RobSin: A New Tool for Robust Design of PID and Three-Term Controllers based on Singular Frequencies

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***Abstract*— This paper introduces a new Matlab Toolbox called RobSin, which for a given plant (multi-)model provides a fast computation of the stabilizing region in the coordinates of PID parameters. The algorithms use the concept of singular frequencies for computation of convex polygonal slices on planes $k_P = const$. In addition to time-continuous systems, the method described in this paper is especially convenient for the design of time-discrete PID and three-term controllers and time-delay systems.**

I. INTRODUCTION

It has been shown that stability regions of PID controllers with fixed proportional gain consist of convex polygons. Different approaches which prove this result are based on a generalization of the Hermite-Biehler theorem [5], calculation of the real-axis intersections of the Nyquist plot, [7] and singular frequencies [1], [2]. It turns out that for variations of two specific controller parameters (or linear combinations thereof) the eigenvalues can cross the imaginary axis (or a parallel to it) or circles with arbitrary radii and center on real axis only at singular frequencies, that are determined as the roots of a polynomial. At these singular frequencies the stability boundaries are straight lines that bound the stable convex polygons. Thereby the goal is to find simultaneous stabilizers for a finite set of given plants (multi-model uncertainty). It is shown, that the orientation of the polygon slices depends only on the Γ-stability region[1] and not on family plant members. Therefore it is easy to find the set of all simultaneous Γ-stabilizers by the intersection of convex polygons. Notice that the method applies for a wide range of multi-model uncertain systems including time-delay systems and discrete-time systems.

This paper presents the new tool RobSin (Robust Design based on Singular Frequencies), which basically includes design algorithms based on mapping of singular frequencies into controller parameter space for fast controller design or/and fine tuning. RobSin provides a Command Line, as well as a GUI (Graphical-User-Interface) working environment. The presentation of the features of the toolbox is associated with a short recapitulation of theoretical background. Thereby the proofs of theorems are avoided, but the interested reader is referred to the literature for further reading, [1], [2], [3], and [4]. The usability of the tool is illustrated by the robust control benchmark problem proposed in [8].

[1]The enclosing region of the system eigenvalues

II. THE SOFTWARE TOOL ROBSIN

A. Graphical-user-interface

The GUI of RobSin includes four interactive windows, [6]. The *main window* discriminates between the synthesis and analysis mode. The synthesis mode defines the control problem, including the multi-model, controller structure, and the $\Gamma-$region of specifications, whereas the analysis mode provides the interactive analysis with common methods such as eigenvalues, Bode-diagrams, Nyquist-plots, etc. (compare Fig. 2 and 3).

The multi-model plant may be defined in two ways: (a) implicitly by settings the min-max limits of parameter uncertainties (see Fig. 2), or (b) explicitly by entering the finite set of plants. The controller structure may be defined to be a PID or a three-term one. Depending on the specifications the $\Gamma-$region may be chosen to be (a) Hurwitz, (b) $\sigma < \sigma_o$ or (3) a circle with arbitrary radius and center on the real axis.

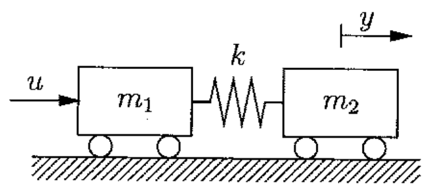

Fig. 1. The benchmark problem

Example: A Benchmark Problem: Consider the problem of robust stabilization of the fourth-order mechanical system depicted in Fig. 1, whereby the mass m_1 and the spring k are considered to be uncertain within the limits $m_1 \in [0.5, 1]$ and $k \in [0.5, 2]$ ($m_2 = 1 = const.$). For illustration purposes a multi-model containing five operating points will be investigated. Besides the four edges of the uncertainty domain in parameters m_1 and k, an additional operating point at $m_1 = 0.75$ and $k = 1$ is included, see Fig. 2.

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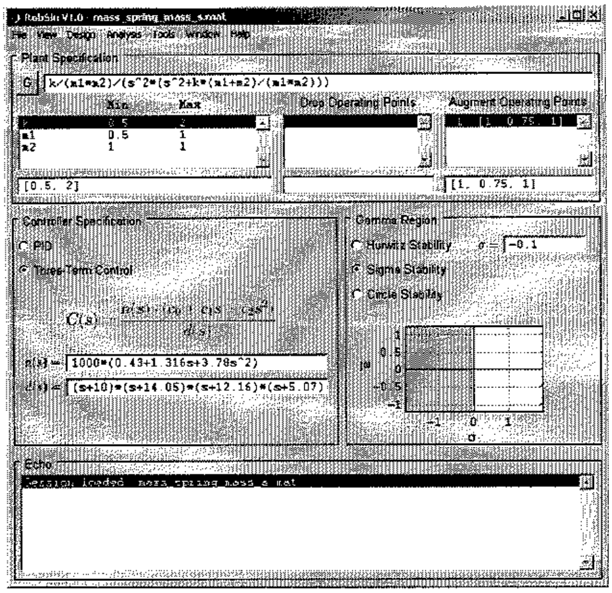

Fig. 2. The main RobSin window: synthesis mode

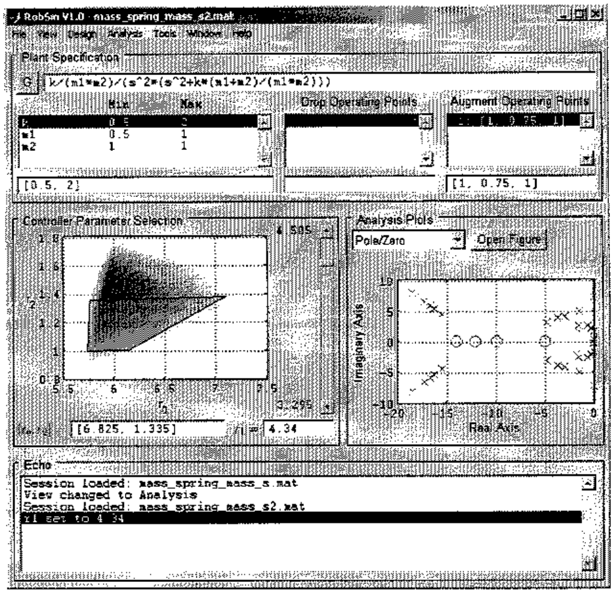

Fig. 3. The main RobSin window: analysis mode.

The transfer function from the input force u to the output position y is easily checked to be

$$G(s,k,m_1,m_2) = \frac{y(s)}{u(s)} = \frac{k/m_1m_2}{s^2\left(s^2 + k\frac{m_1+m_2}{m_1m_2}\right)}. \tag{1}$$

It can be shown that no PID controller can stabilize the system in (1). Therefore a three-term controller structure will be investigated, e.g.

$$C(s,c) = \frac{1000\,(0.43+1.316s+3.78s^2)\,(c_0+c_1s+c_2s^2)}{(s+10)(s+14.05)(s+12.16)(s+5.07)}. \tag{2}$$

Further, the requirements are set by the Γ–region, $\sigma < -0.1$, see Fig. 2. Now the task is to find the whole region in parameter space $c = [c_0, c_1, c_2]^{\mathrm{T}}$, which copes with the latter requirements.

B. Command-line framework

Equivalently, in the command-line framework, the problem may be defined by using the command

```
>> sq = singf(argin);
```

whereby the parameter `argin` represents a structure with two fields: (a) Γ-region and (b) the multi-model. `sq` represents an object of the class `singf`, with different properties and methods. Some of most important properties and methods will be addressed in the course of this article.

III. Singular Frequencies and Γ-regions

Consider a characteristic polynomial (c.p.) expressed in vectorial form at a complex frequency $s = \sigma + j\omega$,

$$P(s,r) \doteq \begin{bmatrix} R_P(\sigma,\omega,r) \\ I_P(\sigma,\omega,r) \end{bmatrix}. \tag{3}$$

whereby r is a vector of real parameters, which enter linearly in the coefficients of (3).

Definition 1: A frequency $s = s_o$ is said to be singular iff the following two conditions apply:

$$\text{Rank-Condition:}\quad \mathrm{Rank}(\partial P/\partial r)\big|_{s_o} = 1 \tag{4}$$

$$\text{Root-Condition:}\quad P(s_o, r^*) = 0. \tag{5}$$

Thereby r^* is a fixed point in r-parameter space. Notice that the rank-condition does not depend on parameters r.

Definition 2: A closed region in complex plane (so-called Γ*-region*) bounded by the contour $\partial\Gamma$ is said to be singular iff

$$\mathrm{Rank}(\partial P/\partial r) = 1 \qquad \text{for all } s \in \partial\Gamma. \tag{6}$$

An important result regarding robustness is the following theorem. Consider the family of c.p.

$$P_\nu(s,r) = A_\nu(s)\,Q(s,r) + B_\nu(s), \tag{7}$$

with

$$\nu = 1, 2, \cdots, N.$$

Theorem 1: The rank-condition (4) for the c.p. (7) in r-parameter space is set by the polynomial $Q(s,r)$ and does not depend on the polynomials $A_\nu(s)$ and $B_\nu(s)$.

Let $A_\nu(s)$ and $B_\nu(s)$ include the uncertainties of the control loop and r be the design parameters. The above theorem assures that uncertainties do not impact the singularity of a Γ*-region*. However, they certainly influence the distribution of singular frequencies over the $\partial\Gamma$ boundary.

IV. Hurwitz-, σ- and Circle Stability

Basically this paper considers a finite family of polynomials or/and quasipolynomials of the form

$$P_\nu(s) = A_\nu(s)Q(s) + B'_\nu(s, L_d), \tag{8}$$

whereby,

$$Q(s) = c_0 + c_1 s + c_2 s^2, \tag{9}$$

and

$$B'_v(s, L_d) = B_v(s) e^{L_{d,v}\ s}. \tag{10}$$

A_v und B_v are real polynomials of Laplace, s, or discrete, z, variable. L_d corresponds to the the dead-time of a continuous system, which in (8) may be zero or positive. (For time-discrete systems $L_d = 0$).

The method of singular frequencies is convenient for determination of all simultaneous stabilizers $c = [c_0, c_1, c_2]^{\mathrm{T}}$ of the family (8). The polynomial $Q(s)$ in (9) may correspond to a PID controller, or to a zero or pole of a three-term controller.

It is easy to show that the rank-condition (4) for the polynomial (8) applies on each point of the imaginary axis $j\omega$, i.e. Hurwitz Γ-region is singular. However, for σ-stability (all eigenvalues lie in the half-plane $\sigma \leq \sigma_o$) and circle stability (the eigenvalues are enclosed by a circle with arbitrary radius r and real center m) a linear transformation T from c- to r-parameter space is required,

$$c = Tr = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix}. \qquad \det T \neq 0 \tag{11}$$

It can be shown that for σ-stability,

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ -2\sigma_o t_{31} & t_{22} & -2\sigma_o t_{33} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}, \tag{12}$$

and for circle-stability,

$$T = \begin{bmatrix} -t_{21}m + t_{31}(r^2 - m^2) & t_{12} & -t_{23}m + t_{33}(r^2 - m^2) \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}, \tag{13}$$

whereby the arbitrary parameters t should be chosen such that $\det(T) \neq 0$.

Example (continued): It can be easily shown that c.p. of the problem presented in Section II belongs to the class of the polynomials (8) with $L_d = 0$. Recall that the design requirements are set by the Γ-region $\sigma < -0.1$. Therefore, according to (12) the transformation matrix is chosen to be,

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{14}$$

V. Generator of Singular Frequencies and Straight Lines

The following equations are used to parametrize a σ- and a circle singular Γ-region (i.e. their boundary $\partial\Gamma$):

$$s(\alpha) = \sigma^* + j\alpha, \qquad \alpha \in [0; \infty). \tag{15}$$

($\sigma^* = const$; for Hurwitz region $\sigma^* = 0$),

$$z(\alpha) = m + r \cdot e^{j\alpha}, \qquad \alpha \in [0; \pi]. \tag{16}$$

Substitution of this parametrization into the original c.p. in (3) yields,

$$\mathscr{P}(\alpha, r) = \alpha\ F(\alpha) \begin{bmatrix} r_0 \\ r_2 \end{bmatrix} + G(\alpha)\ r_1 + H(\alpha), \tag{17}$$

with

$$F(\alpha) = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \ G(\alpha) = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, \ H(\alpha) = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}. \tag{18}$$

Note further that due to rank-condition (4),

$$\det(F(\alpha)) = 0, \ \ \forall \alpha \in \partial\Gamma. \tag{19}$$

Thus the root-condition (5) may be decoupled into two conditions,

$$\begin{aligned} r_1 &= \frac{h_{11} f_{22} - h_{21} f_{12}}{f_{12} f_{21} - g_{11} f_{22}}, \\ f_{21} r_0 + f_{22} r_2 &= \frac{g_{11} f_{22} h_{21} - f_{21} h_{11} f_{22}}{\alpha (f_{12} f_{21} - g_{11} f22)}. \end{aligned} \tag{20}$$

The first equation will be referred to as the *generator of the singular frequencies* (GSF), since for a fixed $r_1 = r_1^*$ all solutions for α represent singular frequencies, see Fig. 4. On the other hand, such a solution $\alpha = \alpha_k$ defines via the second equation in (20) a straight line $\lambda(\alpha_k)$ in the (r_0, r_2)–plane, which will be termed as *singular line*. The second equation itself will be referred as the *generator of singular lines* (GSL).

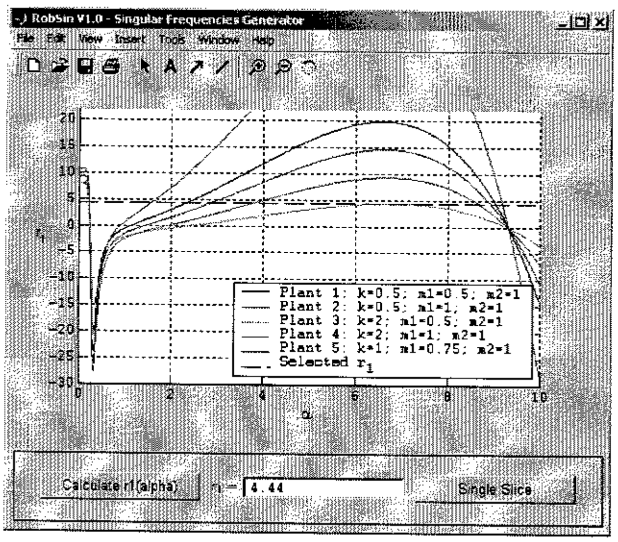

Fig. 4. The GSF for the benchmark example

If all singular frequencies α_k are mapped to a plane $r_1 = r_1^*$, a finite (i.e. infinite for quasipolynomials) set of singular lines $\lambda(\alpha_k)$ appear, which in turn define a finite (i.e. infinite) set of convex polygons. Because of the continuity such a polygon will be stable iff a single point within it is stable. Such a polygon will be called as a *stable polygon*.

Example (continued): In RobSin the singular frequencies for the class `sq`, are computed using the command `gensfs`. E.g. to compute the singular frequencies lying on

$\sigma = -0.1$ for the example from Section II for $r_1 = 0.5$ use the command

```
>> sq.gensfs(0.5);
```

To check the singular frequencies the property `SingularFrequencies` should be called. E.g. the singular frequencies of the object `sq` corresponding to the second operating point, i.e $k = 0.5$, $m_1 = 1$ and $m_2 = 1$, are

```
>> sq.SingularFrequencies{2};
ans =
          0
     0.2301
     1.5608
     6.4804
```

The RobSin command for the generation of the singular lines at the plane $r_1 = 0.5$ is

```
>> sq.gensls(0.5);
```

and finally, the generator of singular frequencies may be plotted by the command

```
>> sq.plotgsf;
```

VI. Inner Polygons

The concept of inner polygons is introduced in [1] for automatic detection of stable polygons. The essential advantage of this method is that it can be applied in general, including quasipolynomials.

Definition 3: A polygon is said to be an *inner polygon* if it is defined by a set of singular frequencies $\{s_o^i\}$ and lines $\{\lambda(s_o^i)\}$, such that any transition over $\{\lambda(s_o^i)\}$ inside the polygon causes an eigenvalue to enter the Γ–region at $\{s_o^i\}$.

It is clear that all stable polygons are inner polygons, i.e. this is just a necessary condition for stability of a polygon.

In order to detect an inner polygon each singular line $\lambda(s_o)$, will be assigned a "transition" function e: it is positive if the transition $[\delta r_0, \delta r_2]$ over the singular line causes an eigenvalue to enter the Γ–region and negative in other cases.

Define a normal vector $\vec{n}$ at a singular frequency s_o on $\partial\Gamma$ and let it point outside the Γ-region. Assume that $\partial\Gamma$ can be described by an implicit function $F(\sigma, \omega) = 0$ such that

$$\vec{n} := \operatorname{grad}(F)\big|_{s_o} = \begin{bmatrix} \partial F/\partial\sigma \\ \partial F/\partial\omega \end{bmatrix}_{s_o}. \tag{21}$$

The following two theorems are essential.

Theorem 2: For a transition $[\delta r_0, 0]$, resp. $[0, \delta r_2]$ over a singular line $\lambda(z_S)$ at a singular frequency s_o the transition function can be calculated by the formula

$$e_{0/2} := \left(\frac{\partial F}{\partial\sigma} \cdot \det\left[\frac{\partial(R_P, I_P)}{\partial(\omega, r_{0/2})}\right] + \frac{\partial F}{\partial\omega} \cdot \det\left[\frac{\partial(R_P, I_P)}{\partial(r_{0/2}, \sigma)}\right] \right)_{s_o}. \tag{22}$$

Theorem 3: The transition function $e_{0/2}$ does not depend on where the singular line $\lambda(s_o)$ is crossed at.

The algorithm for automatic detection of inner polygons is shown in the following. The polygon Π consists of N edges, each with a start point r^S and an end point r^E.

```
for i=1 to N
   if abs(r^S_{0,i}-r^E_{0,i}) > abs(r^S_{2,i}-r^E_{2,i}) then
      e_i = e_0
      Δ_i = r^E_{0,i} - r^S_{0,i}
   else
      e_i = e_2
      Δ_i = r^E_{2,i} - r^S_{2,i}
   end
end
if any(sign(e_i) ≠sign(Δ_i))
   Π is not inner polygon
else
   Π is an inner polygon
end
```

Once the set of all inner polygons for a fixed r_1 is detected, RobSin picks up those which posses the maximal number of Γ-stable eigenvalues. Thereby no eigenvalues are indeed calculated, but using the transition function, one can compute the relative number of Γ-stable eigenvalues w.r.t. a fixed polygon.

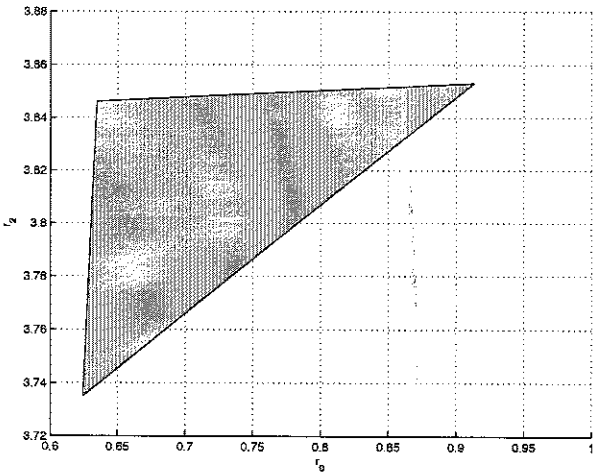

Fig. 5. Robust stable polygon for the uncertain mass-spring-mass system

Example (continued): The RobSin command for the detection of an inner polygon lying at the plane $r_1 = 0.5$ reads

```
>> sq.genslice(0.5);
```

The result is shown in Fig. 5.

VII. Stable Gridding Intervals

In all considerations up to now the parameter r_1 was assumed to be fixed. To complete the theory a rule is needed, which would discriminate r_1–intervals, such that stable PID controllers may exist therein. It has been tempted to extrapolate such a simple rule from the generator of singular frequencies, (20), since its minima and maxima define intervals in r_1 with different number of singular frequencies. Especially critical is the case of quasipolynomials ($L_d \neq 0$) with infinite such intervals. Recently, in the paper [4] two criteria have been proposed, which relate the number of singular frequencies with stability conditions. In the sequel the two criteria are presented without proofs.

Consider a polynomial of the form

$$P(s) = A(s)(r_0 + r_1 s + r_2 s^2) + B(s), \qquad (23)$$

where $A(s)$ has no zeros on the axis $j\omega$ and
N: order of the polynomial $P(s)$
M: order of the polynomial $A(s)$
P: unstable zeros of $A(s)$.

Theorem 4: If the polynomial $P(s)$ is Hurwitz-stable for a fixed $r_1 = r_1^\star$, then the number of singular frequencies, Z, on the interval $-\infty < \omega < +\infty$, corresponding to $r_1^\star$ is

$$Z \geq N - M + 2P, \qquad (24)$$

including the singular frequency at zero, $s = 0$, and if any at infinity, $s = \infty$.
Notice that a singular frequency exists always at $s = 0$, but not necessarily at $s = \infty$. Using this theorem the user can directly read from GSF-curve, 4, the r_1-interval(s) which should be gridded.

Further, consider a quasipolynomial with principal term,

$$P(s) = A(s)(r_0 + r_1 s + r_2 s^2) + B(s)e^{L_d s}, \qquad (25)$$

where the same definitions for N, M, P hold.

Theorem 5: If the quasipolynomial $P(s)$ is Hurwitz-stable for a fixed $r_1 = r_1^\star$, then a $k \in \mathbb{N}$ exists, such that for $l \geq k$, $l \in \mathbb{N}$, the number of singular frequencies, Z, on the interval $(-2l\pi + \delta)/L_d < \omega < (2l\pi + \delta)/L_d$, corresponding to $r_1^\star$ is

$$Z \geq 4l + N - M + 2P, \qquad (26)$$

including the singular frequency at zero, $s = 0$, whereby δ is chosen such that the principal term of $P(s)$ does not vanish.

However, the bounds which are defined by this theorem may be conservative if the so-called *null*-polygons appear, see [4], i.e. an inner-polygon disappears or emerges at a point, within an r_1-interval which satisfies conditions (24) or (26), see Fig. 6.

Example (continued): If the parameter r_1 is gridded within its stable interval, the three-dimensional region of simultaneous stabilizers is built. The corresponding RobSin command for the benchmark example treated in this paper is,

```
>> sq.genall(linspace(0.34, 0.69, 21));
```

The resulting region of simultaneous stabilizers after back-transformation into $c-$parameter space is shown in Fig.6.

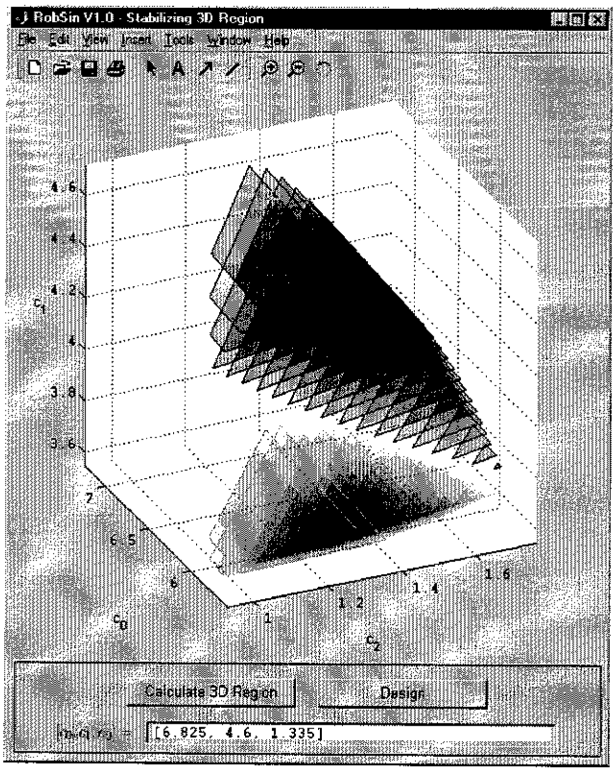

Fig. 6. Simultaneous stabilizers in the original $c-$parameter space for the benchmark example

VIII. SUMMARY

This paper introduces the Matlab Toolbox RobSin, a program for fast calculation of the three dimensional region of all robust stabilizing PID or three-term controller parameters. Its algorithm, based on the method of singular frequencies, uses the fact that 2D-slices of the stable region consist of polygons for Hurwitz-, σ- and circle-stability. Thus, an extremely fast calculation of 3-D stable regions in (k_P, k_I, k_D)-parameter space is feasible. In addition to time-continuous systems, the method described in this paper is especially convenient for the design of time-discrete PID and three-term controllers and time-delay systems. Furthermore, RobSin supports analysis of the calculated region with on-line visualization of the analysis plots. There are two diffenent ways of using RobSin: by command-line and by GUI. With the GUI the usage of RobSin is very comfortable and additional functionalities are accessible.

RobSin can be obtained for free in the internet at http://www.robotic.de/control/robsin.

REFERENCES

[1] BAJCINCA, N. *The method of singular frequencies for robust design in an affine parameter space* 9th Mediterranean Conference on Control and Automation, Dubrovnik, 2001.

[2] ACKERMANN, J.; KAESBAUER, D.; BAJCINCA, N. *Discrete-time robust PID and three-term control* XV IFAC World Congress, Barcelona, 2002.

[3] BAJCINCA, N.; KOEPPE, R.; ACKERMANN, J. *Design of robust stable master-slave systems with uncertain dynamics and time-delay* XV IFAC World Congress, Barcelona, July 2002.

[4] BAJCINCA, N: *Design of robust PID controllers based on singular frequencies* sub. in Automatica, 2004.

[5] DATTA,A. AND HO,M. T. AND BHATTACHARYYA,S. P.: *Structure and synthesis of PID controllers.* Springer–Verlag, London, 2000.

[6] HULIN, T.: *Software-Werkzeug zur Auslegung von robusten PID-Reglern.* Munich, Technical University of Munich, Institute of Automatic Control Engineering, diploma thesis, 2003.

[7] SOYLEMEZ, M. T., MUNRO, N. AND BAKI,H.: *Fast calculation of stabilizing PID controllers.* Automatica 39, pp. 121-126, 2003.

[8] WIE, B. ; BERNSTEIN, D. S.: *A benchmark problem for robust control design* San Diego, CA, Mai 1990. – American Control Conference, pp. 961–962