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Computation of least order solutions of linear rational equations

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Abstract

We propose a numerically reliable approach for computing solutions of least McMillan order of linear equations with rational matrix coefficients. The main computational ingredients are the orthogonal reduction of the associated system matrix pencil to a certain Kronecker-like staircase form and the solution of a minimal dynamic cover design problem. For these computations we discuss numerically reliable algorithms relying on matrix pencil and descriptor system techniques.

1 Introduction

For given $p \times m$ and $p \times q$ rational matrices $G(\lambda)$ and $F(\lambda)$, respectively, we consider the problem to solve a linear rational system

$$G(\lambda)X(\lambda) = F(\lambda) \tag{1}$$

where the resulting solution $X(\lambda)$ must have the least possible McMillan degree. It is a well known fact that the system (1) has a solution provided the rank condition

$$\operatorname{rank} G(\lambda) = \operatorname{rank} [\, G(\lambda) \; F(\lambda) \,] \tag{2}$$

is fulfilled. We assume throughout the paper that this condition holds.

The solution of the linear rational system of the form (1) has many applications in control theory. One important application is the minimum degree design problem for which a solution method based on polynomial manipulations has been proposed by Wang and Davison [20]. This approach has been extended by Forney [4] into a general polynomial bases computation approach. Related problems like the computation of least order inverses or least order solution of the model matching problem can be elegantly solved using this computational framework.

The main computational problem when applying polynomial manipulation techniques is the computation of a polynomial basis for the right nullspace of a certain rational matrix. The solution method frequently advocated to be used is based on a polynomial echelon form [6]. However, this method, as well as similar polynomial manipulation based approaches, are in general numerically unstable. Even the numerically more reliable algorithm of [2] has limitations from numerical point of view, because it involves repeated matrix multiplications and inversions. Although there exist reliable algorithms for many basic polynomial computations, there are two basic limitations for the usefulness of the polynomial approach to solve large order problems.

The first limitation is the *intrinsic ill-conditioning* of polynomial representations because of possible extremely wide range of polynomial coefficients. It is not uncommon to arrive to polynomial matrices for which the range of magnitudes of the coefficients exceeds the intervals $(\varepsilon_M, 1)$, or $(1, 1/\varepsilon_M)$, or both of them, where ε_M is the relative machine precision (e.g., $\varepsilon_M \approx 10^{-16}$ for double precision computations on many machines). For such matrices, applying *any algorithm* (including numerically stable ones) can lead to a complete loss of accuracy, thus to a complete failure.

The second limitation is that many algorithms based on polynomial manipulations are *numerically unstable*. The reason for that is simple: typical operations like choosing pivots are determined by powers of the polynomial indeterminate rather than by the numerical values of coefficients. Thus, algorithms to compute minimal polynomial bases (e.g., algorithms based on the Hermite normal form or on the polynomial echelon form [6]), frequently lead to numerical instability. Therefore, the computed results for large order systems tend to be very inaccurate.

Avoiding the above mentioned difficulties was our main motivation to investigate alternative state-space methods to compute least order solutions of rational equations. We propose a numerically reliable approach for computing solutions of least McMillan degree of linear equations of form (1). The main computational ingredient in solving such equations is the orthogonal reduction of the system matrix pencil of the associated descriptor system realization of $[\, G(\lambda) \;\; F(\lambda) \,]$ to a certain Kronecker-like staircase form. Using this form a solution can be easily constructed, without the need to explicitly invert any rational or polynomial matrix. To determine solutions of least dynamical orders, minimal dynamic cover design techniques are employed. For all these computations we discuss numerically reliable algorithms relying on matrix pencil and descriptor system techniques. The proposed approach has been implemented using the robust numerical tools available in the DESCRIPTOR SYSTEMS Toolbox developed by the author [13].

2 Solving rational equations

The general solution of (1) can be expressed as

$$X(\lambda) = X_0(\lambda) + X_N(\lambda)Y(\lambda), \tag{3}$$

where $X_0(\lambda)$ is any particular solution of (1) and $X_N(\lambda)$ is a rational basis matrix for the right nullspace of $G(\lambda)$. In the case when both $X_0(\lambda)$ and $X_N(\lambda)$ are proper a straightforward approach to compute a solution $X(\lambda)$ of least McMillan degree is to determine suitable proper $Y(\lambda)$ to achieve this goal. A geometric control theoretic method for this purpose has been developed in [7], based on computing minimum dynamic covers. This method has been turned recently into an efficient and numerically reliable state-space computational approach in [17], which can be readily used to solve this subproblem.

Since $X_N(\lambda)$ can always be chosen proper (see subsection 2.2), the main difficulty using the above approach is the computation of appropriate $Y(\lambda)$ in the case when there is no proper solution of (1), and thus $X_0(\lambda)$ can not be chosen proper. To overcome this difficulty we can determine $X_0(\lambda)$ so that its polynomial part corresponds to a minimal number of infinite poles. These infinite poles originate from the intrinsic improper nature of any solution of (1) and are related to the common infinite zeros of $G(\lambda)$ and $F(\lambda)$. How to determine such an $X_0(\lambda)$ is shown in the subsection 2.1. By exploiting the problem structure, we can devise an approach similar to that of [7] to determine a proper $Y(\lambda)$ to reduce the McMillan degree of the proper part of $X_0(\lambda)$. This approach is described in subsection 2.3 and relies on the generalized minimum cover algorithm of [18].

In what follows we discuss computational methods based on descriptor system techniques to perform the main steps of the above approach, namely: (1) computation of a special particular solution $X_0(\lambda)$ with minimum number of infinite poles; (2) computation of a rational basis $X_N(\lambda)$ for the right nullspace of $G(\lambda)$; (3) computation of a solution $X(\lambda)$ of least McMillan degree using minimum cover design techniques.

2.1 Computation of $X_0(\lambda)$

Let assume that the compound rational matrix $[\, G(\lambda)\ F(\lambda)\,]$ has a minimal descriptor realization of order n of the form

$$\begin{array}{rcl} E\lambda x(t) & = & Ax(t) + B_G u(t) + B_F \nu(t) \\ \xi(t) & = & Cx(t) + D_G u(t) + D_F \nu(t) \end{array} \tag{4}$$

satisfying

$$[\, G(\lambda)\ F(\lambda)\,] = C(\lambda E - A)^{-1}[\, B_G\ B_F\,] + [\, D_G\ D_F\,] \tag{5}$$

where

$$G(\lambda) := \left[\begin{array}{c|c} A-\lambda E & B_G \\ \hline C & D_G \end{array}\right], \quad F(\lambda) := \left[\begin{array}{c|c} A-\lambda E & B_F \\ \hline C & D_F \end{array}\right]$$

denotes the state space realizations of $G(\lambda)$ and $F(\lambda)$, respectively. In accordance with the type of the system (4), λ can represent either the differential operator $\lambda x(t) = \dot{x}(t)$ in the case of a continuous-time system or the advance operator $\lambda x(t) = x(t+1)$ in the case of a discrete-time system. Note that for most of practical applications $[\, G(\lambda)\ F(\lambda)\,]$ is proper, thus we can always choose a realization such that $E = I$. However, for the sake of generality, we only assume that the pencil $A - \lambda E$ is regular, without assuming E is nonsingular. In this way, we will also cover the most general case of solving improper rational linear systems.

Let $S_G(\lambda)$ and $S_F(\lambda)$ be the system matrix pencils associated to the realizations of $G(\lambda)$ and $F(\lambda)$

$$S_G(\lambda) = \left[\begin{array}{cc} A-\lambda E & B_G \\ C & D_G \end{array}\right], \quad S_F(\lambda) = \left[\begin{array}{cc} A-\lambda E & B_F \\ C & D_F \end{array}\right]$$

Using the straightforward relations

$$\begin{bmatrix} A-\lambda E & B_G \\ O & G(\lambda) \end{bmatrix} = \begin{bmatrix} I_n & O \\ -C(A-\lambda E)^{-1} & I_p \end{bmatrix} S_G(\lambda)$$

$$\begin{bmatrix} A-\lambda E & B_F \\ O & F(\lambda) \end{bmatrix} = \begin{bmatrix} I_n & O \\ -C(A-\lambda E)^{-1} & I_p \end{bmatrix} S_F(\lambda)$$

it is easy to see that $X(\lambda)$ is a solution of $G(\lambda)X(\lambda) = F(\lambda)$ if and only if

$$Y(\lambda) = \begin{bmatrix} Y_{11}(\lambda) & Y_{12}(\lambda) \\ Y_{21}(\lambda) & X(\lambda) \end{bmatrix}$$

satisfies

$$S_G(\lambda)Y(\lambda) = S_F(\lambda) \tag{6}$$

The existence of the solution of (6) is guaranteed by (2), which is equivalent to

$$\text{rank}\, S_G(\lambda) = \text{rank}\,[\, S_G(\lambda)\ S_F(\lambda)\,] \tag{7}$$

It follows that instead of solving the rational equation $G(\lambda)X(\lambda) = F(\lambda)$, we can solve the polynomial equation (6) and take

$$X(\lambda) = \begin{bmatrix} O & I_m \end{bmatrix} Y(\lambda) \begin{bmatrix} O \\ I_q \end{bmatrix}$$

In fact, since we are interested in the second block column $Y_2(\lambda)$ of $Y(\lambda)$, we need only to solve

$$\begin{bmatrix} A-\lambda E & B_G \\ C & D_G \end{bmatrix} Y_2(\lambda) = \begin{bmatrix} B_F \\ D_F \end{bmatrix} \tag{8}$$

and compute $X(\lambda)$ as

$$X(\lambda) = \begin{bmatrix} O & I_m \end{bmatrix} Y_2(\lambda)$$

The condition (7) for the existence of a solution becomes

$$\text{rank} \begin{bmatrix} A-\lambda E & B_G \\ C & D_G \end{bmatrix} = \text{rank} \begin{bmatrix} A-\lambda E & B_G & B_F \\ C & D_G & D_F \end{bmatrix} \tag{9}$$

To solve (8), we isolate a full rank part of $S_G(\lambda)$ by reducing it to a particular Kronecker-like form. Let Q and Z be orthogonal matrices to reduce $S_G(\lambda)$ to the Kronecker-like form

$$\overline{S}_G(\lambda) := QS_G(\lambda)Z = \begin{bmatrix} B_r & A_r-\lambda E_r & A_{r,reg}-\lambda E_{r,reg} & * \\ 0 & 0 & A_{reg}-\lambda E_{reg} & * \\ 0 & 0 & 0 & A_l-\lambda E_l \end{bmatrix} \tag{10}$$

where $A_{reg} - \lambda E_{reg}$ is a regular subpencil, the pair $(A_r - \lambda E_r, B_r)$ is controllable with E_r nonsingular and the subpencil $A_l - \lambda E_l$ has full column rank. The above reduction can be computed by employing numerically stable algorithms as those proposed in [11, 1].

If $\overline{Y}_2(\lambda)$ is a solution of the reduced equation

$$\overline{S}_G(\lambda)\overline{Y}_2(\lambda) = Q \begin{bmatrix} B_F \\ D_F \end{bmatrix} \tag{11}$$

then $Y_2(\lambda) = Z\overline{Y}_2(\lambda)$ and thus

$$X(\lambda) = \begin{bmatrix} O & I_m \end{bmatrix} Z\overline{Y}_2(\lambda)$$

is a solution of the equation $G(\lambda)X(\lambda) = F(\lambda)$. Partition

$$Q \begin{bmatrix} -B_F \\ -D_F \end{bmatrix} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \\ \overline{B}_3 \end{bmatrix}$$

in accordance with the row structure of $\overline{S}_G(\lambda)$. Since $A_l - \lambda E_l$ has full column rank, it follows from (9) that $\overline{B}_3 = 0$. Thus, we can choose $\overline{Y}_2(\lambda)$ of the form

$$\overline{Y}_2(\lambda) = \begin{bmatrix} \overline{Y}_{12}(\lambda) \\ \overline{Y}_{22}(\lambda) \\ \overline{Y}_{32}(\lambda) \\ O \end{bmatrix},$$

where the partitioning of $\overline{Y}_2(\lambda)$ corresponds to the column partitioning of $\overline{S}_G(\lambda)$. Choosing $\overline{Y}_{12}(\lambda) = 0$, we obtain

$$\begin{bmatrix} \overline{Y}_{22}(\lambda) \\ \overline{Y}_{32}(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda E_r - A_r & \lambda E_{r,reg} - A_{r,reg} \\ O & \lambda E_{reg} - A_{reg} \end{bmatrix}^{-1} \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}$$

Let partition $[\, O \ I_m \,]Z$ in accordance with the column structure of $S_G(\lambda)$ as

$$[\, O \ I_m \,]Z = [\, D_r \ C_r \ C_{reg} \ C_l \,] \tag{12}$$

and denote

$$\overline{A} - \lambda\overline{E} = \begin{bmatrix} A_r - \lambda E_r & A_{r,reg} - \lambda E_{r,reg} \\ O & A_{reg} - \lambda E_{reg} \end{bmatrix}, \ \ \overline{B} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}, \ \ \overline{C} = [C_r \ C_{reg}\,] \tag{13}$$

Then a particular solution $X_0(\lambda)$ of the equation $G(\lambda)X(\lambda) = F(\lambda)$ can be expressed in form of a descriptor realization

$$X_0(\lambda) := \left[\begin{array}{c|c} \overline{A} - \lambda\overline{E} & \overline{B} \\ \hline \overline{C} & 0 \end{array} \right]$$

Some properties of $X_0(\lambda)$ can be easily deduced from the computed Kronecker-like form. The pair $(\overline{C}, \overline{A} - \lambda\overline{E})$ is always observable, but in general we can not assume that the pair $(\overline{A} - \lambda\overline{E}, \overline{B})$ is controllable. The poles of $X_0(\lambda)$ are among the generalized eigenvalues of the pair $(\overline{A}, \overline{E})$ and are partly freely assignable and partly fixed. The generalized eigenvalues of the pair (A_r, E_r) are called the "spurious" poles, and they originate from the column singularity of $G(\lambda)$. These poles are freely assignable by appropriate choice of a (non-orthogonal) right transformation matrix [14]. The fixed poles are the controllable eigenvalues of the pair $(A_{reg} - \lambda E_{reg}, \overline{B}_2)$. If $G(\lambda)$ and $F(\lambda)$ have no common poles and zeros then the pair $(A_{reg} - \lambda E_{reg}, \overline{B}_2)$ is controllable. In this case $X_0(\lambda)$ has the minimum possible poles at infinity.

In general, there exists a solution $X_0(\lambda)$ without a pole in γ (finite or infinite) if $c_\gamma(G) = c_\gamma([G\ F])$, where $c_\gamma(G)$ is the *content* of $G(\lambda)$ in γ as defined by [19]. Roughly, this is equivalent to say that the pole and zero structures of $G(\lambda)$ and $[G(\lambda)\ F(\lambda)]$ at γ coincide. For practical computations, this implies that some or all of common poles and zeros of $G(\lambda)$ and $[G(\lambda)\ F(\lambda)]$ will cancel. This cancellation can be done explicitly by removing the uncontrollable eigenvalues (finite and infinite) of the pair $(A_{reg} - \lambda E_{reg}, \overline{B}_2)$.

Removing the uncontrollable eigenvalues of the pair $(A_{reg} - \lambda E_{reg}, \overline{B}_2)$ can be done by applying the orthogonal staircase algorithm of [10]. By applying this algorithm, two orthogonal matrices Q_{reg} and Z_{reg} are determined such that all uncontrollable eigenvalues are separated in the trailing part of the transformed regular pencil $Q_{reg}(A_{reg} - \lambda E_{reg})Z_{reg}$, while the corresponding rows of $Q_{reg}\overline{B}_2$ are zero. The uncontrollable part of the triple $(\overline{A} - \lambda\overline{E}, \overline{B}, \overline{C})$ can be thus removed by deleting the trailing rows and/or columns of the transformed triple $(\overline{Q}(\overline{A} - \lambda\overline{E})\overline{Z}, \overline{QB}, \overline{CZ})$, where $\overline{Q} = \text{diag}\,(I, Q_{reg})$ and $\overline{Z} = \text{diag}\,(I, Z_{reg})$. For the sake of simplicity we reuse the same notation (with bar) by assuming that the pair $(\overline{A} - \lambda\overline{E}, \overline{B})$ is already controllable, thus the resulting $X_0(\lambda)$ fulfills the requirement for a minimal number of poles at infinity.

Until now, we employed exclusively orthogonal similarity transformations to determine the matrices of a descriptor realization of a suitable particular solution $X_0(\lambda)$. Therefore, this computation is *numerically stable*, because we can easily show that the computed system matrices in the presence of roundoff errors are exact for an original problem with slightly perturbed data.

To perform the order reduction step in subsection 2.3 we need a block-diagonal descriptor matrix $\overline{E}$ in (13) (i.e., with $E_{r,reg} = 0$). This can be easily achieved by performing an additional non-orthogonal column transformation using the transformation matrix

$$V = \begin{bmatrix} I & R \\ O & I \end{bmatrix}$$

with $R = -E_r^{-1}E_{r,reg}$, where V is partitioned in accordance with the partitioning of system matrices in (13). The transformed system $(\overline{A}V - \lambda\overline{E}V, \overline{B}, \overline{C}V)$, representing also $X_0(\lambda)$, has thus a block-diagonal descriptor matrix $\overline{E}V$. To simplify the presentation we will reuse the notation with bar and assume in what follows that $E_{r,reg} = 0$ in (13).

2.2 Computation of $X_N(\lambda)$

Using the same reduction of $S_G(\lambda)$ to $\overline{S}_G(\lambda)$ as in (10), a nullspace basis $X_N(\lambda)$ of $G(\lambda)$ can be computed from a nullspace basis $\overline{Y}_N(\lambda)$ of $\overline{S}_G(\lambda)$ as

$$X_N(\lambda) = [\,O\; I_m\,]Z\overline{Y}_N(\lambda)$$

We can determine $\overline{Y}_N(\lambda)$ in the form

$$\overline{Y}_N(\lambda) = \left[\begin{array}{c} I \\ (\lambda E_r - A_r)^{-1}B_r \\ O \\ O \end{array}\right].$$

With C_r and D_r defined in (12), we obtain a descriptor realization of $X_N(\lambda)$ as

$$X_N(\lambda) := \left[\begin{array}{c|c} A_r - \lambda E_r & B_r \\ \hline C_r & D_r \end{array}\right]$$

Obviously $X_N(\lambda)$ is proper and it can be shown that has least McMillan degree [16]. Moreover, the poles of $X_N(\lambda)$ are freely assignable by appropriately choosing the transformation matrices Q and Z to reduce the system pencil $S_G(\lambda)$. Note that, to obtain this nullspace basis, we performed exclusively orthogonal transformations on the system matrices. We can prove that all computed matrices are exact for a slightly perturbed original system. It follows that the algorithm to compute the nullspace basis is *numerically stable*.

2.3 Computation of a least order solution $X(\lambda)$

We can represent $X_N(\lambda)$ to have the same state, descriptor and output matrices as $X_0(\lambda)$. Let these realizations of $X_0(\lambda)$ and $X_N(\lambda)$ be

$$\left[\begin{array}{cc} X_0(\lambda) & X_N(\lambda) \end{array}\right] := \left[\begin{array}{c|cc} \overline{A} - \lambda\overline{E} & \overline{B} & \overline{B}_r \\ \hline \overline{C} & \overline{D} & \overline{D}_r \end{array}\right] := \left[\begin{array}{cc|cc} A_r - \lambda E_r & A_{r,reg} & \overline{B}_1 & B_r \\ O & A_{reg} - \lambda E_{reg} & \overline{B}_2 & O \\ \hline C_r & C_{reg} & O & D_r \end{array}\right] \quad (14)$$

where E_r is non-singular.

We consider first the case when $X_0(\lambda)$ is proper, that is, all eigenvalues of the pencil $A_{reg} - \lambda E_{reg}$ are finite and thus $\overline{E}$ is invertible. In this case, it was shown in [7] that a solution with least McMillan degree can be determined as $X(\lambda) = X_0(\lambda) + X_N(\lambda)Y(\lambda)$ by choosing an appropriate proper $Y(\lambda)$. This can be done by determining a suitable feedback matrix $\overline{F}_r$ and a feedforward matrix $\overline{L}_r$ to cancel the maximum number of unobservable and uncontrollable poles of

$$X(\lambda) := \left[\begin{array}{c|c} \overline{A} + \overline{B}_r\overline{F}_r - \lambda\overline{E} & \overline{B} + \overline{B}_r\overline{L}_r \\ \hline \overline{C} + \overline{D}_r\overline{F}_r & \overline{D} + \overline{D}_r\overline{L}_r \end{array}\right] \quad (15)$$

It can be shown that if we start with a minimal realization of $[\, G(\lambda)\ F(\lambda)\,]$, then we can not produce any unobservable poles in $X(\lambda)$ via state-feedback. Therefore, we only need to determine the matrices $\overline{F}_r$ and $\overline{L}_r$ to cancel the maximum number of uncontrollable poles.

This problem has been solved in [7] by reformulating it as a minimal order dynamic cover design problem. Consider the set

$$\mathcal{J} = \{\mathcal{V} : \mathrm{Im}\, \widetilde{B} + \widetilde{A}\mathcal{V} \subset \mathrm{Im}\, \widetilde{B}_r + \mathcal{V}\}$$

where $\widetilde{A} := \overline{E}^{\,-1}\overline{A}$, $\widetilde{B} := \overline{E}^{\,-1}\overline{B}$, and $\widetilde{B}_r := \overline{E}^{\,-1}\overline{B}_r$. Let $\mathcal{J}^*$ denote the set of subspaces in $\mathcal{J}$ of least dimension. If $\mathcal{V} \in \mathcal{J}^*$, then a pair $(\overline{F}_r, \overline{L}_r)$ can be determined such that

$$(\widetilde{A} + \widetilde{B}_r\overline{F}_r)\mathcal{V} + \mathrm{Im}\,(\widetilde{B} + \widetilde{B}_r\overline{L}_r) \subset \mathcal{V}$$

Thus, determining a minimal dimension $\mathcal{V}$ is equivalent to a minimal order cover design problem, and a conceptual geometric approach to solve it has been indicated in [7]. The outcome of his method is, besides $\mathcal{V}$, the pair $(\overline{F}_r, \overline{L}_r)$ which achieves a maximal order reduction by forcing pole-zero cancellations. This approach in the case of standard systems (i.e., $\overline{E} = I$) has been turned into a numerically reliable procedure in [17] and extended recently to the descriptor case with non-singular $\overline{E}$ in [18]. In this latter procedure $\overline{F}_r$ and $\overline{L}_r$ are determined from a special controllability staircase form of the pair $(\overline{A} - \lambda\overline{E}, [\,\overline{B}_r\ \ \overline{B}\,])$ obtained by using a numerically reliable method relying on both orthogonal and non-orthogonal similarity transformations. The implemented minimum cover algorithm (see function `smcover2.m` of the DESCRIPTOR TOOLBOX [13]) determines directly the least order $X(\lambda)$, without explicitly determining $Y(\lambda)$.

It is possible to refine this approach by exploiting the structure of matrices in (14). Assuming $\overline{F}_r = [\, F_r\ F_{reg}\,]$ is partitioned according to the structure of $\overline{A}$, we get from (15)

$$X(\lambda) := \left[\begin{array}{cc|c} A_r + B_rF_r - \lambda E_r & A_{r,reg} + B_rF_{reg} & \overline{B}_1 + B_r\overline{L}_r \\ O & A_{reg} - \lambda E_{reg} & \overline{B}_2 \\ \hline C_r + D_rF_r & C_{reg} + D_rF_{reg} & \overline{D} + D_r\overline{L}_r \end{array}\right]$$

Since the eigenvalues of $A_{reg} - \lambda E_{reg}$ are not controllable via $\overline{B}_r$, the state feedback $\overline{F}_r$ affects only the blocks $A_r - \lambda E_r$ and $A_{r,reg}$. To make a maximum number of eigenvalues of $A_r + B_rF_r - \lambda E_r$ uncontrollable we can alternatively solve a minimum dynamic cover problem of lower dimension for the system

$$\left[\begin{array}{cc} X_{0,r}(\lambda) & X_N(\lambda) \end{array}\right] := \left[\begin{array}{c|cc} A_r - \lambda E_r & [\, A_{r,reg}\ \overline{B}_1\,] & B_r \\ \hline C_r & [\, C_{r,reg}\ \ \overline{D}\,] & D_r \end{array}\right]$$

by determining an appropriate state-feedback matrix F_r and a feedforward matrix $[\, F_{reg}\ \overline{L}_r\,]$. The matrices of the least order solution $X(\lambda)$ can be easily assembled from the matrices of resulting least order $X_r(\lambda)$ for the above problem, which results as

$$X_r(\lambda) = \left[\begin{array}{c|cc} A_r + B_rF_r - \lambda E_r & A_{r,reg} + B_rF_{reg} & \overline{B}_1 + B_r\overline{L}_r \\ \hline C_r + D_rF_r & C_{reg} + D_rF_{reg} & \overline{D} + D_r\overline{L}_r \end{array}\right]$$

Besides lower size of the computational problem, the main advantage of this approach is that it is applicable regarless $A_{reg} - \lambda E_{reg}$ has infinite eigenvalues or not. An interesting open problem in this context is the possibility to extend the minimum cover algorithm of [18] to the case of descriptor systems with singular $\overline{E}$, without the need to make this matrix block-diagonal.

3 Applications

We consider shortly two applications of the proposed method to solve linear rational equations, namely, the computation of least order right or left inverses of rational matrices and the solution of the model matching problem.

3.1 Computation of minimal order right/left inverses

Let $G(\lambda)$ be a $p \times m$ rational matrix. If $\operatorname{rank} G(\lambda) = p$, then $G^R(\lambda)$ is an *right inverse* of $G(\lambda)$ if

$$G(\lambda)G^R(\lambda) = I_p \tag{16}$$

and a proper $G^R(\lambda)$ is a *right L-integral/delay inverse* of $G(\lambda)$ if

$$G(\lambda)G^R(\lambda) = \frac{1}{\lambda^L} I_p \tag{17}$$

Similarly if $\operatorname{rank} G(\lambda) = m$, then $G^L(\lambda)$ is a *left inverse* of $G(\lambda)$ if

$$G^L(\lambda)G(\lambda) = I_m \tag{18}$$

and a proper $G^L(\lambda)$ is an *left L-integral/delay inverse* of $G(\lambda)$ if

$$G^L(\lambda)G(\lambda) = \frac{1}{\lambda^L} I_m \tag{19}$$

Numerous applications of these inverses may be found in the control literature (see [9, 8] and cited references therein). These include, but are not restricted to, solutions to decoupling problems, new approaches to the design of controllers, decoding of convolutional codes, solving fault detection and isolation problems, etc.

To compute a right inverse of $G(\lambda)$, we can directly solve equation (16) for $G^R(\lambda)$. Using the proposed approach we can determine a proper right inverse provided $G(\lambda)$ has no infinite zeros. Furthermore, the right inverse can be obtained stable if $G(\lambda)$ has no unstable zeros as well. Clearly, the least McMillan degree of the resulting right inverse is bounded below by the number of zeros (finite and infinite) of $G(\lambda)$. In general, there is no guarantee that there exists a least order right inverse which is also stable, even if $G(\lambda)$ has no unstable and no infinite zeros. However, occasionally this condition can be fulfilled (see Example) and the algorithm of [17] is able to cope with this situation. To compute a left inverse we can solve the transposed equation (18) for $(G^L(\lambda))^T$. The related problem to determine least order right/left integral/delay inverses with smallest number L of integrators/delays can be solved using the approach described in the next subsection. Note that the smallest L can be easily determined from the maximum multiplicity of infinite eigenvalues of the pencil $A_{reg} - \lambda E_{reg}$ (resulting after eliminating the noncontrollable ones).

3.2 Proper and stable solution of the model matching problem

Consider the more general linear rational system of the form

$$G(\lambda)X(\lambda) = F(\lambda)M(\lambda) \tag{20}$$

where $G(\lambda)$ and $F(\lambda)$ are given $p \times m$ and $p \times q$ rational matrices, respectively, and where we need to choose an invertible $M(\lambda)$ such that the resulting solution $X(\lambda)$ is proper, stable and has the least possible McMillan degree. In contrast to the problem (1), here we have the additional freedom of choosing $M(\lambda)$ to impose the properness and stability of the computed solution. Solving equation (20) has many applications. Choosing appropriate $M(\lambda)$ is frequently used to formulate the *exact model matching problem* such that a physically realizable solution exists [5]. Minimal functional observer design with stability constraint or the design of fault detection and isolation filters also require the choice of appropriate diagonal $M(\lambda)$ to guarantee physical realizability. A straightforward approach is to solve (20) is to compute first a solution of (1) employing the method proposed in this paper and then compute a proper and stable right coprime factorization of the solution using methods as those proposed in [12]. An alternative approach to solve (20) with diagonal $M(\lambda)$ has been proposed in [15], where the determination of $M(\lambda)$ is embedded in the solution approach.

4 Example

Consider the transposed 2×3 full row rank rational matrix from [20]

$$G(\lambda) = \begin{bmatrix} \dfrac{1}{\lambda+2} & \dfrac{\lambda+3}{\lambda^2+3\lambda+2} & \dfrac{\lambda^2+3\lambda}{\lambda^2+3\lambda+2} \\ \dfrac{1}{\lambda+1} & \dfrac{\lambda}{\lambda+1} & 0 \end{bmatrix}$$

for which we compute a least order righ-inverse by solving (16).

A state space realization $(A, [\, B_G \; B_F \,], C, [\, D_G \; D_F \,])$ for $[\, G(\lambda) \; I \,]$ is given by

$$A = \begin{bmatrix} -3 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B_G = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & -2 \\ 1 & -1 & 0 \end{bmatrix}, \quad B_F = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad D_G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad D_F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To allow an easy reproducibility of the results, we illustrate our method by using non-orthogonal transformation matrices Q and Z to obtain the Kronecker-like form of the system pencil $S_G(\lambda)$ with ”nice” numbers. With the following transformation matrices

$$Q = \left[\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad Z = \left[\begin{array}{cccccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

we obtain the reduced pencil $\overline{S}_G(\lambda) = QS_G(\lambda)Z$ such that

$$\overline{S}_G(\lambda) = \left[\begin{array}{c||c|c} B_r & A_r - \lambda I & A_{r,reg} \\ \hline O & 0 & A_{reg} \end{array}\right] = \left[\begin{array}{c||ccc|cc} 1 & -\lambda & 0 & 0 & -1 & 0 \\ 0 & -3 & -\lambda & 0 & 4 & -2 \\ 0 & 0 & -2 & -3-\lambda & -2 & 2 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] := [\,\overline{B}_r || \overline{A} - \lambda \overline{E}\,]$$

and

$$Q\left[\begin{array}{c} -B_F \\ \hline -D_F \end{array}\right] := \overline{B} = \left[\begin{array}{c} \overline{B}_1 \\ \hline \overline{B}_2 \end{array}\right] \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & -1 \\ -1 & 0 \end{array}\right],$$

$$[\,O\ I_3\,]Z := [\,D_r\,||\,\overline{C}\,] := [\,D_r\,||\,C_r\mid C_{reg}\,] = \left[\begin{array}{c||ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

The particular solution $X_0(\lambda) = \overline{C}(\lambda\overline{E} - \overline{A})\overline{B}$ and the nullspace $X_N(\lambda) = C_r(\lambda E_r - A_r)B_r + D_r$ of $G(\lambda)$ are

$$X_0(\lambda) = \left[\begin{array}{cc} 0 & 0 \\ 0 & \frac{\lambda+1}{\lambda} \\ \frac{\lambda^2+3\lambda+2}{\lambda^2+3\lambda} & -\frac{\lambda+1}{\lambda^2} \end{array}\right], \quad X_N(\lambda) = \left[\begin{array}{c} 0 \\ -\frac{1}{\lambda} \\ \frac{-\lambda^2+3}{\lambda^3+3\lambda^2} \end{array}\right]$$

Notice that $X_0(\lambda)$ has McMillan degree three and is proper.

For simplicity we eliminate the non-dynamical part from the representation of $X_0(\lambda)$ to obtain a 3rd order realization of $[\,X_0(\lambda)\ \ X_N(\lambda)\,]$ given by

$$\left[\begin{array}{cc} X_0(\lambda) & X_N(\lambda) \end{array}\right] := \left[\begin{array}{c|cc} A_r - \lambda I & \overline{B}_1 & B_r \\ \hline C_r & \overline{D} & D_r \end{array}\right]$$

where

$$A_r = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & -2 & -3 \end{bmatrix}, \quad \overline{B}_1 = \begin{bmatrix} 0 & -1 \\ -2 & 4 \\ 2 & -2 \end{bmatrix}, \quad B_r = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_r = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad D_r = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

It is easy to observe that by choosing

$$\overline{F}_r = 0, \quad \overline{L}_r = [\,0 \;\; 1\,]$$

we can make the pair $(A_r + B_r F_r, \overline{B}_1 + B_r L_r)$ uncontrollable. The corresponding solution of McMillan degree 2 is

$$X(\lambda) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ \dfrac{\lambda^2+3\lambda+2}{\lambda^2+3\lambda} & -\dfrac{2\lambda+4}{\lambda^2+3\lambda} \end{bmatrix}$$

This solution is however unstable (both in continuous-time and discrete-time sense) having poles at 0 and -3.

Using the approach suggested in [17] we can perform first a similarity transformation with a special transformation matrix to move the controllable eigenvalues to desired locations and then determine the feedback and feedforward matrices to make the remaining eigenvalue uncontrollable. For reference purposes we give the result of such a computation by which we moved the pole in the origin to -3 to make the solution stable in a continuous-time sense. The corresponding feedback and feedforward matrices are

$$\overline{F}_r = [\,-3 \;\; 0 \;\; 0\,], \quad \overline{L}_r = [\,-2 \;\; 5\,]$$

The corresponding stable solution of McMillan degree 2 is

$$X(\lambda) = \begin{bmatrix} -\dfrac{2\lambda}{\lambda+3} & \dfrac{5\lambda+3}{\lambda+3} \\ \dfrac{2}{\lambda+3} & \dfrac{\lambda-1}{\lambda+3} \\ \dfrac{\lambda^2+8\lambda+11}{\lambda^2+6\lambda+9} & -\dfrac{6\lambda+10}{\lambda^2+6\lambda+9} \end{bmatrix}$$

5 Conclusions

We proposed numerically reliable approaches to solve several basic computational problems encountered when solving linear rational equations: (1) computation of an appropriate particular solution of a rational linear equation; (2) the computation of rational nullspace bases of rational matrices; and (3) the reduction of the dynamical orders of the solutions by employing minimal dynamic cover design techniques. Each of these computations can be performed using numerically stable or numerically reliable algorithms. The proposed approach in combination with special rational factorizations techniques can be employed to solve the more general model matching problem with stability and properness constraints.

For the implementation of the proposed approach, all necessary basic numerical software is available in the DESCRIPTOR SYSTEMS Toolbox for MATLAB [13], as for example, the computation of Kronecker-like staircase forms, computation of standard and special controllability forms (required in minimum cover design), computation of poles and zeros of descriptor systems, determination of minimal realizations, etc. The basic computational tools in this toolbox are several functionally rich *MEX*-functions, representing MATLAB interfaces to powerful and numerically robust Fortran subroutines partly available in the control and systems library SLICOT [3].

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