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Reliable algorithms for computing minimal dynamic covers for descriptor systems

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Abstract

Minimal dimension dynamic covers play an important role in solving the structural synthesis problems of minimum order functional observers or fault detectors, or in computing minimal order inverses or minimal degree solutions of rational equations. We propose numerically reliable algorithms to compute two basic types of minimal dimension dynamic covers for a linear system. The proposed approach is based on a special controllability staircase condensed form of a structured descriptor pair $(A - \lambda E, [B_1, B_2])$, which can be computed using exclusively orthogonal similarity transformations. Using such a condensed form minimal dimension covers and corresponding feedback/feedforward matrices can be easily computed. The overall algorithm has a low computational complexity and is provably numerically reliable.

1 Introduction

Our motivation to address the computational aspects of determining minimal dimension dynamic covers is the following concrete problem encountered in the design of minimal order fault detectors [8]: for a given linear descriptor system $(A - \lambda E, [B_1, B_2], C, [D_1, D_2])$ with invertible E, one wants to compute a state feedback matrix F and a feedforward matrix G to achieve the cancellation of a maximum number of uncontrollable poles of the transfer-function matrix

$$R(\lambda) = (C + D_1 F)(\lambda E - A - B_1 F)^{-1}(B_1 G + B_2) + (D_1 G + D_2) \tag{1}$$

Different instances of this problem for the standard case $E = I$ appear in solving various structural synthesis problems, as for example, the design of minimum order functional observers [3], determining minimal order inverses [1] or computation of minimal degree solutions of rational equations [4]. The proposed solution procedures reformulate these problems as minimum dynamic cover problems, which can be solved using the "standard" method of [12] relying on subspace manipulation techniques employed in the geometric theory of linear systems [11]. This approach has been turned recently into an efficient and numerically reliable algorithm [9] and can be also employed in the case of a general invertible E by replacing the matrices E, A, B_1 and B_2 by I, $E^{-1}A$, $E^{-1}B_1$ and $E^{-1}B_2$, respectively.

From a numerical point of view it is advisable to avoid any matrix inversions in early phases of computational algorithms, especially if rank decisions follow in a later stage. For our problem the explicit inversion of an ill-conditioned E can lead to severe loss of accuracy of the computed results. Since rank decisions are performed later on the transformed data, such an accuracy loss can even lead to a complete failure of computations. It follows that employing the algorithms of [9] on the transformed matrices is not a satisfactory computational approach to determine minimal dynamic covers in the case of a general E. Therefore, it is important to develop more general algorithms able to address such computational problems without inverting E.

In this paper we propose a numerically reliable and computationally efficient approach to compute a feedback matrix F and a possibly nonzero feedforward matrix G to achieve the desired cancellation of maximum number of uncontrollable poles in (1). We solve the problems of determining both F and G or only F which lead to cancellation of maximum number of uncontrollable poles. Solving these problems involves to compute bases for subspaces representing minimal dimension dynamic covers of *Type II* and *Type I*, respectively (see [3]). The main computational ingredient in these computations is bringing the system matrices in condensed forms which exhibit the structural information necessary to solve the problem. For the matrices in the resulting condensed forms the computation of appropriate F and G is a simple, almost trivial task. The algorithm to compute the condensed form has two stages: (1) an orthogonal reduction of the structured descriptor pair $(A - \lambda E, [B_1, B_2])$ to a special controllability staircase form followed by special row/column block permutations; and (2) a non-orthogonal transformation to zero additionally a minimum number of elements. The orthogonal reduction part is based on employing techniques similar to that used in the controllability staircase form algorithms for descriptor systems [5]. This part involves many rank decisions which can be computed by using reliable techniques (e.g., singular values based rank evaluations). The non-orthogonal part of the reduction does not involve any rank computations and is performed to allow an easy computation of appropriate feedback/feedforward matrices. The overall algorithm has a low computational complexity and is provably numerically reliable.

In the last part we also address shortly the solution of minimum cover problems with stability constraints. In the case the minimum cover problem with stabilization is solvable, we propose a reliable computational solution to this problem by exploiting the existing parametric freedom in the cover determination problem.

2 Computation of Type II minimal dynamic covers

The computational problem which we solve is the following: given the descriptor pair $(A - \lambda E, B)$ with $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and B partitioned as $B = [\, B_1 \;\; B_2 \,]$ with $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, determine the matrices F and G such that the pair $(A + B_1F - \lambda E, B_1G + B_2)$ has maximal number of uncontrollable eigenvalues. This problem is essentially equivalent [4] to compute a subspace $\mathcal{V}$ having least possible dimension satisfying

$$(\overline{A} + \overline{B}_1F)\mathcal{V} \subset \mathcal{V}, \quad \text{span}\,(\overline{B}_1G + \overline{B}_2) \subset \mathcal{V} \tag{2}$$

where $\overline{A} = E^{-1}A$, $\overline{B}_1 = E^{-1}B_1$, and $\overline{B}_2 = E^{-1}B_2$. If we denote $\overline{\mathcal{B}}_1 = \text{span}\,\overline{B}_1$ and $\overline{\mathcal{B}}_2 = \text{span}\,\overline{B}_2$, then the above condition can be rewritten also as a condition defining a *Type II* minimum dynamic cover [2, 3] of the form

$$\begin{array}{rcl} \overline{A}\mathcal{V} & \subset & \mathcal{V} + \overline{\mathcal{B}}_1 \\ \overline{\mathcal{B}}_2 & \subset & \mathcal{V} + \overline{\mathcal{B}}_1 \end{array} \tag{3}$$

The computation of the minimal dynamic covers relies on the reduction of the descriptor pair $(A - \lambda E, [B_1, B_2])$ to a particular condensed form, for which the solution of the problem is simple. This reduction is performed in two stages. The first stage is an orthogonal reduction which represents a particular instance of the descriptor controllability staircase procedure of [5] applied to the descriptor pair $(A - \lambda E, [B_1, B_2])$. This procedure can be seen as a generalized orthogonal variant of the basis selection approach of [3] and therefore will be useful to construct both *Type II* and *Type I* minimal covers. In the second stage, additional zero blocks are generated in the reduced matrices using non-orthogonal transformations and by applying appropriate feedback and feedforward matrices. In what follows we present in detail these two stages.

Stage I: Special Controllability Staircase Algorithm

0. Compute an orthogonal matrix Q such that $Q^T E$ is upper triangular;
   compute $A \leftarrow Q^T A$, $E \leftarrow Q^T E$, $B_1 \leftarrow Q^T B_1$, $B_2 \leftarrow Q^T B_2$.
   Comment. Set $Q = I_n$ for a standard system.

1. Set $j = 1$, $r = 0$, $k = 2$, $\nu_1^{(0)} = m_1$, $\nu_2^{(0)} = m_2$, $A^{(0)} = A$, $E^{(0)} = E$, $B_1^{(0)} = B_1$, $B_2^{(0)} = B_2$, $Z = I_n$.

2. Compute the orthogonal matrix U_1 to compress the matrix $B_1^{(j-1)} \in \mathbb{R}^{(n-r)\times\nu_1^{(j-1)}}$ to a full row rank matrix

$$U_1^T B_1^{(j-1)} := \underset{\nu_1^{(j-1)}}{\left[\begin{array}{c} A_{k-1,k-3} \\ 0 \end{array}\right]} \begin{array}{l} \nu_1^{(j)} \\ \rho_1^{(j)} \end{array}$$

3. Compute $U_1^T B_2^{(j-1)}$ and partition it in the form

$$U_1^T B_2^{(j-1)} := \underset{\nu_2^{(j-1)}}{\left[\begin{array}{c} A_{k-1,k-2} \\ X \end{array}\right]} \begin{array}{l} \nu_1^{(j)} \\ \rho_1^{(j)} \end{array}$$

4. Compute the orthogonal matrix U_2 to compress the matrix $X \in \mathbb{R}^{(n-r-\nu_1^{(j)})\times\nu_2^{(j-1)}}$ to a full row rank matrix

$$U_2^T X := \underset{\nu_2^{(j-1)}}{\left[\begin{array}{c} A_{k,k-2} \\ 0 \end{array}\right]} \begin{array}{l} \nu_2^{(j)} \\ \rho_2^{(j)} \end{array}$$

   and compute the orthogonal matrix W_1 such that $\text{diag}(I, U_2^T)U_1^T E^{(j-1)} W_1$ is upper triangular.
   Comment. Set $W_1 = U_1\text{diag}(I, U_2)$ in the standard case.

5. Compute $\mathrm{diag}(I, U_2^T)U_1^T A^{(j-1)} W_1$, $\mathrm{diag}(I, U_2^T)U_1^T E^{(j-1)} W_1$ and partition them in the form

$$\begin{bmatrix} I & O \\ O & U_2^T \end{bmatrix} U_1^T A^{(j-1)} W_1 := \begin{array}{c} \left[\begin{array}{ccc} A_{k-1,k-1} & A_{k-1,k} & A_{k-1,k+1} \\ A_{k,k-1} & A_{k,k} & A_{k,k+1} \\ B_1^{(j)} & B_2^{(j)} & A^{(j)} \end{array}\right] \begin{array}{c} \nu_1^{(j)} \\ \nu_2^{(j)} \\ \rho_2^{(j)} \end{array} \\ \begin{array}{ccc} \nu_1^{(j)} & \nu_2^{(j)} & \rho_2^{(j)} \end{array} \end{array}$$

$$\begin{bmatrix} I & O \\ O & U_2^T \end{bmatrix} U_1^T E^{(j-1)} W_1 := \begin{array}{c} \left[\begin{array}{ccc} E_{k-1,k-1} & E_{k-1,k} & E_{k-1,k+1} \\ O & E_{k,k} & E_{k,k+1} \\ O & O & E^{(j)} \end{array}\right] \begin{array}{c} \nu_1^{(j)} \\ \nu_2^{(j)} \\ \rho_2^{(j)} \end{array} \\ \begin{array}{ccc} \nu_1^{(j)} & \nu_2^{(j)} & \rho_2^{(j)} \end{array} \end{array}$$

6. $A_{i,k-1}W_1 := \begin{array}{c} [\, A_{i,k-1} \;\; A_{i,k} \;\; A_{i,k+1} \,] \\ \begin{array}{ccc} \nu_1^{(j)} & \nu_2^{(j)} & \rho_2^{(j)} \end{array} \end{array}$ and $E_{i,k-1}W_1 := \begin{array}{c} [\, E_{i,k-1} \;\; E_{i,k} \;\; E_{i,k+1} \,] \\ \begin{array}{ccc} \nu_1^{(j)} & \nu_2^{(j)} & \rho_2^{(j)} \end{array} \end{array}$, for $i = 1, \ldots, k-2$.

7. $Q \leftarrow Q\,\mathrm{diag}(I_r, U_1)\,\mathrm{diag}(I_{r+\nu_1^{(j)}}, U_2)$, $Z \leftarrow Z\,\mathrm{diag}(I_r, W_1)$.

8. $r \leftarrow r + \nu_1^{(j)} + \nu_1^{(j)}$; if $\rho_2^{(j)} = 0$ then $\ell = j$ and **Exit 1**.

9. If $\nu_1^{(j)} + \nu_2^{(j)} = 0$ then $\ell = j - 1$, **Exit 2**; else, $j \leftarrow j + 1$, $k \leftarrow k + 2$, and go to Step 2.

At the end of this algorithm $\widehat{A} - \lambda \widehat{E} = Q^T(A - \lambda E)Z$ and $\widehat{B} = Q^T B$ have the following form

$$\widehat{A} - \lambda\widehat{E} = \begin{array}{c} \left[\begin{array}{cc} A_c - \lambda E_c & * \\ O & A_{\overline{c}} - \lambda E_{\overline{c}} \end{array}\right] \begin{array}{c} r \\ n-r \end{array} \\ \begin{array}{cc} r & n-r \end{array} \end{array}, \quad \widehat{B} = \begin{bmatrix} B_c \\ O \end{bmatrix} \begin{array}{c} r \\ n-r \end{array}$$

where the pair $(A_c - \lambda E_c, B_c)$ has only controllable finite eigenvalues, $A_{\overline{c}} - \lambda E_{\overline{c}}$ contains the uncontrollable finite eigenvalues of $A - \lambda E$, and $\widehat{E}$ is upper triangular. The pair (A_c, B_c) is in the special staircase form

$$\left[\begin{array}{c|c} B_c & A_c \end{array}\right] = \left[\begin{array}{cc|cccccccc} A_{1,-1} & A_{1,0} & A_{11} & A_{12} & \cdots & A_{1,2\ell-3} & A_{1,2\ell-2} & A_{1,2\ell-1} & A_{1,2\ell} \\ O & A_{2,0} & A_{21} & A_{22} & \cdots & A_{2,2\ell-3} & A_{2,2\ell-2} & A_{2,2\ell-1} & A_{2,2\ell} \\ O & O & A_{31} & A_{32} & \cdots & A_{3,2\ell-3} & A_{3,2\ell-2} & A_{3,2\ell-1} & A_{3,2\ell} \\ O & O & O & A_{42} & \cdots & A_{4,2\ell-3} & A_{4,2\ell-2} & A_{4,2\ell-1} & A_{4,2\ell} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & O & \cdots & A_{2\ell-1,2\ell-3} & A_{2\ell-1,2\ell-2} & A_{2\ell-1,2\ell-1} & A_{2\ell-1,2\ell} \\ O & O & O & O & \cdots & O & A_{2\ell,2\ell-2} & A_{2\ell,2\ell-1} & A_{2\ell,2\ell} \end{array}\right] \quad (4)$$

where $A_{2j-1,2j-3} \in \mathbb{R}^{\nu_1^{(j)} \times \nu_1^{(j-1)}}$ and $A_{2j,2j-2} \in \mathbb{R}^{\nu_2^{(j)} \times \nu_2^{(j-1)}}$ are full row rank matrices for $j = 1, \ldots, \ell$. The resulting upper triangular matrix E_c has a similar block partitioned form

$$E_c = \begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1,2\ell-1} & E_{1,2\ell} \\ O & E_{22} & \cdots & E_{2,2\ell-1} & E_{2,2\ell} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & E_{2\ell-1,2\ell-1} & E_{2\ell-1,2\ell} \\ O & O & \cdots & O & E_{2\ell,2\ell} \end{bmatrix}$$

Note that in the standard case $Q = Z$ and $\widehat{E} = I$.

To compute a Type II minimal cover, in the second reduction stage we use non-orthogonal upper triangular left and right transformation matrices $W = \operatorname{diag}(W_c, I_{n-r})$ and $U = \operatorname{diag}(U_c, I_{n-r})$, respectively, to annihilate a minimum set of blocks in A_c and E_c. Assume W_c and U_c have block structures identical to E_c. The following procedure performs the second reduction stage by exploiting the full rank of submatrices $A_{2j-1,2j-3}$ and $E_{2j,2j}$ to introduce zero matrices in the block row $2j-1$ of A_c and block column $2j$ of E_c, respectively.

Stage II: Special reduction for Type II Covers

Set $W = I, U = I$.

for $k = \ell, \ell-1, \ldots, 1$

Comment. Annihilate blocks $E_{2k-1,2j}$, for $j = k, k+1, \ldots, \ell$.

for $j = k, k+1, \ldots, \ell$

Compute $W_{2k-1,2j}$ such that $W_{2k-1,2j}E_{2j,2j} + E_{2k-1,2j} = 0$.

$A_{2k-1,i} \leftarrow A_{2k-1,i} + W_{2k-1,2j}A_{2j,i}$, $i = 2j-2, 2j-1, \ldots, 2\ell$.

$E_{2k-1,i} \leftarrow E_{2k-1,i} + W_{2k-1,2j}E_{2j,i}$, $i = 2j, 2j+1, \ldots, 2\ell$.

end

if $k > 1$ **then**

Comment. Annihilate blocks $A_{2k-1,2j}$, for $j = k-1, k, \ldots, \ell$.

for $j = k-1, k, \ldots, \ell$

Compute $U_{2k-3,2j}$ such that $A_{2k-1,2k-3}U_{2k-3,2j} + A_{2k-1,2j} = 0$.

$A_{i,2j} \leftarrow A_{i,2j} + A_{i,2k-3}U_{2k-3,2j}$, $i = 1, 2, \ldots, 2k-1$.

$E_{i,2j} \leftarrow E_{i,2j} + E_{i,2k-3}U_{2k-3,2j}$, $i = 1, 2, \ldots, 2k-3$.

end

end if

end

At the end of Stage II, the upper triangular matrices W and U contain the accumulated non-orthogonal transformations performed in the reduction. Let $\widetilde{A} := W\widehat{A}U$, $\widetilde{E} := W\widehat{E}U$, and $\widetilde{B} = [\,\widetilde{B}_1\ \widetilde{B}_2\,] := W\widehat{B}$ be the system matrices resulted at the end of Stage II. Define also the feedback matrix $\widetilde{F} \in \mathbb{R}^{m_1 \times n}$ partitioned column-wise compatibly with $\widehat{E}$

$$\widetilde{F} = [\, O \; F_2 \; \cdots \; F_{2\ell-2} \; O \; F_{2\ell} \; O \,]$$

where F_{2j} are such that $A_{1,-1}F_{2j} + A_{1,2j} = 0$ for $j = 1, \ldots, \ell$. Choose also G such that $A_{1,-1}G + A_{1,0} = 0$.

With these matrices, we achieved that

$$\left[\widetilde{B}_1 \; \widetilde{B}_1 G + \widetilde{B}_2 \middle| \widetilde{A} + \widetilde{B}_1 \widetilde{F} \right] = \left[\begin{array}{cc|cccccccc} A_{1,-1} & O & \overline{A}_{11} & O & \cdots & \overline{A}_{1,2\ell-3} & O & \overline{A}_{1,2\ell-1} & O \\ O & A_{2,0} & A_{21} & \overline{A}_{22} & \cdots & \overline{A}_{2,2\ell-3} & \overline{A}_{2,2\ell-2} & \overline{A}_{2,2\ell-1} & \overline{A}_{2,2\ell} \\ O & O & A_{31} & O & \cdots & \overline{A}_{3,2\ell-3} & O & \overline{A}_{3,2\ell-1} & O \\ O & O & O & A_{42} & \cdots & \overline{A}_{4,2\ell-3} & \overline{A}_{4,2\ell-2} & \overline{A}_{4,2\ell-1} & \overline{A}_{4,2\ell} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & O & \cdots & A_{2\ell-1,2\ell-3} & O & \overline{A}_{2\ell-1,2\ell-1} & O \\ O & O & O & O & \cdots & O & A_{2\ell,2\ell-2} & \overline{A}_{2\ell,2\ell-1} & \overline{A}_{2\ell,2\ell} \end{array} \right] \tag{5}$$

$$\overline{E} = \begin{bmatrix} E_{11} & O & \overline{E}_{13} & \cdots & \overline{E}_{1,2\ell-1} & O \\ O & E_{22} & \overline{E}_{23} & \cdots & \overline{E}_{2,2\ell-1} & \overline{E}_{2,2\ell} \\ O & O & E_{33} & \cdots & \overline{E}_{3,2\ell-1} & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & E_{2\ell-1,2\ell-1} & O \\ O & O & O & \cdots & O & E_{2\ell,2\ell} \end{bmatrix}$$

where the elements without bars have not been modified in Stage II.

Consider now the permutation matrix defined by

$$P^T = \left[\begin{array}{cc|cc|c|cc|c} O & I_{\nu_2^{(1)}} & O & O & \cdots & O & O & O \\ \hline O & O & O & I_{\nu_2^{(2)}} & \cdots & O & O & O \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline O & O & O & O & \cdots & O & I_{\nu_2^{(\ell)}} & O \\ \hline I_{\nu_1^{(1)}} & O & O & O & \cdots & O & O & O \\ \hline O & O & I_{\nu_1^{(2)}} & O & \cdots & O & O & O \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline O & O & O & O & \cdots & I_{\nu_1^{(\ell)}} & O & O \\ \hline O & O & O & O & \cdots & O & O & I_{n-r} \end{array} \right] \tag{6}$$

If we define $V = ZUP$, $L = P^T W Q^T$ and $F = \widetilde{F} V^{-1}$, then overall we achieved that

$$L(A + B_1 F - \lambda E)V = \left[\begin{array}{c|c|c} \breve{A}_1 - \lambda \breve{E}_1 & * & * \\ \hline O & \breve{A}_2 - \lambda \breve{E}_2 & * \\ \hline O & O & A_{\overline{c}} - \lambda E_{\overline{c}} \end{array} \right], \quad L(B_2 G + B_1) = \left[\begin{array}{c} \breve{B}_1 \\ \hline O \\ \hline O \end{array} \right] \tag{7}$$

where

$$\left[\ \breve{B}_1 \mid \breve{A}_1 - \lambda\breve{E}_1\ \right] = \left[\begin{array}{c|ccccc} A_{2,0} & \overline{A}_{2,2} - \lambda E_{2,2} & \overline{A}_{2,4} - \lambda\overline{E}_{2,4} & \cdots & \overline{A}_{2,2\ell} - \lambda\overline{E}_{2,2\ell} \\ O & A_{4,2} & \overline{A}_{4,4} - \lambda E_{4,4} & \dots & \overline{A}_{4,2\ell} - \lambda\overline{E}_{4,2\ell} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & O & A_{2\ell,2\ell-2} & \overline{A}_{2\ell,2\ell} - \lambda E_{2\ell,2\ell} \end{array}\right]$$

$$\breve{A}_2 - \lambda\breve{E}_2 = \left[\begin{array}{c|ccccc} A_{1,-1} & \overline{A}_{1,1} - \lambda E_{1,1} & \overline{A}_{1,3} - \lambda\overline{E}_{1,3} & \cdots & \overline{A}_{1,2\ell-1} - \lambda\overline{E}_{1,2\ell-1} \\ O & A_{3,1} & \overline{A}_{3,3} - \lambda E_{3,3} & \dots & \overline{A}_{3,2\ell-1} - \lambda\overline{E}_{1,2\ell-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & O & A_{2\ell-1,2\ell-3} & \overline{A}_{2\ell-1,2\ell-1} - \lambda E_{2\ell-1,2\ell-1} \end{array}\right]$$

In the standard case we take $W = U^{-1}$ to ensure $W\widehat{E}U = I$. The resulting $\widetilde{A}$ and $\widetilde{B}$ satisfy (5).

It follows by inspection that the pair $(\breve{A}_1 - \lambda\breve{E}_1, \breve{B}_1)$ is controllable. Thus, by the above choice of F and G, we made $\sum_{i=1}^{\ell} \nu_1^{(i)}$ of eigenvalues of the pair $(A + B_1F - \lambda E, B_2G + B_1)$ uncontrollable, additionally to the $n - r$ uncontrollable original eigenvalues. The first $n_c = \sum_{i=1}^{\ell} \nu_2^{(i)}$ columns of V_1 satisfy

$$\overline{A}V_1 = V_1\breve{E}_1^{-1}\breve{A}_1 - \overline{B}_1FV_1, \quad \overline{B}_2G = V_1\breve{E}_1^{-1}\breve{B}_1 - \overline{B}_1$$

and thus, according to (3), span a *Type II* dynamic cover of dimension n_c for the pair $(\overline{A}, [\,\overline{B}_1\ \ \overline{B}_2\,])$. The following result can be shown using the results of [3]:

Theorem 1 *The Type II dynamic cover $\mathcal{V} =$ span V_1 has minimum dimension.*

3 Computation of Type I minimal dynamic covers

The computational problem which we solve in this section is the following: given the descriptor pair $(A - \lambda E, B)$ with $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and B partitioned as $B = [\,B_1\ \ B_2\,]$ with $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, determine the matrix F such that the pair $(A + B_2F - \lambda E, B_1)$ has maximal number of uncontrollable eigenvalues. This problem is essentially equivalent [12] to compute a subspace $\mathcal{V}$ having least possible dimension satisfying

$$(\overline{A} + \overline{B}_2F)\mathcal{V} \subset \mathcal{V}, \quad \text{span}\,\overline{B}_1 \subset \mathcal{V} \tag{8}$$

This condition can be rewritten also as a condition defining a *Type I* minimum dynamic cover [2, 3] of the form

$$\begin{array}{rcl} \overline{A}\mathcal{V} & \subset & \mathcal{V} + \overline{\mathcal{B}}_2 \\ \overline{\mathcal{B}}_1 & \subset & \mathcal{V} \end{array} \tag{9}$$

To compute Type I covers, we perform first the Stage I orthogonal reduction on the pair $(A - \lambda E, [B_1, B_2])$, as done in the previous section. However, at Stage II the non-orthogonal reduction annihilates a different set of blocks in A_c and E_c. The following procedure performs the second reduction stage by exploiting the full rank of submatrices $A_{2j,2j-2}$ and $E_{2j-1,2j-1}$ to introduce zero matrices in the block row $2j$ of A_c and block column $2j - 1$ of E_c, respectively.

Stage II: Special reduction for Type I Covers

Set $W = I, U = I$.

for $k = \ell, \ell-1, \ldots, 2$

Comment. Annihilate blocks $A_{2k,2j-1}$, for $j = k, k+1, \ldots, \ell$.

for $j = k, k+1, \ldots, \ell$

Compute $U_{2k-2,2j-1}$ such that $A_{2k,2k-2}U_{2k-2,2j-1} + A_{2k,2j-1} = 0$.

$A_{i,2j-1} \leftarrow A_{i,2j-1} + A_{i,2k-2}U_{2k-2,2j-1}$, $i = 1, 2, \ldots, 2k$.

$E_{i,2j-1} \leftarrow E_{i,2j-1} + E_{i,2k-2}U_{2k-2,2j-1}$, $i = 1, 2, \ldots, 2k-2$.

end

Comment. Annihilate blocks $E_{2k-2,2j-1}$, for $j = k, k+1, \ldots, \ell$.

for $j = k, k+1, \ldots, \ell$

Compute $W_{2k-2,2j-1}$ such that $W_{2k-2,2j-1}E_{2j-1,2j-1} + E_{2k-2,2j-1} = 0$.

$A_{2k-2,i} \leftarrow A_{2k-2,i} + W_{2k-2,2j-1}A_{2j-1,i}$, $i = 2j-2, 2j-1, \ldots, 2\ell$.

$E_{2k-2,i} \leftarrow E_{2k-2,i} + W_{2k-2,2j-1}E_{2j-1,i}$, $i = 2j, 2j+1, \ldots, 2\ell$.

end

end

Let $\widetilde{A} := W\widehat{A}U$, $\widetilde{E} := W\widehat{E}U$, and $\widetilde{B} = [\,\widetilde{B}_1\ \widetilde{B}_2\,] := W\widehat{B}$ be the system matrices resulted at the end of Stage II. Define also the feedback matrix $\widetilde{F} \in \mathbb{R}^{m_2 \times n}$ partitioned column-wise compatibly with $\widehat{E}$

$$\widetilde{F} = [\,F_1\ O\ F_3\ \cdots\ O\ F_{2\ell-1}\ O\ \ O\,]$$

where F_{2j-1} are such that $A_{2,0}F_{2j-1} + A_{2,2j-1} = 0$ for $j = 1, \ldots, \ell$.

Consider now the permutation matrix defined by

$$P^T = \left[\begin{array}{cc|cc|c|cc|c}
I_{\nu_1^{(1)}} & O & O & O & \cdots & O & O & O \\ \hline
O & O & I_{\nu_1^{(2)}} & O & \cdots & O & O & O \\ \hline
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline
O & O & O & O & \cdots & I_{\nu_1^{(\ell)}} & O & O \\ \hline
O & I_{\nu_2^{(1)}} & O & O & \cdots & O & O & O \\ \hline
O & O & O & I_{\nu_2^{(2)}} & \cdots & O & O & O \\ \hline
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline
O & O & O & O & \cdots & O & I_{\nu_2^{(\ell)}} & O \\ \hline
O & O & O & O & \cdots & O & O & I_{n-r}
\end{array}\right]$$

If we define $V = ZUP$, $L = P^TWQ^T$ and $F = \widetilde{F}V^{-1}$, then overall we achieved that

$$L(A + B_2F - \lambda E)V = \left[\begin{array}{c|c|c}
\breve{A}_1 - \lambda\breve{E}_1 & * & * \\ \hline
O & \breve{A}_2 - \lambda\breve{E}_2 & * \\ \hline
O & O & A_{\overline{c}} - \lambda E_{\overline{c}}
\end{array}\right], \quad LB_1 = \left[\begin{array}{c}
\breve{B}_1 \\ \hline
O \\ \hline
O
\end{array}\right] \qquad (10)$$

where

$$\left[\,\breve{B}_1 \,\middle|\, \breve{A}_1 - \lambda \breve{E}_1\,\right] = \left[\begin{array}{c|ccccc} A_{1,-1} & \overline{A}_{1,1} - \lambda E_{1,1} & \overline{A}_{1,3} - \lambda \overline{E}_{1,3} & \cdots & \overline{A}_{1,2\ell-1} - \lambda \overline{E}_{1,2\ell-1} \\ O & A_{3,1} & \overline{A}_{3,3} - \lambda E_{3,3} & \cdots & \overline{A}_{3,2\ell-1} - \lambda \overline{E}_{1,2\ell-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & O & A_{2\ell-1,2\ell-3} & \overline{A}_{2\ell-1,2\ell-1} - \lambda E_{2\ell-1,2\ell-1} \end{array}\right]$$

$$\breve{A}_2 - \lambda \breve{E}_2 = \left[\begin{array}{c|ccccc} A_{2,0} & \overline{A}_{2,2} - \lambda E_{2,2} & \overline{A}_{2,4} - \lambda \overline{E}_{2,4} & \cdots & \overline{A}_{2,2\ell} - \lambda \overline{E}_{2,2\ell} \\ O & A_{4,2} & \overline{A}_{4,4} - \lambda E_{4,4} & \cdots & \overline{A}_{4,2\ell} - \lambda \overline{E}_{4,2\ell} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & O & A_{2\ell,2\ell-2} & \overline{A}_{2\ell,2\ell} - \lambda E_{2\ell,2\ell} \end{array}\right]$$

It follows by inspection that the pair $(\breve{A}_1 - \lambda \breve{E}_1, \breve{B}_1)$ is controllable. Thus, by the above choice of F, we made $\sum_{i=1}^{\ell} \nu_2^{(i)}$ of eigenvalues of the pair $(A + B_2F - \lambda E, B_1)$ uncontrollable, additionally to the $n - r$ uncontrollable original eigenvalues. The first $n_c = \sum_{i=1}^{\ell} \nu_1^{(i)}$ columns of V_1 satisfy

$$\overline{A}V_1 = V_1\breve{E}_1^{-1}\breve{A}_1 - \overline{B}_2FV_1, \quad \overline{B}_1 = V_1\breve{E}_1^{-1}\breve{B}_1$$

and thus span a dynamic cover *Type I* of dimension n_c for the pair $(\overline{A}, [\,\overline{B}_1\ \overline{B}_2\,])$. The following result can be shown using the results of [3]:

Theorem 2 *The Type I dynamic cover $\mathcal{V} = span\, V_1$ has minimum dimension.*

4 Numerical aspects

The key reduction of system matrices to the special controllability form can be performed by using exclusively orthogonal similarity transformations. It can be shown that the computed condensed matrices $\widehat{A}$, $\widehat{E}$, and $\widehat{B}$ are exact for matrices which are nearby to the original matrices A, E, and B, respectively. Thus this part of the reduction is *numerically backward stable*. In implementing the algorithm, the row compressions are usually performed using rank revealing QR-factorizations with column pivoting. To make rank determinations even more reliable, QR-decompositions and singular value decompositions can be combined.

To achieve an $O(n^3)$ computational complexity in Stage I reduction, it is essential to perform the row compressions simultaneously with maintaining the upper triangular shape of E during reductions. The basic computational technique, described in details in [5], consists in employing elementary Givens transformations from left to introduce zero elements in the rows of B, while applying from right appropriate Givens transformations to annihilate the generated nonzero subdiagonal elements in E. By performing the rank revealing QR-decomposition in this way (involving also column permutations), we can show that the overall worst-case computational complexity of the special staircase algorithm is $O(n^3)$. In fact, when $m \ll n$, then the maximum number of required floating-point operations (flops) is essentially the same as that required to compute the generalized Hessenberg form of the pair (A, E), by accumulating only the left transformation Z. This amounts to about $13/2n^3$ flops. Note that for solving the problem (1), the accumulation of Z is not even necessary, since all right transformations can be directly applied to C.

The computations at Stage II to determine a basis for the minimal dynamic cover and the computation of feadback/feedforward matrices involve the solution of many, generally overdetermined, linear equations. For the computation of the basis for $\mathcal{V}$, we can estimate the condition numbers of the overall transformation matrices by computing $\|V\|_F^2 = \|U\|_F^2$ and $\|L\|_F^2 = \|W\|_F^2$. If these norms are relatively small (e.g., ≤ 10000) then practically there is no danger for a significant loss of accuracy due to nonorthogonal reduction. Note that it is very important to compute these condition numbers, since large values of them provide a clear hint of *possible* accuracy losses. In practice, it suffices to look at the largest magnitudes of elements of W and U used at Stage II to obtain equivalent information. For the computation of the feedback/feedforward matrices, condition numbers for solving the underlying equations can be also easily estimated. For the Stage II reduction, a simple operation count is possible by assuming all blocks 1×1 and this indicates a computational complexity of $O(n^3)$.

5 Minimum covers with stabilization

In some applications it is important to achieve simultaneously that the resulting feedback is stabilizing. For a Type II cover, this amounts to determine F, G and V such that the resulting $\breve{A}_1 - \lambda\breve{E}_1$ has all eigenvalues in an appropriate stability domain $\mathbb{C}^-$. This goal can not always be achieved, but it is always possible to move a maximum number of eigenvalues in this domain. To show how this is possible, consider the pair $(P^T(\widetilde{A} - \lambda\widetilde{E})P, P^T\widetilde{B})$, where $\widetilde{A}$, $\widetilde{E}$, and $\widetilde{B}$ are the resulting matrices at the end of Stage II and P^T is the permutation matrix (6). The matrices of the above pair have the form

$$P^T\widetilde{B} = \left[\begin{array}{cc} O & \widetilde{B}_{12} \\ \widetilde{B}_{21} & \widetilde{B}_{22} \\ O & O \\ \hline O & O \end{array}\right]$$

$$P^T(\widetilde{A} - \lambda\widetilde{E})P = \left[\begin{array}{ccc|c} \widetilde{A}_{11} - \lambda\widetilde{E}_{11} & \widetilde{A}_{12} & \widetilde{A}_{13} - \lambda\widetilde{E}_{13} & * \\ \widetilde{A}_{21} & \widetilde{A}_{22} - \lambda\widetilde{E}_{22} & \widetilde{A}_{23} - \lambda\widetilde{E}_{23} & * \\ O & \widetilde{A}_{32} & \widetilde{A}_{33} - \lambda\widetilde{E}_{33} & * \\ \hline O & O & O & A_{\bar{c}} \end{array}\right]$$

where the pair $(\widetilde{A}_{11} - \lambda\widetilde{E}_{11}, \widetilde{B}_{12})$ is controllable, and $\widetilde{B}_{21}$ and has full row rank. Note that the Stage II special reduction achieves basically to zero the blocks $\widetilde{A}_{31}$ and $\widetilde{E}_{12}$, while the feedback matrix F and feedforward matrix G achieve additionally to zero $\widetilde{A}_{21}$ and $\widetilde{B}_{22}$, respectively, by exploiting the full rank property of $\widetilde{B}_{21}$.

Consider the transformation matrices

$$T_Y = \left[\begin{array}{ccc|c} I & O & O & O \\ Y & I & O & O \\ O & O & I & O \\ \hline O & O & O & I \end{array}\right], \quad T_X = \left[\begin{array}{ccc|c} I & O & O & O \\ X & I & O & O \\ O & O & I & O \\ \hline O & O & O & I \end{array}\right]$$

partitioned in accordance with the structure of $P^T\widetilde{A}P$. It follows that

$$T_Y P^T \widetilde{B} = \left[\begin{array}{cc} O & \widetilde{B}_{12} \\ \widetilde{B}_{21} & \overline{B}_{22} \\ O & O \\ \hline O & O \end{array}\right]$$

$$T_Y P^T(\widetilde{A}-\lambda\widetilde{E})PT_X \left[\begin{array}{ccc|c} \widetilde{A}_{11}+\widetilde{A}_{12}X-\lambda\widetilde{E}_{11} & \widetilde{A}_{12} & \widetilde{A}_{13}-\lambda\widetilde{E}_{13} & * \\ \overline{A}_{21}-\lambda(\widetilde{E}_{22}X+Y\widetilde{E}_{11}) & \overline{A}_{22}-\lambda\widetilde{E}_{22} & \overline{A}_{23}-\lambda\overline{E}_{23} & * \\ \widetilde{A}_{32}X & \widetilde{A}_{32} & \widetilde{A}_{33}-\lambda\widetilde{E}_{33} & * \\ \hline O & O & O & A_{\bar{c}} \end{array}\right]$$

where we denoted with bars the changed quantities. If we choose X such that $\widetilde{A}_{32}X = 0$, and determine Y such that $\widetilde{E}_{22}X + Y\widetilde{E}_{11} = 0$, then we can preserve the structure of the original pair $(P^T(\widetilde{A}-\lambda\widetilde{E})P, P^T\widetilde{B})$. Thus, defining V as $V = ZUPT_X$, and $L = T_Y P^T W Q^T$, we can compute the feedback and feedforward matrices F and G exactly as before.

With T_X and T_Y chosen as above, the resulting $\breve{A}_1 - \lambda\breve{E}_1$ is $\widetilde{A}_{11} + \widetilde{A}_{12}X - \lambda\widetilde{E}_{11}$ and we can try to exploit this parametric freedom to move the eigenvalues of this pencil to stable locations. The following straightforward computations are necessary for this purpose:

1. Compute X_N with orthonormal columns such that span X_N is the right nullspace of $\widetilde{A}_{32}$.

2. Compute $\widetilde{F}$ to place a maximum number of eigenvalues of $\widetilde{A}_{11} + \widetilde{A}_{12}X_N\widetilde{F} - \lambda\widetilde{E}_{11}$ into the stability domain $\mathbb{C}^-$.

3. Define $X = X_N\widetilde{F}$ and $Y = -\widetilde{E}_{22}X\widetilde{E}_{11}^{-1}$.

All steps of this algorithms can be performed using numerically reliable computations. The computation of X_N is straightforward, since $\widetilde{A}_{32}$ is part of a staircase form. Thus, no further rank determination is necessary and X_N results from an RQ-like decomposition of $\widetilde{A}_{32}$ which exploits the full row rank of its leading nonzero rows. To determine $\widetilde{F}$, the most appropriate method is to apply a partial pole assignment technique like that of [6]. This approach can easily accommodate with non-stabilizable pairs, by moving only the controllable unstable generalized eigenvalues of the pair $(\widetilde{A}_{11}, \widetilde{E}_{11})$ into $\mathbb{C}^-$. If the pair $(\widetilde{A}_{11} - \lambda\widetilde{E}_{11}, \widetilde{A}_{12}X_N)$ is stabilizable then this algorithm can assign all unstable eigenvalues to arbitrary stable locations using minimum norm local feedbacks. In this way, the norm of X is minimized as well and thus also the condition number of the transformation matrix T_X and implicitly that of T_Y. A similar approach can be devised for determining Type I minimal covers with stabilization.

A specific aspect of determining minimal dynamic covers is the non-uniqueness of the resulting solution triple (F, G, V). This non-uniqueness manifests at several points of the proposed approach and can have negative or positive influence on the stabilizability properties determined by the triple $(\widetilde{A}_{11} - \lambda\widetilde{E}_{11}, \widetilde{A}_{12}, \widetilde{A}_{32})$. For example, selecting differently at Stage I the linearly independent columns in $B_1^{(j-1)}$ and $B_2^{(j-1)}$ or computing differently the blocks of U at Stage II when solving the underdetermined linear systems can lead to different minimal covers and different stabilizability properties. For numerical implementations, we recommend those solutions which ensure the best numerical properties of the proposed approach (e.g., selecting independent columns using column pivoting or determining least-norm solutions of all underdetermined linear systems).

6 Conclusions

We proposed efficient algorithms to compute two types of minimal dynamic covers, which have many important applications in various structural synthesis problems of linear systems. The proposed algorithms rely on the extensive use of orthogonal transformations. The use of non-orthogonal transformations at the final step of the reduction process allows to also obtain a precise estimation of possible accuracy losses induced by the overall reduction. Thus the proposed algorithm, although not numerically stable, can be considered numerically reliable. An interesting open problem is how to determine F and G to ensure the maximum number of uncontrollable poles cancellation in the case when E is singular. This problem is relevant to computing least McMillan degree solutions of linear rational equations [10]. A solution of this problem in a particular setting has been provided in [10].

The Stage I algorithm has been implemented in Fortran 77 and can be used via a mex-file interface from MATLAB. Furthermore, the Stage II of the proposed approach has been implemented in MATLAB and underlies the implementation of methods to compute least order left or right inverses and least order solutions of linear rational equations [10]. All this software is part of the DESCRIPTOR SYSTEMS Toolbox for MATLAB developed by the author [7].

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