minimum lies near one for small times, then from about t_{20} to t_{109} near zero on the edge of the set, and then from that point on again near one. The optimal solution for the mechanical control α found by IPOPT is already known and the function $1 - \alpha$ is shown in Fig. 2. As a result, the function α is near one for all time steps. This is a contradiction to the solution which is found by using the minimum principle. In this approach the minimum of the Hamiltonian function is on the edge of the admissible set for certain times that means maximal control in this case. Fig. 10 shows the differences between the two results for α, corresponding to the right hand side plots in Figs. 5 and 7 for the other two controls. Since no analytical result for α is available, Fig. 10 compares the result of IPOPT with the values of α at which the minimum of $\mathcal{H}_3$ is reached, for both $\gamma_E = 0.25$ and $\gamma_E = 0.30$. These γ_E values correspond to Figs. 8 and 9. For some time steps the difference is nearly 100%. Since the minimum principle provides a necessary optimality condition which is violated here, the solution provided by IPOPT for the discretized NLP turns out to be not the minimum of the underlying optimal control problem. The proposed numerical solution provides an optimal solution for c_m and c_A, but does not for α. Due to the good coding of the algorithm we are sure that we have at least found a local minimum (or at least a KKT point) of the NLP problem. In practice the suboptimality might not be a concern since the function values of the cost functional for the presented numerical solution and the true solution might not be very different due to the mixing principle. Note that the NLP problem (and the optimal control problem) is non-convex in the control α. Since we have to deal with a non-regular Hamiltonian, it is known that difficulties are to be expected and finding a global minimum of the optimal control problem in this case is hard.

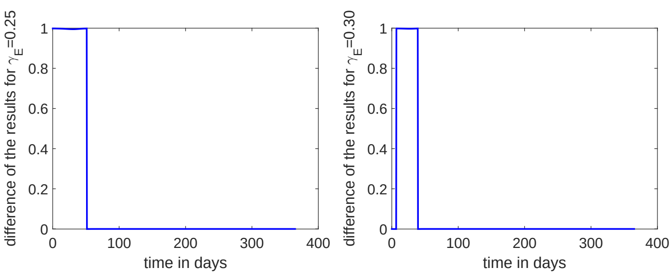

Fig. 10. Difference of two results for α: obtained from IPOPT and value of α at the minimum of $\mathcal{H}_3(\alpha)$ at each time step, with $\gamma_E = 0.25$ resp. $\gamma_E = 0.30$

4. CONCLUSION

In this paper a mathematical model about the dynamical development of dengue fever is described and measures to control the transmission vector are introduced. To find the optimal extent of the measures, and to give recommendations about the amount of their use in affected areas, an optimal control approach is applied. The numerical results using AMPL and IPOPT are presented and a verification of these results using the minimum principle of optimal control theory is performed. The simulations with the results of IPOPT are very promising, because they show that even a little use of the insecticides and the mechanical control can reduce the number of infected individuals. The numerical computed solution does fulfill a local minimum principle of the Hamiltonian but unfortunately not the global minimum principle of the Hamiltonian. Therefore the numerical computed solution seems not to be the best possible one. On the other hand, the new numerical solution gives a better value of the cost functional in contrast to the original papers Rodrigues et al. (2013b,a), which is a good motivation of pursuing our approach. However, one has to keep in mind in this comparison that the discretization error of the ODE as well as the error in the discretized constraints might be different. The observed discrepancies have to be addressed in future research.

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