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Computation of Kalman decompositions of periodic systems

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Abstract

We consider the numerically reliable computation of reachability and observability Kalman decompositions of a periodic system with time-varying dimensions. These decompositions generalize the controllability/observability Kalman decompositions for standard state space systems and have immediate applications in the structural analysis of periodic systems. We propose a structure exploiting numerical algorithm to compute the periodic controllability form by employing exclusively orthogonal state-space similarity transformations. The new algorithm is computationally efficient and backward stable, thus fulfils all requirements for a satisfactory algorithm for periodic systems.

Keywords: Periodic systems, discrete-time systems, time-varying systems, Kalman decomposition, numerical methods.

1 Introduction

Among the important open computational problems which we listed in a recent survey [15], the computation of periodic reachability and observability Kalman decompositions is one which has many useful applications. Besides characterizing the structural properties of the system (reachability/controllability, observability/reconstructibility), properties as stabilizability and detectability can be checked by computing the non-reachable and non-observable characteristic multipliers. Furthermore, by computing the periodic reachability form and the dual periodic observability form of the reachable subsystem, minimal realizations of periodic systems can be easily computed. This computation is a basic step in a recently developed algorithm to evaluate the transfer-function matrix of a periodic system [13].

We consider periodic time-varying systems of the form

$$\begin{array}{rcl} x(k+1) & = & A_k x(k) + B_k u(k) \\ y(k) & = & C_k x(k) \end{array} \tag{1}$$

where the matrices $A_k \in \mathbb{R}^{n_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{n_{k+1}\times m_k}$, $C_k \in \mathbb{R}^{p_k\times n_k}$, are periodic with period $N \geq 1$. This periodic system will be alternatively denoted by the periodic triple (A_k, B_k, C_k).

For the definition of the periodic reachability/observability Kalman decompositions it is important to consider the more general case of time-varying dimensions. Note that the Kalman decompositions even of constant dimension periodic systems may lead to reachable/observable and unreachable/unobservable subsystems with time-varying dimensions [4]. Thus, the minimal realizations of periodic systems (i.e., reachable and observable) have, in general, time-varying state dimensions [2, 3]. Periodic systems with time varying input and output vector dimensions have been considered in [6] and arise in a natural way in some computational problems [13].

Periodic systems with time-varying state dimensions have been already considered earlier in [5, 3]. However, numerically reliable algorithms for systems with time-varying dimensions have been developed only very recently. Notable examples are the algorithms for the computation of minimal realizations [12], the evaluation of the transfer-function matrix of a periodic system [13], and the numerically stable algorithms to compute the zeros of periodic systems [16, 14]. Note that, the development of general algorithms able to address the case of time-varying dimensions, is one of the requirements which we formulated for a *satisfactory* numerical algorithm for periodic systems [15].

The computation of Kalman decompositions by using orthogonal similarity transformations was one of the first numerically stable algorithms developed to solve system theoretic problems. In a survey [9], six distinct groups of authors are cited who proposed around 1981, almost simultaneously, numerically reliable algorithms to compute the Kalman controllability decomposition via the so-called *controllability staircase form*. Although the corresponding theoretical results have been extended to the periodic case by Grasselli already in 1984 [4], and subsequently have been refined in the works of various authors [8, 3, 2], until now there exists no computation oriented algorithm to compute the periodic Kalman decompositions.

In this paper, we propose a structure exploiting numerically reliable algorithm to compute the Kalman reachability decomposition for discrete-time periodic systems using exclusively orthogonal state-space similarity transformations. A dual algorithm can be used to compute the Kalman observability decomposition. With these two algorithms, the minimal realization problem of periodic systems can be solved in a numerically reliable way. The new algorithm is computationally efficient and backward stable, thus fulfils all requirements for a satisfactory algorithm for periodic systems.

2 Periodic Kalman decompositions

The transition matrix of the system (1) is defined by the $n_j \times n_i$ matrix $\Phi_A(j,i) = A_{j-1}A_{j-2}\cdots A_i$, where $\Phi_A(i,i) := I_{n_i}$. In the case of a null dimension, say n_j, by convention the product A_jA_{j-1} is a $n_{j+1}\times n_{j-1}$ matrix of zeros. The state transition matrix over one period $\Phi_A(j+N,j) \in \mathbb{R}^{n_j\times n_j}$ is called the *monodromy matrix* of system (1) at time j and its eigenvalues are called the *characteristic multipliers* at time j. Note that the spectrum of $\Phi_A(j+K,j)$ contains always at least $n_j - \underline{n}$ zero elements, where $\underline{n} := \min_k\{n_k\}$. The rest of $\underline{n}$ eigenvalues are independent of time j and form the *core characteristic multipliers* [5]. The periodic system (1) is *asymptotically stable* if all characteristic multipliers belong to the open unit disk.

For the definitions of reachability, observability and minimality of periodic systems we rely on [1] (see also [3] for a more detailed exposition).

Definition 1 *The periodic system* (1) *is* reachable at time k *if*

$$\operatorname{rank}\mathcal{R}_k = n_k, \tag{2}$$

where $\mathcal{R}_k$ *is the infinite columns reachability matrix*

$$\mathcal{R}_k = [\, B_{k-1} \;\; A_{k-1}B_{k-2} \;\cdots\; \Phi_A(k,i+1)B_i \;\cdots]. \tag{3}$$

The periodic system (1) *is* completely reachable *if* (2) *holds for all* k.

Definition 2 *The periodic system* (1) *is* observable at time k *if*

$$\operatorname{rank}\mathcal{O}_k = n_k, \tag{4}$$

where $\mathcal{O}_k$ is the infinite rows observability matrix

$$\mathcal{O}_k = \begin{bmatrix} C_k \\ C_{k+1}A_k \\ \vdots \\ C_i\Phi_A(i,k) \\ \vdots \end{bmatrix}. \tag{5}$$

The periodic system (1) *is* completely observable *if* (4) *holds for all k.*

Definition 3 *The periodic system* (1) *is* minimal *if it is completely reachable and completely observable.*

Let $S_k \in \mathbb{R}^{n_k \times n_k}$ be an N-periodic nonsingular matrix. The reachability, observability and minimality properties are invariant under a *state-space similarity transformation* of the form

$$\widetilde{A}_k = S_{k+1}^{-1}A_kS_k, \quad \widetilde{B}_k = S_{k+1}^{-1}B_k, \quad \widetilde{C}_k = C_kS_k$$

The reachability and observability Kalman decompositions of periodic systems have been introduced in [4] for systems with constant dimensions and extended recently to the case of time-varying dimensions [6]. We recall below the main results of [4, 6]:

Theorem 1 *Every N-periodic system (A_k, B_k, C_k) is state-space equivalent to an N-periodic system $(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k)$, with*

$$\widetilde{A}_k = \begin{bmatrix} A_k^r & * \\ 0 & A_k^{\overline{r}} \end{bmatrix}, \quad \widetilde{B}_k = \begin{bmatrix} B_k^r \\ 0 \end{bmatrix}, \quad \widetilde{C}_k = [\, C_k^r \;\; C_k^{\overline{r}} \,] \tag{6}$$

where $A_k^r \in R^{r_{k+1}\times r_k}$, $r_k = rank\mathcal{R}_k$ and the periodic pair (A_k^r, B_k^r) is completely reachable.

The decomposition of the system matrices in the form (6) is called the *periodic Kalman reachability decomposition* (PKRD). The transfer-function matrices of the corresponding linear time-invariant lifted representations (see [7]) of the reachable subsystem (A_k^r, B_k^r, C_k^r) and of the original periodic system (A_k, B_k, C_k) are the same [2]. The unreachable characteristic multipliers of the system (1) are the eigenvalues of $\Phi_{A^{\overline{r}}}(N, 0)$.

Definition 4 *The periodic system* (1) *is* completely controllable *if all unreachable characteristic multipliers are zero.*

It is interesting to note that every periodic system having at least one null state vector dimension is completely controllable. This follows easily, since the set of core characteristic eigenvalues being empty, only zero characteristic multipliers may appear.

The dual result to Theorem 1 is the following one:

Theorem 2 *Every N-periodic system (A_k, B_k, C_k) is state-space equivalent to an N-periodic system $(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k)$, with*

$$\widetilde{A}_k = \begin{bmatrix} A_k^o & 0 \\ * & A_k^{\overline{o}} \end{bmatrix}, \quad \widetilde{B}_k = \begin{bmatrix} B_k^o \\ B_k^{\overline{o}} \end{bmatrix}, \quad \widetilde{C}_k = \begin{bmatrix} C_k^o & 0 \end{bmatrix} \tag{7}$$

where $A_k^o \in R^{q_{k+1}\times q_k}$, $q_k = rank\mathcal{O}_k$, and the periodic pair (A_k^o, C_k^o) is completely observable.

The transfer-function matrices of the corresponding lifted systems of the observable periodic subsystem (A_k^o, B_k^o, C_k^o) and of the original periodic system (A_k, B_k, C_k) are the same [2]. The unobservable characteristic multipliers of the system (1) are the eigenvalues of $\Phi_{A^{\overline{o}}}(N, 0)$.

Definition 5 *The periodic system* (1) *is* completely reconstructible *if all unobservable characteristic multipliers are zero.*

3 PKRD algorithm

In this section we show that the periodic reachability form (6) can be computed using orthogonal state-space similarity transformations and we develop an efficient computational algorithm which generalizes the algorithm of [10] and similar algorithms cited in [9].

To justify our approach, we consider the periodic pair (A_k, B_k) and let U_k be periodic orthogonal state-space transformations such that each U_{k+1} compresses B_k to a full row rank matrix. If ν_{k+1} is the rank of B_k, then we can write

$$U_{k+1}^T B_k := \begin{array}{cc} \left[\begin{array}{c} A_{k,10} \\ 0 \end{array} \right] & \begin{array}{c} \nu_{k+1} \\ \rho_{k+1} \end{array} \\ m_k & \end{array} , \tag{8}$$

where $A_{k,10}$ has full row rank. We apply the transformation to A_k and partition $U_{k+1}^T A_k U_k$ as follows

$$U_{k+1}^T A_k U_k := \begin{array}{cc} \left[\begin{array}{cc} A_{k,11} & A_{k,12} \\ \widetilde{B}_k & \widetilde{A}_k \end{array} \right] & \begin{array}{c} \nu_{k+1} \\ \rho_{k+1} \end{array} \\ \begin{array}{cc} \nu_k & \rho_k \end{array} & \end{array} \tag{9}$$

Note that some dimensions can be zero, depending on the ranks of the matrices B_k, $k = 1, \ldots, N$.

We now apply to the reduced pairs a second state-space transformation V_k of the form $V_k = \operatorname{diag}(I_{\nu_k}, \widetilde{U}_k)$. These transformations will affect only $\widetilde{B}_k$, $\widetilde{A}_k$ and $A_{k,12}$. This time we choose $\widetilde{U}_{k+1}$ to compress the rows of $\widetilde{B}_k$ to a full row rank matrix and repeat the partitioning in form (8) and (9) for the matrices $\widetilde{U}_{k+1}^T \widetilde{B}_k$ and $\widetilde{U}_{k+1}^T \widetilde{A}_k \widetilde{U}_k$. We obtain globally

$$V_{k+1}^T U_{k+1}^T A_k U_k V_k := \begin{array}{cc} \left[\begin{array}{ccc} A_{k,11} & A_{k,12} & A_{k,13} \\ A_{k,21} & A_{k,22} & A_{k,23} \\ 0 & \widehat{B}_k & \widehat{A}_k \end{array} \right] & \begin{array}{c} \nu_{k+1} \\ \tilde{\nu}_{k+1} \\ \tilde{\rho}_{k+1} \end{array} \\ \begin{array}{ccc} \nu_k & \tilde{\nu}_k & \tilde{\rho}_k \end{array} & \end{array} \tag{10}$$

where some submatrices have been redefined. This reduction process continues until $\tilde{\nu}_k = 0$ for $k = 1, \ldots, N$, that is, all $\widehat{B}_k = 0$, or $\tilde{\rho}_k = 0$, for $k = 1, \ldots, N$, that is all $\widehat{B}_k$ have full row rank.

The following implementable algorithm formalizes the above ideas:

PKRD Algorithm: Periodic Kalman Reachability Decomposition

Given $A_k \in \mathbb{R}^{n_{k+1} \times n_k}$, $B_k \in \mathbb{R}^{n_{k+1} \times m_k}$ and $C_k \in \mathbb{R}^{p_k \times n_k}$ for $k = 1, \ldots, N$, this algorithm computes the orthogonal matrices Q_k, $k = 1, \ldots, N$, such that the transformed periodic system $(Q_{k+1}^T A_k Q_k, Q_{k+1}^T B_k, C_k Q_k)$ is in the periodic Kalman reachability form (6).

1. Set $j = 1$ and $r_k = 0$, $\nu_k^{(0)} = m_k$, $A_k^{(0)} = A_k$, $B_k^{(0)} = B_k$, $Q_k = I_{n_k}$ for $k = 1, \ldots, N$.
2. For $k = 1, \ldots, N$, compute the orthogonal matrices U_{k+1} to compress the matrix $B_k^{(j-1)} \in \mathbb{R}^{(n_{k+1} - r_{k+1}) \times \nu_k^{(j-1)}}$ to a full row rank matrix

$$U_{k+1}^T B_k^{(j-1)} := \begin{array}{cc} \left[\begin{array}{c} A_{k;j,j-1} \\ 0 \end{array} \right] & \begin{array}{c} \nu_{k+1}^{(j)} \\ \rho_{k+1}^{(j)} \end{array} \\ \nu_k^{(j-1)} & \end{array}$$

3. For $k = 1, \ldots, N$, compute $U_{k+1}^T A_k^{(j-1)} U_k$ and partition it in the form

$$U_{k+1}^T A_k^{(j-1)} U_k := \begin{array}{cc} \left[\begin{array}{cc} A_{k;j,j} & A_{k;j,j+1} \\ B_k^{(j)} & A_k^{(j)} \end{array}\right] & \begin{array}{c} \nu_{k+1}^{(j)} \\ \rho_{k+1}^{(j)} \end{array} \\ \begin{array}{cc} \quad\nu_k^{(j)} & \qquad\rho_k^{(j)} \end{array} & \end{array}$$

4. For $k = 1, \ldots, N$ and $i = 1, \ldots, j-1$, compute

$$A_{k;i,j} U_k := \begin{array}{c} [\, A_{k;i,j} \;\; A_{k;i,j+1} \,] \\ \begin{array}{cc} \nu_k^{(j)} & \quad\rho_k^{(j)} \end{array} \end{array}$$

5. $Q_k \leftarrow Q_k \mathrm{diag}(I_{r_k}, U_k)$, $C_k \leftarrow C_k \mathrm{diag}(I_{r_k}, U_k)$, for $k = 1, \ldots, N$.
6. $r_k \leftarrow r_k + \nu_k^{(j)}$, for $k = 1, \ldots, N$;
   if $\rho_k^{(j)} = 0$ for $k = 1, \ldots, N$, then $\ell = j$, **Exit 1**.
7. If $\nu_k^{(j)} = 0$ for $k = 1, \ldots, N$, then $\ell \leftarrow j-1$, **Exit 2**; else, $j \leftarrow j+1$ and go to Step 2.

After performing the **PKRD Algorithm**, each pair (A_k, B_k) is in the periodic reachability form (6), where the pair (A_k^r, B_k^r) is in a staircase form

$$[\, B_k^r \mid A_k^r \,] = \left[\begin{array}{c|cccc} A_{k;1,0} & A_{k;1,1} & A_{k;1,2} & \cdots & A_{k;1,\ell} \\ O & A_{k;2,1} & A_{k;2,2} & \cdots & A_{k;2,\ell} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & O & A_{k;\ell,\ell-1} & A_{k;\ell,\ell} \end{array}\right] \tag{11}$$

and $A_k^{\overline{r}} \in \mathbb{R}^{\rho_{k+1}^{(\ell)} \times \rho_k^{(\ell)}}$. In (11), $A_{k;i,i} \in \mathbb{R}^{\nu_{k+1}^{(i)} \times \nu_k^{(i)}}$, $i = 1, \ldots, \ell$; $A_{k;i,i-1} \in \mathbb{R}^{\nu_{k+1}^{(i)} \times \nu_k^{(i-1)}}$ and $\mathrm{rank}\, A_{k;i,i-1} = \nu_{k+1}^{(i)}$ for $i = 1, \ldots, \ell$.

The above algorithm basically constructs, in a step-by-step manner, orthogonal bases for the images of the reachability matrices $\mathcal{R}_k$. These bases are formed from the leading r_k columns of the resulting orthogonal transformation matrices Q_k. The periodic system (1) is reachable at time k if $r_k = n_k$. The following result summarizes this important fact.

Theorem 3 *For each periodic pair (A_k, B_k) there exists a periodic orthogonal matrix Q_k such that the transformed periodic pair $(\widetilde{A}_k, \widetilde{B}_k) := (Q_{k+1}^T A_k Q_k, Q_{k+1}^T B_k)$ is in the periodic Kalman reachability form (6).*

Proof. We apply the **PKRD Algorithm** to the periodic pair (A_k, B_k) (assuming $C_k = 0$) and obtain orthogonal periodic Q_k such that the periodic pair $(Q_{k+1}^T A_k Q_k, Q_{k+1}^T B_k)$ is in the form (6) with each pair (A_k^r, B_k^r) in the form (11). We need to show that this periodic pair is reachable. Consider the matrix pair $(\mathcal{A}^r, \mathcal{B}^r)$, where

$$\mathcal{A}^r = \left[\begin{array}{cccc} 0 & \cdots & 0 & A_N^r \\ A_1^r & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{N-1}^r & 0 \end{array}\right], \quad \mathcal{B}^r = \left[\begin{array}{cccc} B_N^r & 0 & \cdots & 0 \\ 0 & B_1^r & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{N-1}^r \end{array}\right]$$

We can easily extend the proof of Lemma 1 of [8] to the case of time-varying dimensions and show that the periodic pair (A_k^r, B_k^r) is completely reachable if and only if the pair $(\mathcal{A}^r, \mathcal{B}^r)$ is reachable. Thus, by using the Popov-Belevich-Hautus test, to prove reachability we need only to show that the pencil $[\, \mathcal{A}^r - zI \;\; \mathcal{B}^r \,]$ has full row rank $\sum_{k=1}^N r_k$ for all $z \in \mathbb{C}$.

By column and row permutations we can bring this pencil in the form

$$R(z) = \begin{bmatrix} S_1 & -zT_1 & O & \cdots & O \\ O & S_2 & -zT_2 & \cdots & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & & & S_{N-1} & -zT_{N-1} \\ -zT_N & O & \cdots & O & S_N \end{bmatrix} \tag{12}$$

where, for $k = 1, \ldots, N$,

$$S_k := [\, B_k^r \;\; A_k^r \,], \quad T_k := [\, O \;\; I_{r_{k+1}} \,]$$

Note that by construction (see (11)), each S_k has full row rank $r_{k+1} = \sum_{i=1}^{\ell} \nu_{k+1}^{(i)}$. Thus, by performing ℓ cyclic block column reductions using block column operations, we can bring $R(z)$, for any finite z, in the form $\operatorname{diag}(S_1, \ldots, S_N)$. However, this matrix has full row rank, because each S_k has full row rank. □

Remark 1. For each pair (A_k^r, B_k^r) we can define the time-varying reachability index μ_k as the largest value of i such that $\nu_k^{(i)} \neq 0$. Let h_k be the least integer such that $\nu_{k+1}^{(h_k)} = 0$. Then, it is easy to see that the trailing $\sum_{i=h_k}^{\ell} \nu_{k+1}^{(i)} \times \sum_{i=h_k}^{\ell} \nu_k^{(i)}$ block of A_k^r is in a block upper trapezoidal form with all diagonal blocks having full row rank. It follows that the resulting matrices $\widetilde{B}_k$ and $\widetilde{A}_k$ of the PKRD (6) have, in general, the forms

$$\widetilde{B}_k = \left[\begin{array}{c} B_k^r \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} B_{k,1}^r \\ 0 \\ \hline 0 \end{array}\right], \quad \widetilde{A}_k = \left[\begin{array}{c|c} A_k^r & * \\ \hline 0 & A_k^{\overline{r}} \end{array}\right] = \left[\begin{array}{cc|c} A_{k,11}^r & A_{k,12}^r & * \\ 0 & A_{k,22}^r & * \\ \hline 0 & 0 & A_k^{\overline{r}} \end{array}\right],$$

where $A_{k,22}^r$ has full row rank. Note that in the single-input case, the leading block $A_{k,11}^r$ is in an unreduced upper Hessenberg form, while the trailing block $A_{k,22}^r$ is full row rank upper trapezoidal. □

Remark 2. It is possible to further reduce $A_k^{\overline{r}}$ by separating the zero and nonzero characteristic multipliers in the product $\Phi_{A^{\overline{r}}}(N, 0)$. This can be done once again by employing exclusively orthogonal state-space transformations. The resulting periodic matrix after this separation has the form

$$A_k^{\overline{r}} = \begin{bmatrix} A_k^0 & * \\ 0 & A_k^{\overline{c}} \end{bmatrix},$$

where all characteristic values of the periodic matrix A_k^0 are zero, and the periodic matrix $A_k^{\overline{c}} \in \mathbb{R}^{n_{\overline{c}} \times n_{\overline{c}}}$ has constant dimension, is square and nonsingular. The eigenvalues of $\Phi_{A_{\overline{c}}}$ represents the uncontrollable characteristic multipliers of the periodic system (1). It follows that the periodic system (1) is *stabilizable* if all uncontrollable characteristic multipliers belong to the open unit disk. □

Remark 3. The **PKRD Algorithm** can be extended to periodic descriptor systems of the form

$$\begin{array}{rcl} E_k x(k+1) & = & A_k x(k) + B_k u(k) \\ y(k) & = & C_k x(k) \end{array} \tag{13}$$

where the matrices A_k, B_k, and C_k are the same as in (1) and $E_k \in \mathbb{R}^{n_{k+1} \times n_{k+1}}$ is an N-periodic invertible matrix. The similarity transformation used in this case has the form

$$\widetilde{E}_k = T_{k+1} E_k S_{k+1}, \quad \widetilde{A}_k = T_{k+1} A_k S_k, \quad \widetilde{B}_k = T_{k+1} B_k, \quad \widetilde{C}_k = C_k S_k$$

with S_k and T_k N-periodic nonsingular matrices.

After a preliminary reduction of E_k to an upper triangular form using suitable orthogonal S_k and T_k, we perform, as in the **PKRD Algorithm**, the row compression on B_k using an orthogonal transformation U_{k+1}. The only difference in the descriptor case is that the upper triangular form of

E_k is preserved while reducing B_k. This can be done by computing an appropriate V_{k+1} such that $U_{k+1}^T E_k V_{k+1}$ remains upper triangular. In fact, the compression of B_k and maintaining the upper triangular form of E_k can be done simultaneously, in a similar way as done in [11] for standard descriptor systems. The combined reduction and restoring of triangular form can efficiently be done by employing orthogonal Givens transformations. $\square$

4 Numerical aspects

To estimate the *floating-point operations* (*flops*) necessary to compute the periodic reachability Kalman decomposition, we assume for simplicity constant dimensions: $n = n_k$, $m = m_k$, $p = p_k$. The worst-case operations count result if the periodic system is reachable. In this case, if we use Householder transformations based QR decompositions with column pivoting for the row compressions in the **PKRD Algorithm**, then we can easily give an estimate of the total number of *flops* necessary to compute the PKRD as

$$N_{flops} = N\left(\frac{5}{3}n^3 + (p+m)n^2\right)$$

To accumulate the transformations the algorithm needs additionally Nn^3 *flops*. Thus, the computational complexity of this algorithm is $O(Nn^3)$. The same computational complexity can be achieved also in the descriptor case.

All computations can be performed in place, thus the required memory of $(n+m+p)nN$ storage locations is minimal if the transformations are not accumulated. The information on the performed transformations can be compactly stored in the generated zero submatrices during the reduction, and in additional N n-vectors. To form the transformation matrices explicitly, Nn^2 additional storage locations are necessary. These figures are valid also for time-varying dimensions, where n, m and p are now the maximum values of state, input and output vector dimensions, respectively.

The backward stability of the **PKRD Algorithm** can be easily proved. The basic idea is that each transformation U_k can be computed and applied in a numerically stable way. A sequence of such transformations can be also performed in a numerically stable way, since each orthogonal matrix has unity norm. For details see [17]. It follows that the results computed with the **PKRD Algorithm** are exact for slightly perturbed initial matrices $\overline{A}_k$, $\overline{B}_k$, $\overline{C}_k$, which satisfy

$$\|\overline{X} - X\| \leq \varepsilon_X \|X\|, \quad X = A_k, B_k, C_k$$

where, in each case, ε_X is a modest multiple of the relative machine precision ε_M.

5 Applications

5.1 Computation of PKOD

To compute the PKOD, an **PKOD Algorithm** analogous **PKRD Algorithm** can be devised. Instead row compressions, this algorithm performs column compressions on the matrices $C_k^{(i)}$ in the successively generated pairs $(A_k^{(i)}, C_k^{(i)})$. The resulting algorithm can be seen as the application of the **PKRD Algorithm** to a certain dual periodic system. The following procedure formalizes the main steps of this approach:

PKOD Algorithm: Periodic Kalman Observability Decomposition

Given $A_k \in \mathbb{R}^{n_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{n_{k+1}\times m_k}$ and $C_k \in \mathbb{R}^{p_k\times n_k}$ for $k = 1,\ldots,N$, this algorithm computes the orthogonal matrices Q_k, $k = 1,\ldots,N$, such that the transformed system $(Q_{k+1}^T A_k Q_k, Q_{k+1}^T B_k, C_k Q_k)$ is in the periodic Kalman observability form (7).

1. For $k = 1,\ldots,N$, form the dual system matrices

$$\widehat{A}_k = A_{N-k+1}^T, \quad \widehat{B}_k = C_{N-k+1}^T, \quad \widehat{C}_k = B_{N-k+1}^T$$

2. Apply the **PKRD Algorithm** to the periodic triple $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k)$ to determine the orthogonal N-periodic transformation matrices $\widehat{Q}_k$ such that

$$\widehat{A}_k \leftarrow Q_{k+1}^T \widehat{A}_k Q_k, \quad \widehat{B}_k \leftarrow Q_{k+1}^T \widehat{B}_k, \quad \widehat{C}_k \leftarrow \widehat{C}_k Q_k$$

   with the resulting periodic pair $(\widehat{A}_k, \widehat{B}_k)$ in the periodic reachability form

$$\widehat{A}_k = \left[\begin{array}{cc} \widehat{A}_k^r & * \\ 0 & \widehat{A}_k^{\bar{r}} \end{array}\right], \quad \widehat{B}_k = \left[\begin{array}{c} \widehat{B}_k^r \\ 0 \end{array}\right]$$

3. For $k = 1,\ldots,N$, form the system matrices of the PKOD

$$A_k \leftarrow \widehat{A}_{N-k+1}^T, \quad B_k \leftarrow \widehat{C}_{N-k+1}^T, \quad C_k \leftarrow \widehat{B}_{N-k+1}^T$$

4. Set $Q_N = \widehat{Q}_N$, and $Q_k = \widehat{Q}_{N-k}$, for $k = 1,\ldots,N-1$.

Using the computed results of this algorithm, the *reconstructibility* and *detectability* properties can be analyzed in a similar way as indicated in **Remark 2** for the dual properties of controllability and stabilizability, respectively.

5.2 Computation of minimal realizations

The numerical computation of minimal realizations of periodic systems has been addressed in [12], where a balancing-related approach was proposed. This algorithm relies on the computation of the extended periodic Schur form of the periodic matrix A_k, and involves the solution of two non-negative definite periodic Lyapunov equations. This algorithm is numerically reliable, since each computational step relies on backward stable algorithms. The main advantage of this algorithm is that the N rank decisions necessary to obtain the state-vector dimensions of a minimal realization are performed only once at the end of the algorithm. Thus, this approach is very reliable in determining the order of the minimal realizations.

In some applications, as for example when computing the TFM of a periodic system [13], the algorithmic efficiency aspects play an important role. Thus, instead employing the above algorithm, we can alternatively use a significantly more efficient procedure to compute minimal realizations by eliminating successively the unreachable and unobservable parts. A two step procedure is formalized below:

Minimal realization procedure

1. Apply the **PKRD Algorithm** to the periodic system (A_k, B_k, C_k) to compute the reachable periodic realization (A_k^r, B_k^r, C_k^r).
2. Apply the **PKOD Algorithm** to the reachable system (A_k^r, B_k^r, C_k^r) to compute the minimal realization as the observable part $(A_k^{ro}, B_k^{ro}, C_k^{ro})$.

This algorithm is backward stable and has a computational complexity of $O(Nn^3)$.

6 Numerical examples

Example 1. To show in detail the result obtained by the proposed algorithm, consider the 3-periodic single-input single-output system with the constant dimension system matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 2 & 4 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}, \; B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_3 = \begin{bmatrix} 3 & 1 \end{bmatrix}$$

By applying the **PKRD Algorithm** we obtain

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and the transformed system matrices in the periodic Kalman reachability form

$$\left[\begin{array}{c|c} A_1^r & * \\ \hline 0 & * \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right], \; \left[\begin{array}{c} B_1^r \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} 3 \\ \hline 0 \end{array}\right], \; \left[\begin{array}{c|c} C_1^r & * \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \end{array}\right]$$
$$\left[\begin{array}{c|c} A_2^r & * \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ 1 & 2 \end{array}\right], \; B_2^r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \left[\begin{array}{c|c} C_2^r & * \end{array}\right] = \left[\begin{array}{c|c} 2 & 4 \end{array}\right]$$
$$\left[\begin{array}{c} A_3^r \\ \hline 0 \end{array}\right] = \left[\begin{array}{cc} 4 & 1 \\ \hline 0 & 0 \end{array}\right], \quad \left[\begin{array}{c} B_3^r \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right], \; C_3^r = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

Thus, the reachable part has time-varying state dimensions, $r_1 = 1$, $r_2 = 1$, $r_3 = 2$. Note that this part is also observable, thus $(\mathcal{A}^r, \mathcal{B}^r, \mathcal{C}^r)$ represents a minimal realization of the original system.

Example 2. This example illustrates that, in general, randomly generated periodic systems with random state and input dimensions are generically nonreachable. Consider a randomly generated system with period $N = 5$ and state vector and input vector dimensions $n_1 = 19$, $n_2 = 15$, $n_3 = 0$, $n_4 = 6$, $n_5 = 10$; and $m_1 = 1$, $m_2 = 3$, $m_3 = 1$, $m_4 = 4$, $m_5 = 1$, respectively. Note that the state vector has zero dimension at time moments $3 + 5k$. It follows that this system has only zero characteristic multipliers and thus is completely controllable.

By applying the **PKRD Algorithm** we obtain that the reachable subsystem has state dimensions $r_1 = 6$, $r_2 = 7$, $r_3 = 0$, $r_4 = 1$, $r_5 = 5$. Thus excepting the time moments $3 + 5k$, the system is unreachable at all other time instants. It is interesting to look at the computational details when performing this algorithm. The algorithm has finished after $\ell = 4$ steps and determined the following dimensions at successive steps:

Step	$\nu_1^{(j)}$	$\nu_2^{(j)}$	$\nu_3^{(j)}$	$\nu_4^{(j)}$	$\nu_5^{(j)}$
$j = 0$	1	3	1	4	1
$j = 1$	1	1	0	1	4
$j = 2$	4	1	0	0	1
$j = 3$	1	4	0	0	0
$j = 4$	0	1	0	0	0

Thus, the reachability indices of the reachable part are $\mu_1 = 3$, $\mu_2 = 4$, $\mu_3 = 0$, $\mu_4 = 1$, $\mu_5 = 2$.

Example 3. To illustrate the numerical performance of the proposed algorithm, we generated random systems with constant dimensions $n_k = 50, 100, 200$; $m_k = 5$, $p_k = 2$, with periods $N = 5, 10, 20, 50, 100$. In Table 1, we give the execution times of the **PKRD Algorithm** on a Pentium III 933MHz machine under Windows 2000. The results have been obtained with MATLAB 6.5 via a *mex*-function interface to a Fortran 95 implementation of this algorithm.

$n_k \setminus N$	5	10	20	50	100
50	0.05	0.05	0.08	0.20	0.41
100	0.11	0.23	0.45	1.12	2.26
200	1.25	2.47	4.98	12.37	25.95

Table 1: Execution times (in seconds) for the **PKRD Algorithm**.

It is easy to observe that for each fixed dimension n_k, the execution times vary almost linearly with the period N and this confirms the expected low computational complexity of the proposed algorithm with respect to the period. Also the cubic dependence on the dimensions is visible, especially by comparing the times for $n_k = 100$ and $n_k = 200$.

To evaluate the effects of the roundoff errors, we computed the backward errors by evaluating

$$err = \max \left\{ \|\overline{X} - X\|/\|X\| \mid X = A_k, B_k, C_k \right\}$$

over all generated examples. Specifically, we computed $\overline{A}_k = Q_{k+1}\widetilde{A}_k Q_k^T$, $\overline{B}_k = Q_{k+1}\widetilde{B}_k$, and $\overline{C}_k = \widetilde{C}_k Q_k^T$, where Q_k and $(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k)$ are the transformation and system matrices computed by the **PKRD Algorithm**, respectively. Note that $\overline{A}_k$, $\overline{B}_k$, $\overline{C}_k$ contains the cumulated errors due to roundoff *reflected back* into the original system matrices. To assess numerical stability, the backward error err must be of the order of the machine precision $\varepsilon_M \approx 2.22 \cdot 10^{-16}$ (for IEEE double precision floating point arithmetic). The resulting value $err = 1.2 \cdot 10^{-15}$ is therefore a clear indication for the backward stability of the proposed method.

7 Conclusion

In this paper we proposed a backward stable algorithm to compute the periodic Kalman reachability decomposition of a periodic system. This algorithm can be applied to compute the periodic Kalman observability decomposition as well, and thus can be used to compute minimal realizations of periodic systems. The algorithm works for system matrices with time-varying dimensions and can be easily extended to descriptor periodic systems. By fully exploiting the problem structure, an acceptable computational complexity can be achieved, which is linear in the period N and cubic in the maximum dimension of the state vector. Thus, the new algorithm fulfils all requirements which we formulated in [15] for a satisfactory algorithm for periodic systems.

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