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Computation of Kronecker-like forms of periodic matrix pairs

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Abstract

We propose a computationally efficient and numerically reliable algorithm to compute Kronecker-like forms of periodic matrix pairs. The eigenvalues and Kronecker indices are defined via the Kronecker structure of an associated lifted matrix pencil. The proposed reduction method relies on structure preserving manipulations of this pencil to extract successively lower complexity subpencils which contains the finite and infinite eigenvalues as well as the left and right Kronecker structures. The new algorithm uses exclusively orthogonal transformations and for the overall reduction the backward numerical stability can be proved.

1 Introduction

The invariants of a matrix pencil $A - zE$ under strict equivalence transformation are contained in the *Kronecker canonical form* (KCF) of this pencil [3]. Specifically, it is possible to find two invertible matrices Q and Z such that

$$Q(A - zE)Z = \mathrm{diag}\,\{J_f - zI, I - zJ_\infty, L_{\epsilon_1}, \ldots, L_{\epsilon_s}, L^T_{\eta_1}, \ldots, L^T_{\eta_t}\} \tag{1}$$

where J_f and J_∞ are in Jordan form with J_∞ nilpotent, and L_k is the bidiagonal matrix pencil of dimension $k \times (k+1)$:

$$L_k = \begin{bmatrix} -z & 1 & & & \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{bmatrix}$$

The matrices J_f and J_∞ describes the finite and infinite eigenvalues, respectively, while the index sets $\{\epsilon_i, i = 1, \ldots, s\}$ and $\{\eta_j, j = 1, \ldots, t\}$ are the right and left minimal indices of $A - zE$, respectively.

To compute these structural invariants, generally there is no need to compute the KCF, (this would involve using possibly ill-conditioned transformations), but it is possible to determine, using exclusively orthogonal transformations, so-called *Kronecker-like forms* (KLFs) which contain a part or the complete information on the structural invariants. For example, using two orthogonal transformation matrices Q and Z, we can determine using the method of [10] the KLF

$$Q(A - zE)Z = \begin{bmatrix} B^r & A^r - zE^r & * & * & * \\ O & O & A^\infty - zE^\infty & * & * \\ O & O & O & A^f - zE^f & * \\ O & O & O & O & A^l - zE^l \\ O & O & O & O & C^l \end{bmatrix}$$

where: (1) the pair $(A^r - zE^r, B^r)$ is controllable with E^r invertible; (2) the pair $(C^l, A^l - zE^l)$ is observable with E^l invertible; (3) $A^\infty - zE^\infty$, with A^∞ invertible and E^∞ nilpotent, contains the infinite eigenvalues; and (4) $A^f - zE^f$, with E^f invertible, contains the finite eigenvalues. Since the subpencils $[\, B^r \;\; A^r - zE^r \,]$ and $\begin{bmatrix} A^l - zE^l \\ C^l \end{bmatrix}$ are obtained in special staircase forms, the left and right Kronecker indices can be easily deduced from the dimensions of the full-row rank and full-column rank diagonal blocks of these subpencils, respectively. The subpencil $A^\infty - zE^\infty$ is also in a special staircase form and the dimensions of the diagonal blocks of E^∞ determines the multiplicity of infinite eigenvalues. Similar algorithms to compute KLFs have been proposed in [8, 1, 2, 6]. These algorithms mainly differ by their computational complexities (i.e., $O(n^3)$ or $O(n^4)$, where n represents the maximal dimension of A and E), the shape of the resulting submatrices, and the employed rank determination strategies.

Let $S_k \in \mathrm{R}^{\mu_k \times \nu_k}$ and $T_k \in \mathrm{R}^{\mu_k \times \nu_{k+1}}$ be periodic matrices with period $N \geq 1$. The index k can be freely associated with a time instant and thus the matrices S_k and T_k can also be interpreted as periodically time-varying matrices. Note that the dimensions of these matrices are time-varying as well. The time related interpretation is especially relevant in connection with linear periodic discrete-time systems where many structural analysis problems can be formulated in terms of periodic matrix pairs [16].

In this paper we extend the structural invariant concepts for linear pencils to study analogous structural invariants of a periodic matrix pair (S_k, T_k) under periodic similarity transformations. Two N-periodic pairs (S_k, T_k) and $(\widetilde{S}_k, \widetilde{T}_k)$ are *equivalent* if there exist invertible N-periodic matrices Q_k and Z_k such that

$$\widetilde{S}_k = Q_k S_k Z_k, \quad \widetilde{T}_k = Q_k T_k Z_{k+1} \tag{2}$$

The transformation (2) is called a *periodic similarity transformation*. We propose an algorithm to determine *orthogonal* periodic transformation matrices Q_k and Z_k such that

$$Q_k S_k Z_k = \begin{bmatrix} B_k^r & A_k^r & * & * & * \\ O & O & A_k^\infty & * & * \\ O & O & O & A_k^f & * \\ O & O & O & O & A_k^l \\ O & O & O & O & C_k^l \end{bmatrix}, \quad Q_k T_k Z_{k+1} = \begin{bmatrix} O & E_k^r & * & * & * \\ O & O & E_k^\infty & * & * \\ O & O & O & E_k^f & * \\ O & O & O & O & E_k^l \\ O & O & O & O & O \end{bmatrix} \tag{3}$$

where:

(a) E_k^r is invertible and the periodic pair $\big((E_k^r)^{-1}A_k^r, (E_k^r)^{-1}B_k^r\big)$ is completely reachable;
(b) E_k^l is invertible and the periodic pair $\big(C_k^l, (E_k^l)^{-1}A_k^l\big)$ is completely observable;
(c) A_k^∞ is invertible and the product $(A_k^\infty)^{-1}E_k^\infty \ldots (A_{k+N-1}^\infty)^{-1}E_{k+N-1}^\infty$ is nilpotent; and
(d) E_k^f is non-singular.

In (3), $Q_kS_kZ_k$ and $Q_kT_kZ_{k+1}$ have the same row partition which however generally depends on k. For a fixed column partitioning of $Q_kS_kZ_k$, the corresponding column partitioning of $Q_kT_kZ_{k+1}$ is uniquely determined by the conditions (a)-(d) above. As we will show in the next section, the periodic pair (A_k^∞, E_k^∞) specifies the structure at infinity of an associated lifted pencil, while the pair (A_k^f, E_k^f) specifies its finite structure. Similarly, the periodic triples (A_k^r, E_k^r, B_k^r) and (A_k^l, E_k^l, C_k^l) specify the right and left Kronecker structure of this pencil, respectively.

Notation. For an N-periodic matrix $X_i \in \mathrm{R}^{m_k\times n_k}$ we use alternatively the *script* notation

$$\mathcal{X}_k := \mathrm{diag}\,(X_k, X_{k+1}, \ldots, X_{k+N-1}),$$

which associates the block-diagonal matrix $\mathcal{X}_k$ to the cyclic matrix sequence X_i, $i = k, \ldots, k+N-1$ starting at time moment k. To simplify the notation for the case $k = 1$, we drop the usage of index used for the matrices and dimensions. For an N-periodic matrix pair (A_k, E_k) with $A_k \in \mathrm{R}^{n_{k+1}\times n_k}$ and $E_k \in \mathrm{R}^{n_{k+1}\times n_{k+1}}$ invertible, we denote the $n_j \times n_i$ *generalized transition matrix* $\Phi_{A,E}(j,i) := E_{j-1}^{-1}A_{j-1}E_{j-2}^{-1}A_{j-2}\cdots E_i^{-1}A_i$, where $\Phi_{A,E}(i,i) := I_{n_i}$.

2 Basic definitions and results

Consider the lifted pencil at time k associated with the periodic pair (S_k, T_k)

$$P_k(z) = \begin{bmatrix} S_k & -T_k & O & \cdots & O \\ O & S_{k+1} & -T_{k+1} & \cdots & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & & & S_{k+N-2} & -T_{k+N-2} \\ -zT_{k+N-1} & O & \cdots & O & S_{k+N-1} \end{bmatrix} \tag{4}$$

This pencil represents a generalization to rectangular pairs with time-varying dimensions of the lifted pencil introduced in [4] to study periodic systems with constant state dimensions. The same lifting has been used in [16, 11] to define and compute the zeros of periodic descriptor systems using algorithms which exploits and respectively, preserve the special *cyclic* structure of the pencil $P_k(z)$.

The following definitions generalize corresponding notions for standard linear pencils.

Definition 1 *The finite eigenvalues at time moment k of the N-periodic pair (S_k, T_k) are the finite eigenvalues of the pencil $P_k(z)$ in (4).*

From the above definition it follows that the finite eigenvalues of the periodic pair (S_k, T_k) are those values of z (counting multiplicities) where the rank of the lifted pencil $P_k(z)$ drops below its normal rank. This definition generalizes the definition of *characteristic multipliers* of a single periodic matrix $S_k \in \mathrm{R}^{\nu_{k+1}\times\nu_k}$ (defined as the eigenvalues of the product $S_{k+N-1}\cdots S_{k+1}S_k$). These are precisely the finite eigenvalues of the periodic pair $(S_k, I_{\nu_{k+1}})$.

Definition 2 *The infinite eigenvalues at time moment k of the N-periodic pair (S_k, T_k) are the infinite eigenvalues of the pencil $P_k(z)$ in (4) excepting $\sum_{i=1}^{N-1}\mathrm{rank}\, T_{k+i-1}$ simple infinite eigenvalues.*

The infinite eigenvalues of $P_k(z)$ include $\sum_{i=1}^{N-1}\mathrm{rank}\, T_{k+i-1}$ simple eigenvalues at ∞ of the pencil $P_k(z)$ which originate from the lifting. These should not play any role when counting the ”true” infinite eigenvalues, and therefore must be discarded from the total count. Since the multiplicities of infinite zeros of a pencil are by definition in excess one with respect to the multiplicities of infinite eigenvalues, we have the following simpler definition of zeros.

Definition 3 *The zeros (finite and infinite) at time moment k of the N-periodic pair (S_k, T_k) are the zeros of the pencil $P_k(z)$ in (4).*

Definition 4 *The left/right minimal indices at time moment k of the N-periodic pair (S_k, T_k) are the left/right minimal indices of the pencil $P_k(z)$ in (4).*

Definition 5 *The N-periodic pair (S_k, T_k) is called regular if it has no left or right Kronecker indices.*

Using transformations as in (2) to reduce the periodic pair (S_k, T_k) to the form (3) is equivalent to compute $\widetilde{P}_k(z) = \mathcal{Q}_k P_k(z)\mathcal{Z}_k$, where $\widetilde{P}_k(z)$ has the same *cyclic* structure as $P_k(z)$. By using appropriate permutation matrices Π_1 and Π_2 we can reorder the blocks of $\widetilde{P}_k(z)$ such that

$$\Pi_1 \widetilde{P}_k(z)\Pi_2 = \begin{bmatrix} P_k^r(z) & * & * & * \\ O & P_k^\infty(z) & * & * \\ O & O & P_k^f(z) & * \\ O & O & O & P_k^l(z) \end{bmatrix} \tag{5}$$

with each nonzero block having the same cyclic structure as $P_k(z)$. For example, the diagonal blocks have the form

$$P_k^x(z) = \begin{bmatrix} S_k^x & -T_k^x & O & \cdots & O \\ O & S_{k+1}^x & -T_{k+1}^x & \cdots & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & & & S_{k+N-2}^x & -T_{k+N-2}^x \\ -zT_{k+N-1}^x & O & \cdots & O & S_{k+N-1}^x \end{bmatrix}$$

where the upper index x stays for r, ∞, f, or l. For $k = 1, \ldots, N$, the submatrices of the diagonal subpencils of (5) have the structures

$$\begin{array}{lllllll} S_k^r & := & \left[\begin{array}{cc} B_k^r & A_k^r \end{array} \right], & T_k^r & := & \left[\begin{array}{cc} O & E_k^r \end{array} \right] \\ S_k^\infty & := & A_k^\infty, & T_k^\infty & := & E_k^\infty \\ S_k^f & := & A_k^f, & T_k^f & := & E_k^f \\ S_k^l & := & \left[\begin{array}{c} A_k^l \\ C_k^l \end{array} \right], & T_k^l & := & \left[\begin{array}{c} E_k^l \\ O \end{array} \right] \end{array}$$

The main issue when relating the eigenvalues and minimal indices of the reduced pair $(\widetilde{S}_k, \widetilde{T}_k)$ in (3) to the structures of submatrices in (4) is to discard the simple infinite eigenvalues of the pencil $P_k(z)$ which are introduced via the lifting. Provided the submatrices of reduced pair $(\widetilde{S}_k, \widetilde{T}_k)$ in (3) satisfy the properties (a), (b), (c), (d), then we can easily prove the following results:

Proposition 1 *The finite eigenvalues at time moment k of the N-periodic pair (S_k, T_k) are the eigenvalues of the generalized monodromy matrix $\Phi_{A^f,E^f}(k+N,k)$.*

Proposition 2 *The infinite eigenvalues at time moment k of the N-periodic pair (S_k, T_k) are the generalized eigenvalues of $P_k^\infty(z)$, excepting $\sum_{i=1}^{N-1}$rank E_{k+i-1}^∞ simple infinite eigenvalues.*

Proposition 3 *The right minimal indices at time moment k of the N-periodic pair (S_k, T_k) are given by the right minimal indices of $P_k^r(z)$.*

Proposition 4 *The left minimal indices at time moment k of the N-periodic pair (S_k, T_k) are given by the left minimal indices of $P_k^l(z)$.*

In what follows, we propose a computational approach which ensures by construction the properties (a), (b), (c), (d) of the submatrices of the reduced pair $(\widetilde{S}_k, \widetilde{T}_k)$ in (3). We also show how the proposed algorithm allows to determine directly from the structures of the reduced matrices the left and right Kronecker minimal indices.

3 Computational approach

In this section we propose a computational approach to determine the KLF (3) of a given periodic pair (S_k, T_k). To simplify notation we describe only the computation for $k = 1$, but the same algorithm is evidently applicable for arbitrary k after a suitable permutation of the order of matrices. The proposed algorithm has several main steps, which we discuss in the subsequent subsections.

3.1 Computation of compressed form

In the first step we reduce the problem to an equivalent one, but with a special structure of matrices. Let $Q^{(1)}$ and $Z^{(1)}$ be orthogonal periodic matrices such that

$$S_k^{(1)} := Q_k^{(1)} S_k Z_k^{(1)} = \begin{bmatrix} B_k & A_k \\ D_k & C_k \end{bmatrix}, \quad T_k^{(1)} := Q_k^{(1)} T_k Z_{k+1}^{(1)} = \begin{bmatrix} O & E_k \\ O & O \end{bmatrix} \tag{6}$$

where, for $k = 1, \ldots, N$: $E_k \in \mathrm{R}^{n_{k+1}\times n_{k+1}}$ is invertible, $A_k \in \mathrm{R}^{n_{k+1}\times n_k}$, $B_k \in \mathrm{R}^{n_{k+1}\times m_k}$, $C_k \in \mathrm{R}^{p_k\times n_k}$, $D_k \in \mathrm{R}^{p_k\times m_k}$, with $m_k := \nu_k - n_k$ and $p_k := \mu_k - n_{k+1}$. The compression of each T_k to a non-singular E_k can be done by computing a full orthogonal decomposition $Q_k^{(1)} T_k V_{k+1} = \mathrm{diag}\,(E_k, O)$ using either the singular-value decomposition (SVD) or a rank-revealing QR-decomposition followed by an RQ-decomposition. Finally, the form in (6) is obtained by applying column permutations with an appropriate permutation matrix Π_k. For both rank determination techniques, we can freely assume that each resulting E_k is upper triangular.

Define $Z_k^{(1)} = V_k \Pi_k$. The pencil $P(z)$ and the transformed pencil $P^{(1)}(z) = \mathcal{Q}^{(1)} P(z) \mathcal{Z}^{(1)}$ corresponding to the reduced periodic pair $(S_k^{(1)}, T_k^{(1)})$ have the same Kronecker-canonical form, thus this pair has the same eigenvalues, as well as left and right minimal indices.

Interestingly, the "compressed pair", specified by the quintuple $(\mathcal{E}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, defines a periodic descriptor system

$$\begin{aligned} E_k x(k+1) &= A_k x(k) + B_k u(k) \\ y(k) &= C_k x(k) + D_k u(k) \end{aligned} \tag{7}$$

where the input vector $u(k)$, state vector $x(k)$ and output vector $y(k)$ have in general time-varying dimensions. The system matrices E_k, A_k, B_k, C_k, and D_k have time-varying dimensions as well and E_k is invertible. Note that the original periodic pair (S_k, T_k) frequently arises in an already compressed form when analyzing periodic systems properties [11].

3.2 Basic reduction

The goal of this reduction step is to separate the right-infinite ($r\infty$) and finite-left (fl) structures of the compressed pencil $P^{(1)}(z)$. In this step we compute orthogonal periodic matrices $\mathcal{Q}^{(2)}$ and $\mathcal{Z}^{(2)}$ such that

$$S_k^{(2)} := Q_k^{(2)} S_k^{(1)} Z_k^{(2)} = \begin{bmatrix} A_k^{r\infty} & * \\ O & A_k^{fl} \end{bmatrix}, \quad T_k^{(2)} := Q_k^{(2)} T_k^{(1)} Z_{k+1}^{(2)} = \begin{bmatrix} E_k^{r\infty} & * \\ O & E_k^{fl} \end{bmatrix} \tag{8}$$

where, for $k = 1, \ldots, N$, $A_k^{r\infty} \in \mathrm{R}^{m_k^{r\infty}\times n_k^{r\infty}}$ is in a staircase form with full row rank matrices on its main diagonal, $E_k^{r\infty} \in \mathrm{R}^{m_k^{r\infty}\times n_{k+1}^{r\infty}}$ has the part formed from the trailing non-zero columns of full-column rank, and $A_k^{fl} \in \mathrm{R}^{m_k^{fl}\times n_k^{fl}}$, and $E_k^{fl} \in \mathrm{R}^{m_k^{fl}\times n_{k+1}^{fl}}$ is upper trapezoidal and full column rank.

For this separation, we can freely apply the **Algorithm PS-REDUCE** proposed in [11] to the compressed pair and accumulate the performed orthogonal transformation. This algorithm can be seen as an extension of the standard pencil reduction technique of [5] to compute the finite eigenvalues of a compressed periodic pair. However, to achieve more symmetry in the structure at infinity, we propose an extension of the basic algorithm of [1] along the lines of improvements suggested in [6]. The following basic reduction algorithm will be used repeatedly to identify and separate different structures of a compressed periodic pair:

Algorithm BASIC_REDUCTION.

For $k = 1, \ldots, N$: set $m_k^{r\infty} = 0$, $n_k^{r\infty} = 0$; set $Q_k^{(2)} = I_{\mu_k}$, $Z_k^{(2)} = I_{\nu_k}$.

step–i

1. For each $k = 1, \ldots, K$, compute (e.g., by performing the QR-decomposition with column pivoting on D_k) an orthogonal matrix W_k and a permutation matrix Π_k such that

$$\left[\begin{array}{c:c|c} B_{k,1} & B_{k,2} & A_k \\ \hline D_{k,1} & D_{k,2} & C_{k,1} \\ \hdashline O & O & C_{k,2} \end{array}\right] := \text{diag}\,(I_{n_{k+1}}, W_k) \left[\begin{array}{c|c} B_k & A_k \\ \hline D_k & C_k \end{array}\right] \text{diag}\,(\Pi_k, I_{n_k}),$$

where $D_{k,1} \in \mathrm{R}^{\tau_k^{(i)} \times \tau_k^{(i)}}$ is invertible and upper-triangular.

2. For each $k = 1, \ldots, N$, compress the rows of $\begin{bmatrix} B_{k,1} \\ D_{k,1} \end{bmatrix}$ with orthogonal X_k such that

$$\left[\begin{array}{c:c|c} B_{k,11} & B_{k,12} & A_{k,1} \\ O & B_{k,22} & A_{k,2} \end{array}\right] := X_k \left[\begin{array}{c:c|c} B_{k,1} & B_{k,2} & A_k \\ \hline D_{k,1} & D_{k,2} & C_{k,1} \end{array}\right], \quad \left[\begin{array}{c} E_{k,1} \\ \hline E_{k,2} \end{array}\right] := X_k \left[\begin{array}{c} E_k \\ \hline O \end{array}\right]$$

with $B_{k,11} \in \mathrm{R}^{\tau_k^{(i)} \times \tau_k^{(i)}}$ full row rank and upper-triangular and $E_{k,2} \in \mathrm{R}^{n_{k+1} \times n_{k+1}}$ invertible and upper-triangular.

3. For $k = 1, \ldots, N$, determine orthogonal U_k to compress the rows of $B_{k,22}$ to a full row rank matrix such that $U_k B_{k,22} = \begin{bmatrix} \widetilde{B}_{k,22} \\ O \end{bmatrix}$, where $\widetilde{B}_{k,22} \in \mathrm{R}^{\rho_{k+1}^{(i)} \times (m_k - \tau_k^{(i)})}$ is of full row rank, and compute orthogonal V_{k+1} such that $U_k E_{k,2} V_{k+1}$ is upper triangular.

4. For $k = 1, \ldots, N$, form the transformation matrices

$$\widetilde{Q}_k = \text{diag}\,\left(I_{\tau_k^{(i)}}, U_k, I_{p_k - \tau_k^{(i)}}\right) \text{diag}\,\left(X_k, I_{p_k - \tau_k^{(i)}}\right) \text{diag}\,\left(I_{n_{k+1}}, W_k\right),$$

$$\widetilde{Z}_k = \text{diag}\,\left(\Pi_k, I_{n_k}\right) \text{diag}\,\left(I_{m_k}, V_k\right)$$

and transform the submatrices and partition them as:

$$\begin{array}{r} \tau_k^{(i)} \\ \rho_{k+1}^{(i)} \\ n_{k+1} - \rho_{k+1}^{(i)} \\ p_k - \tau_k^{(i)} \end{array} \left[\begin{array}{c:c|c:c} B_{k,11} & B_{k,12} & A_{k,11} & A_{k,12} \\ \hdashline O & \widetilde{B}_{k,22} & A_{k,21} & A_{k,22} \\ \hdashline O & O & A_{k,31} & A_{k,32} \\ \hdashline O & O & C_{k,21} & C_{k,22} \end{array}\right] := \widetilde{Q}_k \left[\begin{array}{c|c} B_k & A_k \\ D_k & C_k \end{array}\right] \widetilde{Z}_k,$$
$$\begin{array}{cccc} \tau_k^{(i)} & m_k - \tau_k^{(i)} & \rho_k^{(i)} & n_k - \rho_k^{(i)} \end{array}$$

$$\begin{array}{r} \tau_k^{(i)} \\ \rho_{k+1}^{(i)} \\ n_{k+1}-\rho_{k+1}^{(i)} \\ p_k-\tau_k^{(i)} \end{array}\left[\begin{array}{cc|cc} O & O & E_{k,11} & E_{k,12} \\ O & O & E_{k,21} & E_{k,22} \\ O & O & O & E_{k,32} \\ O & O & O & O \end{array}\right] := \widetilde{Q}_k \left[\begin{array}{c|c} O & E_k \\ O & O \end{array}\right] \widetilde{Z}_{k+1},$$
$$\qquad\qquad \tau_{k+1}^{(i)} \quad m_{k+1}-\tau_{k+1}^{(i)} \quad \rho_{k+1}^{(i)} \quad n_{k+1}-\rho_{k+1}^{(i)}$$

where: $B_{k,11}$ is invertible and upper triangular, $\widetilde{B}_{k,22}$ is of full row rank, and $E_{k,21}$ and $E_{k,32}$ are invertible and upper triangular.

4. For $k = 1, \ldots, N$, update $A_k := A_{k,32}$, $E_k := E_{k,32}$, $B_k := A_{k,31}$, $C_k := C_{k,22}$, $D_k := C_{k,21}$.
5. For $k = 1, \ldots, N$, $Q_k^{(2)} := \mathrm{diag}\left(I_{m_k^{r\infty}}, \widetilde{Q}_k\right) Q_k^{(2)}$, $Z_k^{(2)} := Z_k^{(2)} \mathrm{diag}\left(I_{n_k^{r\infty}}, \widetilde{Z}_k\right)$.
6. For $k = 1, \ldots, N$, update $m_k^{r\infty} := m_k^{r\infty} + \rho_{k+1}^{(i)} + \tau_k^{(i)}$, $n_k^{r\infty} := n_k^{r\infty} + m_k$, $n_k := n_k - \rho_k^{(i)}$, $m_k := \rho_k^{(i)}$, $p_k := p_k - \tau_k^{(i)}$.
7. If $m_k = 0$ for $k = 1, \ldots, K$, then go to **exit**
8. $i := i + 1$ *go to* **step–i**;

exit Compute $S_k^{(2)} := Q_k^{(2)} S_k^{(1)} Z_k^{(2)}$, $T_k^{(2)} := Q_k^{(2)} T_k^{(1)} Z_{k+1}^{(2)}$ and partition them according to (8).

The computation stops when all B_k and D_k have null columns. The resulting periodic pair (A_k^{fl}, E_k^{fl}) has the following form

$$A_k^{fl} = \left[\begin{array}{c} A_k \\ C_k \end{array}\right], \quad E_k^{fl} = \left[\begin{array}{c} E_k \\ O \end{array}\right], \tag{9}$$

where $A_k \in \mathbb{R}^{(\nu_{k+1}-n_{k+1}^{r\infty})\times(\nu_k - n_k^{r\infty})}$, $E_k \in \mathbb{R}^{(\nu_{k+1}-n_{k+1}^{r\infty})\times(\nu_{k+1}-n_{k+1}^{r\infty})}$ is invertible and upper triangular, and $C_k \in \mathbb{R}^{p_k^{lf}\times(\nu_k - n_k^{r\infty})}$, where $p_k^{lf} = (\mu_k - m_k^{r\infty}) - (\nu_{k+1} - n_{k+1}^{r\infty})$. Since the associated lifted pencil has full column rank for almost all values of z (finite and infinite), the pair (A_k^{fl}, E_k^{fl}) can have only finite eigenvalues and/or left Kronecker structure.

The compression at Step 2 can be done by performing a QR-decomposition of $\left[\begin{array}{c} B_{k,1} \\ D_{k,1} \end{array}\right]$ which exploits the upper triangular shape of $D_{k,1}$. This can be achieved by employing sequences of Givens transformations to zero successively elements under the diagonal of $B_{k,1}$. By starting from below (i.e., zeroing first the diagonal element of $D_{k,1}$) the upper triangular structure of $E_{k,2}$ is automatically achieved. For details see [6].

The compression at Step 3 of $B_{k,22}$ to a full row rank matrix can be done simultaneously with maintaining $E_{k,2}$ upper triangular. This compression technique represents the main computational step in determining the periodic controllability staircase form of periodic descriptor systems (see [13] for more details).

At the end of **Algorithm BASIC_REDUCTION** we obtain globally the reduced matrices $S_k^{(2)}$ and $T_k^{(2)}$ in the form (8), where the periodic pair $(A_k^{r\infty}, E_k^{r\infty})$ is in the following staircase form

$$A_k^{r\infty} = \begin{bmatrix} A_{k;1,1}^{r\infty} & A_{k;1,2}^{r\infty} & \cdots & A_{k;1,\ell-1}^{r\infty} & A_{k;1,\ell}^{r\infty} \\ O & A_{k;2,2}^{r\infty} & \cdots & A_{k;2,\ell-1}^{r\infty} & A_{k;2,\ell}^{r\infty} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & A_{k;\ell-1,\ell-1}^{r\infty} & A_{k;\ell-1,\ell}^{r\infty} \\ O & O & \cdots & O & A_{k;\ell,\ell}^{r\infty} \end{bmatrix} \tag{10}$$

$$E_k^{r\infty} = \begin{bmatrix} O & E_{k;1,2}^{r\infty} & \cdots & E_{k;1,\ell-1}^{r\infty} & E_{k;1,\ell}^{r\infty} \\ O & O & \cdots & E_{k;2,\ell-1}^{r\infty} & E_{k;2,\ell}^{r\infty} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & O & E_{k;\ell-1,\ell}^{r\infty} \\ O & O & \cdots & O & O \end{bmatrix} \tag{11}$$

where ℓ is the number of steps performed by the algorithm, $A_{k;i,i}^{r\infty} \in \mathrm{R}^{(\rho_{k+1}^{(i)}+\tau_k^{(i)})\times\rho_k^{(i-1)}}$ is of full row rank and $E_{k;i,i+1}^{r\infty} \in \mathrm{R}^{(\rho_{k+1}^{(i)}+\tau_k^{(i)})\times\rho_{k+1}^{(i)}}$ is of full column rank. Since by construction the associated lifted pencil $P^{r\infty}(z)$ has full row rank for all finite values of z, the pair $(A_k^{r\infty}, E_k^{r\infty})$ has only infinite and/or right Kronecker structures. Furthermore, we have the following relations among the dimensions of the block

$$\rho_{k+i-1}^{(i-1)} \geq \rho_{k+i}^{(i)} + \tau_{k+i-1}^{(i)}, \quad i = 1, \ldots, \ell$$

3.3 Separation of finite and left structures

Let X be an $m \times n$ matrix. Define the pertranspose of X as $X^P = J_n X^T J_m$, where J_j is the $j \times j$ permutation matrix

$$J_j = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & . & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

A possible approach to separate the finite and left structures of the periodic pair (A_k^{fl}, E_k^{fl}) is to apply the **Algorithm BASIC_REDUCTION** to the dual pair $(\widehat{A}_k^{rf}, \widehat{E}_k^{rf})$ defined as

$$\widehat{A}_k^{rf} = (A_{N-k+1}^{fl})^P, \quad k = 1, \ldots, N, \quad \widehat{E}_k^{rf} = (E_{N-k}^{fl})^P, \quad k = 1, \ldots, N-1, \quad \widetilde{E}_N^{rf} = (E_N^{fl})^P$$

Note that the dual pair $(\widehat{A}_k^{rf}, \widehat{E}_k^{rf})$ is already in a particular compressed form as in (6), with D_k and C_k empty matrices.

We perform the **Algorithm BASIC_REDUCTION** to the dual pair to obtain $\widehat{Q}_k$ and $\widehat{Z}_k$ such that

$$\widehat{Q}_k \widehat{A}_k^{rf} \widehat{Z}_k = \begin{bmatrix} \widehat{B}_k^r & \widehat{A}_k^r & * \\ O & O & \widehat{A}_k^f \end{bmatrix}, \quad \widehat{Q}_k \widehat{E}_k^{rf} \widehat{Z}_{k+1} = \begin{bmatrix} O & \widehat{E}_k^r & * \\ O & O & \widehat{E}_k^f \end{bmatrix}$$

where $\widehat{E}_k^r$ and $\widehat{E}_k^f$ are invertible and upper-triangular, and the matrices $[\, \widehat{B}_k^r \ \widehat{A}_k^r \,]$ and $[\, O \ \widehat{E}_k^r \,]$ are in staircase forms similar to (10) and (11), respectively. This separation is equivalent to the recently proposed algorithm to compute the periodic Kalman reachability decomposition of a periodic descriptor system of the form (7) with invertible E_k [13].

By defining

$$Q_k^{(3)} := \operatorname{diag}\left(I_{m_k^{r\infty}}, \widehat{Z}_{N-k+1}^P \right), \quad Z_k^{(3)} := \operatorname{diag}\left(I_{n_k^{r\infty}}, \widehat{Q}_{N-k+1}^P \right)$$

we obtain the matrices of the reduced pair $(S_k^{(3)}, T_k^{(3)}) := (Q_k^{(3)} S_k^{(3)} Z_k^{(3)}, Q_k^{(3)} T_k^{(2)} Z_{k+1}^{(3)})$ in the form

$$S_k^{(3)} := \begin{bmatrix} A_k^{r\infty} & * & * \\ O & A_k^f & * \\ O & O & A_k^l \\ O & O & C_k^l \end{bmatrix}, \quad T_k^{(3)} := \begin{bmatrix} E_k^{r\infty} & * & * \\ O & E_k^f & * \\ O & O & E_k^l \\ O & O & O \end{bmatrix} \tag{12}$$

where the pair (A_k^f, E_k^f) has only finite eigenvalues and the periodic pair $((E_k^l)^{-1} A_k^l, C_k^l)$ is observable. Note that the above approach also ensures that both E_k^f and E_k^l are upper triangular.

Assume that the full row rank diagonal blocks of $[\, \widehat{B}_k^r \ \widehat{A}_k^r \,]$ are $(\widehat{\rho}_{k+1}^{(i)} + \widehat{\tau}_k^{(i)}) \times \widehat{\rho}_k^{(i-1)}$ matrices for $i = 1, ..., \ell^l$ with $\widehat{\tau}_k^{(i)} = 0$, where $\widehat{\rho}_k^{(0)} = p_k^{lf}$ is the row dimension of C_k in (9). The following result, which we give without proof, relates the block sizes of the computed staircase forms $[\, \widehat{B}_k^l \ \widehat{A}_k^l \,]$ to the left minimal Kronecker indices of $P(z)$.

Proposition 5 *The index sets $\{\widehat{\rho}_k^{(i)}\}$, $i = 1, \ldots, \ell^l$ completely determines the left minimal Kronecker indices as follows: there are $\eta_1^{(i)} - \eta_1^{(i+1)}$ Kronecker blocks L_{i-1}^T, for $i = 1, 2, \ldots,$ where*

$$\eta_k^{(i)} = \sum_{j=iN+1}^{\min\{(i+1)N, \ell^l\}} \widehat{\rho}_k^{(j)}$$

3.4 Separation of right and infinite structures

A possible computational approach for this separation is to compress first the pair $(A_k^{r\infty}, E_k^{r\infty})$ as

$$\overline{A}_k^{r\infty} = \begin{bmatrix} B_k & A_k \\ D_k & C_k \end{bmatrix} := U_k A_k^{r\infty}, \quad \overline{E}_k^{r\infty} = \begin{bmatrix} O & E_k \\ O & O \end{bmatrix} := U_k E_k^{r\infty} \tag{13}$$

by using appropriate orthogonal matrices U_k. These matrices can be determined from the QR-decomposition of the trailing non-zero columns of $E_k^{r\infty}$ in (11). By exploiting the full column rank and the staircase structures of these matrices, this computation can be done efficiently.

Then, we form the dual pair $(\widetilde{A}_k^{\infty l}, \widetilde{E}_k^{\infty l})$ defined as

$$\widetilde{A}_k^{\infty l} = (\overline{A}_{N-k+1}^{r\infty})^P, \quad k = 1, \ldots, N, \quad \widetilde{E}_k^{\infty l} = (\overline{E}_{N-k}^{r\infty})^P, \quad k = 1, \ldots, N-1, \quad \widetilde{E}_N^{\infty l} = (\overline{E}_N^{r\infty})^P$$

and apply the **Algorithm BASIC_REDUCTION** to the dual compressed form to obtain the orthogonal transformation matrices $\widetilde{Q}_k$ and $\widetilde{Z}_k$ such that

$$\widetilde{Q}_k \widetilde{A}_k^{\infty l} \widetilde{Z}_k = \begin{bmatrix} \widetilde{A}_k^{\infty} & * \\ O & \widetilde{A}_k^l \\ O & \widetilde{C}_k^l \end{bmatrix}, \quad \widetilde{Q}_k \widetilde{E}_k^{\infty l} \widetilde{Z}_{k+1} = \begin{bmatrix} \widetilde{E}_k^{\infty} & * \\ O & \widetilde{E}_k^l \\ O & O \end{bmatrix}$$

where $\widehat{A}_k^{\infty}$ and $\widehat{E}_k^l$ are invertible and upper-triangular, and the matrices $\widehat{A}_k^{\infty}$ and $\widehat{E}_k^{\infty}$ are in staircase forms similar to (10) and (11), respectively.

By defining

$$Q_k^{(4)} := \operatorname{diag}\left(\widetilde{Z}_{N-k+1}^P U_k, I_{\mu_k - m_k^{r\infty}},\right), \quad Z_k^{(4)} := \operatorname{diag}\left(\widetilde{Q}_{N-k+1}^P, I_{\nu_k - n_k^{r\infty}},\right)$$

we obtain the matrices of the reduced pair $(S_k^{(4)}, T_k^{(4)}) := (Q_k^{(4)} S_k^{(3)} Z_k^{(4)}, Q_k^{(4)} T_k^{(3)} Z_{k+1}^{(4)})$ in the form (3), where the pair $(A_k^{\infty}, E_k^{\infty})$ has only infinite eigenvalues and the periodic pair $\big((E_k^r)^{-1} A_k^r, (E_k^r)^{-1} B_k^r\big)$ is completely reachable. Note that the above approach also ensures that E_k^r and A_k^{∞} are upper triangular. To obtain $[\, B_k^r, \; A_k^r \,]$ in a staircase form like (10), the **Algorithm BASIC_REDUCTION** must be applied once again to the particular compressed pair $([\, B_k^r, \; A_k^r \,], [\, O, \; E_k^r \,]$.

We postulate the existence of a more efficient procedure without computational overheads (e.g., pertransposing) to determine directly the orthogonal matrices U_k and V_k which reduce the pair $(A_k^{r\infty}, E_k^{r\infty})$ to the separated form

$$U_k A_k^{r\infty} V_k = \begin{bmatrix} B_k^r & A_k^r & * \\ O & O & A_k^{\infty} \end{bmatrix}, \quad U_k E_k^{r\infty} V_{k+1} = \begin{bmatrix} O & E_k^r & * \\ O & O & E_k^{\infty} \end{bmatrix}$$

where the matrices of both pairs $([\, B_k^r, \; A_k^r \,], [\, O, \; E_k^r \,]$ and $(A_k^{\infty}, E_k^{\infty})$ are in staircase forms. As basis for such a procedure could serve the Algorithms 3.3.1 and 3.3.3 in [1], suitably extended to exploit the fine structure of matrices $A_k^{r\infty}$ and $E_k^{r\infty}$ in (10) and (11), respectively.

Assume that the full row rank diagonal blocks of $[\, B_k^r \; A_k^r \,]$ are $(\rho_{k+1}^{(i)} + \tau_k^{(i)}) \times \rho_k^{(i-1)}$ matrices for $i = 1, ..., \ell$ with $\tau_k^{(i)} = 0$, where $\rho_k^{(0)}$ is the column dimension of B_k^r. The following result, which we give without proof, relates the block sizes of the computed staircase forms $[\, B_k^r \; A_k^r \,]$ to the right minimal Kronecker indices of $P(z)$.

Proposition 6 *The index sets $\{\rho_k^{(i)}\}$, $i = 1, \ldots, \ell$ completely determines the right minimal Kronecker indices as follows: there are $\epsilon_1^{(i)} - \epsilon_1^{(i+1)}$ Kronecker blocks L_{i-1}, for $i = 1, 2, \ldots,$ where*

$$\epsilon_k^{(i)} = \sum_{j=iN+1}^{\min\{(i+1)N,\ell\}} \rho_k^{(j)}$$

A similar result relating the multiplicity of infinite eigenvalues to the block sizes is still open.

4 Numerical Aspects

For the reduction of the periodic pair (S_k, T_k) to the periodic KLF (3) we employed exclusively orthogonal transformations of the form (2), which can be applied as sequences of Householder and Givens transformations underlying the computation of several QR-decompositions, with or without column pivoting. Thus it possible to prove that the computed matrices in the KLF (3) are exact for slightly perturbed initial matrices $\overline{S}_k$, $\overline{T}_k$, which satisfy

$$\|\overline{X} - X\| \leq \varepsilon_X \|X\|, \quad X = S_k, T_k$$

where, in each case, ε_X is a modest multiple of the relative machine precision ε_M. It follows that the proposed algorithm is *backward stable*.

Regarding the computational complexity of the proposed algorithm, we note that all reductions are performed N times on low order matrices, thus the overall computational complexity is proportional with N. To estimate the worst-case computational complexity in terms of problem dimensions, we assume constant dimensions μ and ν for S_k and T_k, and constant rank n of T_k. The computation of the compressed form (6) can be performed by using either SVD-based or rank-revealing QR-decomposition based reductions. This requires $O(Nn(n+p)(n+m))$ *floating point operations* (flops), where $p = \mu - n$ and $m = \nu - n$. The key computation in the proposed approach is the **Algorithm BASIC_REDUCTION**. The compressions of D_k, $k = 1, \ldots, N$ at Step 1 can be done by computing successively N rank-revealing QR-decompositions of $p \times m$ matrices and applying the transformation to $n \times m$ sub-blocks. This reduction step, performed more than once for decreasing values of p, m and n, has a worst-case computational complexity of $O(N(n+m)pm)$. The compression at Step 2 and the application of transformations to the rest of matrices has a worst-case computational complexity of $O(N(n+p)(n+m)p)$. The compression performed at Step 3 and the application of transformations is the only critical computation of the proposed approach. Note that by just computing V_{k+1} such that $U_k E_{k,2} V_{k+1}$ is upper triangular is an operation of complexity $O(n^3)$. This would make the overall worst-case complexity to maintain $E_{k,2}$ upper triangular for $k = 1, \ldots, N$ to become $O(Nn^4)$. To avoid this, we can perform the compression of $B_{k,22}$ with U_k and restoring the upper-triangular form of $U_k E_{k,2}$ simultaneously, by employing Givens rotations. The reduction technique is entirely similar to that independently developed in [1] and [9]. Using this approach, this computation has per iteration step a complexity at most $O(N\eta n^2)$, where η is small compared to n. Thus, the overall complexity of the compression-restoring algorithm is $O(Nn^3)$. Summing up, the **Algorithm BASIC_REDUCTION** has a worst-case complexity which can be bounded by $O(N(p+n)(m+n)n)$.

5 CONCLUSION

We developed a numerically backward stable algorithm to reduce periodic matrix pairs to Kronecker-like forms. The proposed algorithm allows to determine directly from the structures of the reduced matrices the main Kronecker invariants of the associated lifted pencil. The new algorithm works in the most general setting of periodic matrix pairs with time-varying dimensions. Two key features of this algorithm are: 1) a satisfactory worst-case computational complexity, which is linear in the period N and cubic in the maximum dimension of the blocks; and 2) backward numerical stability achieved by employing exclusively reductions based on orthogonal transformations. According to the requirements we formulated in [15], this is a satisfactory algorithm, well-suited for robust software implementations.

The proposed algorithm has many useful potential applications in the area of analysis and design of multirate and periodic systems. Some examples where the periodic KLF can play a key role are: computation of system zeros and Kronecker structure, computation of generalized periodic reachability/observability decompositions, finite-infinite additive decomposition, computation of generalized inverses of periodic systems [12], computation of left/right annihilators [14], solution of periodic model matching problems, design of fault detectors for periodic systems [14], etc.

The proposed algorithm appears to be equivalent with a recently proposed algorithm to compute a regularizing decomposition for cycles of linear mappings [7, Theorem 6.1]. When applied to the following quiver representation of the periodic pair (S_k, T_k)

$$V_1 \xrightarrow{S_1} W_1 \xleftarrow{T_1} V_2 \xrightarrow{S_2} W_2 \xleftarrow{T_2} \ldots \xleftarrow{T_{N-1}} V_N \xrightarrow{S_N} W_N \xleftarrow{T_N} V_1$$

where V_k and W_k are respectively, ν_k and μ_k dimensional vector spaces, the algorithm of [7] produces via appropriate basis changes, a decomposition of the underlying matrices as a direct sum of elementary canonical constituents. Although both algorithms can produce similar decompositions, establishing an exact equivalence between the structural information determined by the two algorithms is not straightforward. While the algorithm of [7] appears to be rather a conceptual procedure useful mainly for the classification theory of cycles of linear mappings, the algorithm proposed in this paper is a practical approach, directly implementable using existing linear algebra tools. Moreover, the structural information obtained with our approach has a strong system theory relevant interpretation, being useful in addressing several applications of periodic systems (see above), without the need to build the associated lifted representation.

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