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Computation of transfer function matrices of periodic systems

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We present a numerical approach to evaluate the transfer function matrices of a periodic system corresponding to lifted state-space representations as constant systems. The proposed *pole-zero* method determines each entry of the transfer function matrix in a minimal *zeros-poles-gain* representation. A basic computation is the minimal realization of special single-input single-output periodic systems, for which both balancing-related as well as orthogonal periodic Kalman forms based algorithms can be employed. The main computational ingredient to compute poles is the extended periodic real Schur form of a periodic matrix. This form also underlies the solution of periodic Lyapunov equations when computing minimal realizations via balancing-related techniques. To compute zeros and gains, numerically stable fast algorithms are proposed, which are specially tailored to particular single-input single-output periodic systems. The new method relies exclusively on reliable numerical computations and is well suited for robust software implementations. Numerical examples computed with MATLAB-based implementations show the applicability of the proposed method to high-order periodic systems.

1. Introduction

Among the open computational problems listed in a recent survey (Varga and Van Dooren 2001), the computation of the transfer function matrix of a periodic system is one which has some useful applications. For example, the evaluation of frequency-response for a periodic system, can be conveniently done by using the transfer function matrix corresponding to a constant system lifted representation. Furthermore, for the manipulation of periodic systems, the state-space to transfer function matrix conversion is a useful transformation which must be provided by any software toolbox devoted to periodic systems.

In this paper we consider time-varying periodic systems of the form

$$\left.\begin{aligned} x(k+1) &= A_k x(k) + B_k u(k) \\ y(k) &= C_k x(k) + D_k u(k) \end{aligned}\right\} \qquad (1)$$

where the matrices $A_k \in \mathbb{R}^{n_{k+1} \times n_k}$, $B_k \in \mathbb{R}^{n_{k+1} \times m}$, $C_k \in \mathbb{R}^{p \times n_k}$, $D_k \in \mathbb{R}^{p \times m}$ are periodic with period $K \geq 1$. The importance of considering periodic systems with time-varying state dimensions has been revealed by the minimal realization theory of periodic systems (Colaneri and Longhi 1995, Gohberg *et al.* 1992). It is important to note that even for constant dimension periodic systems, the corresponding minimal order (i.e. reachable and observable) state-space realizations have, in general, time-varying state dimensions. Time-varying dimensions are also paramount importance in solving appropriate model reduction problems (Longhi and Orlando 1999, Varga 2000 a). Interestingly, periodic systems with time-varying dimensions have been already considered in earlier papers (Grasselli and Longhi 1991, Gohberg *et al.* 1992), but only recently the development of numerically reliable algorithms has been addressed. Among the first general algorithms for periodic systems with time-varying dimensions are those to compute minimal realizations (Varga 1999), to perform model reduction (Varga 2000 a) and to compute system zeros (Varga and Van Dooren 2002).

Most theoretical results for the analysis of periodic systems (Bittanti and Colaneri 1996) are based on two lifting techniques which allow to reformulate a problem for the time-varying periodic system (1) as an equivalent problem for a time-invariant discrete-time system of increased dimensions. The first lifting approach, proposed by Meyer and Burrus (1975), involves forming products of up to K matrices. Apart from being computationally expensive, the explicit computation of the matrices of the lifted system can lead to numerical difficulties for any algorithm performing on such a model. The second lifting approach appears in the works of several authors (Park and Verriest 1989, Flamm 1991, Grasselli and Longhi 1991) and leads to a large order standard system representation with sparse and highly structured matrices. Although these lifting techniques are useful for their theoretical insight, their sparsity and structure may not be suited for numerical computations. This is why, in parallel to the theoretical developments, numerical methods have been developed that try to exploit efficiently this structure.

In this paper we propose a numerically reliable approach to evaluate the transfer function matrix corresponding to the above-mentioned lifted state space representations. The proposed *pole-zero* method determines each entry of the transfer function matrix in a minimal *zeros-poles-gain* representation. For this

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purpose, a minimal periodic realization for each entry of the transfer-function matrix is necessary to be computed. The corresponding single-input single-output periodic system has a very particular form which can be exploited by devising specially tailored algorithms for all necessary computations. To compute minimal realizations both balancing-related as well as orthogonal periodic Kalman forms based algorithms can be employed. A basic numerical ingredient to compute minimal realizations via balancing-related techniques and also system poles is the *extended periodic real Schur form* of a product of rectangular matrices introduced by Varga (1999). This form represents a generalization of the *periodic real Schur form* (Bojanczyk *et al.* 1992, Hench and Laub 1994) of a product of square matrices. The algorithms to compute these forms are numerically stable and have a low computational complexity. To compute zeros and gains, numerically stable structure exploiting algorithms are developed.

The proposed method is numerically reliable, relying exclusively on using orthogonal and well-conditioned transformations. This is why, all computed quantities, like poles, zeros or gains can be considered as exact for a slightly perturbed original system. This guarantees a certain form of numerical stability for the overall computation. The new method is well suited for robust software implementations. Numerical examples computed with MATLAB-based implementations show the applicability of this method to high order periodic systems.

Notation: To simplify the presentation we use some special notation of periodic matrices. For a K-periodic matrix X_i we use alternatively the *script* notation

$$\mathcal{X}_k := \mathrm{diag}(X_k, X_{k+1}, \ldots, X_{k+K-1})$$

which associates the block-diagonal matrix $\mathcal{X}_k$ to the cyclic matrix sequence X_i, $i=k,\ldots,k+K-1$ starting at time moment k. We reserve the script notation $\mathcal{X}$ (i.e. without subscript) for $\mathcal{X}=\mathcal{X}_1$. The script notation is consistent with the standard matrix operations as for instance addition, multiplication, inversion as well as with several standard matrix decompositions (Cholesky, QR, SVD). We denote with $\sigma\mathcal{X}_k$ the K-cyclic shift

$$\sigma\mathcal{X}_k = \mathrm{diag}(X_{k+1}, \ldots, X_{k+K-1}, X_k)$$

of the cyclic sequence X_i, $i=k,\ldots,k+K-1$.

By using the script notation, the periodic system (1) will be alternatively denoted by the quadruple ($\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$) or ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$) if $k=1$. The transition matrix of the system (1) is defined by the $n_j \times n_i$ matrix $\Phi_A(j,i) = A_{j-1}A_{j-2}\cdots A_i$, where $\Phi_A(i,i) := I_{n_i}$. The state transition matrix over one period $\Phi_A(j+K, j) \in \mathbb{R}^{n_j \times n_j}$ is called the *monodromy matrix* of system (1) at time j and its eigenvalues are called *characteristic multipliers* at time j.

2. Transfer function matrices of periodic systems

To define the *transfer function matrix* (TFM) of the periodic system (1), we consider the time-invariant representations corresponding to the two associated lifted systems introduced in Meyer and Burrus (1975), Park and Verriest (1989) and Flamm (1991), respectively. The lifting technique of Meyer and Burrus (1975) uses the input–output behaviour of the system over time intervals of length K, rather than 1. For a given sampling time k, the corresponding mK-dimensional input and pK-dimensional output vectors are

$$u_k^L(h) = [u^{\mathrm{T}}(k+hK)\cdots u^{\mathrm{T}}(k+hK+K-1)]^{\mathrm{T}}$$
$$y_k^L(h) = [y^{\mathrm{T}}(k+hK)\cdots y^{\mathrm{T}}(k+hK+K-1)]^{\mathrm{T}}$$

and an n_k-dimensional state vector is defined as

$$x_k^L(h) := x(k+hK)$$

The lifted system has the form

$$\left.\begin{aligned} x_k^L(h+1) &= F_k^L x_k^L(h) + G_k^L u_k^L(h) \\ y_k^L(h) &= H_k^L x_k^L(h) + L_k^L u_k^L(h) \end{aligned}\right\} \tag{2}$$

where

$$F_k^L = \Phi_A(k+K, k)$$
$$G_k^L = [\Phi_A(k+K, k+1)B_k \quad \Phi_A(k+K, k+2)B_{k+1} \quad \cdots B_{k+K-1}]$$
$$H_k^L = \begin{bmatrix} C_k \\ C_{k+1}\Phi_A(k+1, k) \\ \vdots \\ C_{k+K-1}\Phi_A(k+K-1, k) \end{bmatrix}$$
$$L_k^L = \begin{bmatrix} D_k & 0 & \cdots & 0 \\ L_{k,2,1} & D_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{k,K,1} & L_{k,K,2} & \cdots & D_k \end{bmatrix}$$

with $L_{k,i,j} = C_{k+i-1}\Phi_A(k+i-1, k+j)B_{k+j-1}$, for $i=2,\ldots,K$, $j=1,2,\ldots,K-1$ and $i>j$.

The system (2) is called the *standard lifted system* at time k of the given K-periodic system (1). The associated TFM is

$$W_k^L(z) = H_k^L(zI_{n_k} - F_k^L)^{-1}G_k^L + L_k^L \tag{3}$$

and depends on the sampling time k. Obviously $W_{k+K}^L(z) = W_k^L(z)$ and the TFMs at two successive

values of k are related by the relation (Grasselli and Longhi 1988)

$$W_{k+1}^L(z) = \begin{bmatrix} 0 & I_{p(K-1)} \\ zI_p & 0 \end{bmatrix} W_k^L(z) \begin{bmatrix} 0 & z^{-1}I_m \\ I_{m(K-1)} & 0 \end{bmatrix}$$

Thus, computing a single TFM for an arbitrary k is sufficient to determine all TFMs at K successive time values.

The second lifted representation we consider is called the *cyclic lifted system* (Park and Verriest 1989, Flamm 1991). Consider the *cyclic shift* matrix

$$Z_k = \begin{bmatrix} 0 & \cdots & 0 & I_{n_{k+K-1}} \\ I_{n_k} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_{n_{k+K-2}} & 0 \end{bmatrix}$$

and define, similarly as done by Park and Verriest (1989) for constant dimensions, the *cyclic lifted system* at time k, with mK inputs and pK outputs, as the time-invariant system

$$\left.\begin{aligned} x_k^C(h+1) &= F_k^C x_k^C + G_k^C u_k^C(h) \\ y_k^C(h) &= H_k^C x_k^C(h) + L_k^C u_k^C(h) \end{aligned}\right\} \tag{4}$$

where

$$(F_k^C, G_k^C, H_k^C, L_k^C) = (Z_k\mathcal{A}_k, Z_k\mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k)$$

The state dimension of this system is $\mu = \sum_{i=1}^K n_i$ and its $pK \times mK$ TFM is

$$W_k^C(z) = H_k^C(zI_\mu - F_k^C)^{-1}G_k^C + L_k^C$$

The relationships between the TFMs of the two lifted systems is (Bittanti and Colaneri 1996)

$$W_k^C(z) = \Delta_p(z^{-1})W_k^L(z^K)\Delta_m(z) \tag{5}$$

where

$$\Delta_j(z) = \mathrm{diag}\{I_j, zI_j, \ldots, z^{K-1}I_j\}$$

Thus, the TFM of the *cyclic lifted system* can be easily determined from the TFM of the *standard lifted system*. Conversely, having the TFM $W_k^C(z)$, we compute $W_k^L(z^K)$ using the relation (5) and then replace z^K by z.

For the computation of the TFM, we can also use the so-called *stacked lifted representation* of Grasselli and Longhi (1991). This is a time-invariant descriptor system representation of the form

$$\left.\begin{aligned} E_k^S x_k^S(h+1) &= F_k^S x_k^S(h) + G_k^S u_k^L(h) \\ y_k^L(h) &= H_k^S x_k^S(h) + L_k^S u_k^L(h) \end{aligned}\right\} \tag{6}$$

where $G_k^S = Z_k\mathcal{B}_k$, $H_k^S = \mathcal{C}_k$, $L_k^S = \mathcal{D}_k$, and

$$F_k^S - \lambda E_k^S = \begin{bmatrix} -\lambda I_{n_k} & O & \cdots & O & A_{k+K-1} \\ A_k & -I_{n_{k+1}} & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & -I_{n_{k+K-2}} & O \\ O & O & \cdots & A_{k+K-2} & -I_{n_{k+K-1}} \end{bmatrix} \tag{7}$$

The TFM of the stacked lifted system is

$$W_k^S(z) = H_k^S(zE_k^S - F_k^S)^{-1}G_k^S + L_k^S$$

and it is easy to show that $W_k^S(z) = W_k^L(z)$, that is, the TFMs of the *stacked* and *standard lifted systems* are the same.

3. Computational approach

In this section we propose an efficient computational approach to determine $W(z) := W_1^L(z)$, the TFM at time $k=1$ of the *standard lifted system* (2). The TFMs for other time moments $k = 2, \ldots, K$ or for the *cycled lifted system* (4) can be easily obtained by simple algebraic manipulations via the relation (5). Before starting our developments, we discuss shortly possible approaches relying on existing algorithms for standard systems.

A straightforward approach to compute the $pK \times mK$ TFM $W(z)$ is to apply to the *standard lifted system* (2) the *pole-zero* method of Varga and Sima (1981) or the *characteristic polynomial* based method of Misra and Patel (1987). This amounts of either computing successively the poles, zeros and gains or the numerator and denominator polynomials corresponding to minimal realizations of the individual input–output channels. However, because the construction of the *standard lifted system* involves matrix multiplications, this approach is certainly not recommendable for numerical computations. To avoid matrix multiplications, we can employ the same approach to the *cyclic lifted system* and compute the TFM of the *standard lifted system* via relation (5). Alternatively, we can compute the TFM of the *stacked lifted system*, which is the same as that of the *standard lifted system*. In this case we can apply a similar *pole-zero* approach but for descriptor systems (Varga 1989). In both cases, the required computational effort and computer storage can be prohibitive for large dimensions or large periods.

The following conceptual procedure serves as the basis to describe the proposed approach to compute an element $w_{ij}(z)$ of $W(z)$ in the *zeros-poles-gain* form

$$w_{ij}(z) = \gamma \frac{\prod_{\iota=1}^{q_z}(z-\mu_\iota)}{\prod_{\iota=1}^{q_f}(z-\rho_\iota)}$$

starting from the *stacked lifted system* (6).

Pole-zero algorithm:

1. Compute a minimal realization $(\hat{A}-z\hat{E},\hat{b},\hat{c},\hat{d})$ of the subsystem corresponding to the (j,i)th input–output channel of the stacked lifted system $(F_1^S - zE_1^S, G_1^S, H_1^S, L_1^S)$.
2. Compute the q_f finite poles $\rho_\iota, \iota = 1,\ldots,q_f$ as the finite generalized eigenvalues of the pair $(\hat{A},\hat{E})$.
3. Compute the q_z finite zeros $\mu_\iota, \iota = 1,\ldots,q_z$ of the descriptor system $(\hat{A}-z\hat{E},\hat{b},\hat{c},\hat{d})$.
4. Choose a real z_0 satisfying $|z_0|>1$, that is neither a pole nor zero and compute the gain

$$\gamma = \left(\hat{c}(z_0\hat{E}-\hat{A})^{-1}\hat{b}+\hat{d}\right)\prod_{\iota=1}^{q_f}(z_0-\rho_\iota)\Big/\prod_{\iota=1}^{q_z}(z_0-\mu_\iota)$$

The basis of this conceptual procedure is the method proposed by Varga (1989) for descriptor systems. This procedure is numerically reliable, since each step can be performed using numerically reliable algorithms. However, because ignoring the structure of the problem, the computational complexity of this approach is too high. To compute a single element of $W(z)$, the computational complexity is, in the worst case, of order $O(\mu^3)$, where $\mu=\sum_{i=1}^K n_i$. For example, in the case of a periodic system with constant state dimensions $n_i=n$, the computational complexity is $O(K^3n^3)$ instead of a desirable complexity of $O(Kn^3)$ as formulated by Varga and Van Dooren (2001) for a satisfactory algorithm for periodic systems. In what follows, we show that such a computational complexity can indeed be achieved by cleverly exploiting the problem structure. For this purpose, we discuss each step of the above algorithm and indicate how the underlying computations can be efficiently done.

3.1. *Computational ingredients*

The use of condensed forms of the system matrices, obtained under orthogonal transformations, is a basic ingredient for solving many computational problems for periodic systems (Varga and Van Dooren 2001). Such condensed forms can be obtained by using a Lyapunov transformation with an orthogonal transformation matrix $\mathcal{Z}$

$$(\tilde{\mathcal{A}},\tilde{\mathcal{B}},\tilde{\mathcal{C}},\tilde{\mathcal{D}}) = (\sigma\mathcal{Z}^{\mathrm{T}}\mathcal{A}\mathcal{Z},\sigma\mathcal{Z}^{\mathrm{T}}\mathcal{B},\mathcal{C}\mathcal{Z},\mathcal{D}) \tag{8}$$

It is well-known that the transformed periodic system $(\tilde{\mathcal{A}},\tilde{\mathcal{B}},\tilde{\mathcal{C}},\tilde{\mathcal{D}})$ has the same TFM as the original system $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$. The main usage of transforming the system matrices to a particular coordinate system in which they are condensed, is that the solutions of many computational problems often become straightforward. For an efficient algorithm to evaluate the TFM of a periodic system two categories of condensed forms will play a central role: the (extended) periodic real Schur form and the periodic reachability/observability Kalman form. We discuss each of these forms and indicate efficient and numerically stable algorithms for their computations below.

3.1.1. *Periodic real Schur form.* For periodic systems with constant dimensions, the *periodic real Schur form* (PRSF) is the basic numerical ingredient to solve periodic Lyapunov equations or to compute the poles of periodic system. According to Bojanczyk *et al.* (1992), given the matrices $A_k\in\mathbb{R}^{n\times n}$, $k=1,\ldots,K$, there exist orthogonal matrices Z_k, $k=1,\ldots,K$, $Z_{K+1}:=Z_1$, such that

$$\tilde{A}_k := Z_{k+1}^{\mathrm{T}}A_kZ_k \tag{9}$$

where $\tilde{A}_K$ is in *real Schur form* (RSF) and the matrices $\tilde{A}_k$ for $k=1,\ldots,K-1$ are upper triangular. Numerically stable algorithms to compute the PRSF have been proposed in Bojanczyk *et al.* (1992) and Hench and Laub (1994). By using these algorithms, we can determine the orthogonal matrices Z_k, $k=1,\ldots,K$ to reduce the cyclic product $A_K\cdots A_2A_1$ to the RSF without forming explicitly this product. For A_k in a PRSF, the computation of the eigenvalues of $A_K\cdots A_2A_1$ becomes very straightforward. The computation of the PRSF is numerically stable and has a computational complexity $O(Kn^3)$.

A main application of the PRSF is the computation of the poles of the TFM $W_k(z)$. For a minimal periodic system, the poles are defined as the eigenvalues of the monodromy matrix $\Phi_A(k+K,k)=A_{k+K-1}\cdots A_{k+1}A_k$. The poles are independent of k and can be easily computed by reducing the K-periodic matrix A_k to a PRSF. Moreover, in the case of a square system (i.e. with same number of inputs and outputs) with an invertible $\mathcal{D}_k$, the zeros of $W_k(z)$ can be determined as the poles of the inverse periodic system $(\mathcal{A}_k-\mathcal{B}_k\mathcal{D}_k^{-1}\mathcal{C}_k,\ -\mathcal{B}_k\mathcal{D}_k^{-1},\mathcal{D}_k^{-1}\mathcal{C}_k,\mathcal{D}_k^{-1})$, thus as the eigenvalues of $\Phi_{A-BD^{-1}C}(k+K,k)$.

For systems with time-varying dimensions, the *extended periodic real Schur form* (EPRSF) represents a generalization of the PRSF which allows to address many problems with varying dimensions. According to Varga (1999), given the matrices $A_k\in\mathbb{R}^{n_{k+1}\times n_k}$, $k=1,\ldots,K$, with $n_{K+1}=n_1$ there exist orthogonal matrices $Z_k\in\mathbb{R}^{n_k\times n_k}$, $k=1,\ldots,K$, $Z_{K+1}:=Z_1$, such that the matrices

$$\tilde{A}_k := A_{k+1}^{\mathrm{T}}A_kZ_k = \begin{bmatrix}\tilde{A}_{k,11} & \tilde{A}_{k,12}\\ 0 & \tilde{A}_{k,22}\end{bmatrix} \tag{10}$$

are block upper triangular, where $\tilde{A}_{k,11}\in\mathbb{R}^{\underline{n}\times\underline{n}}$, $\tilde{A}_{k,22}\in\mathbb{R}^{(n_{k+1}-\underline{n})\times(n_k-\underline{n})}$ for $k=1,\ldots,K$ and $\underline{n}=\min_k\{n_k\}$. Moreover, $\tilde{A}_{K,11}$ is in RSF, $\tilde{A}_{k,11}$ for $k=1,\ldots,K-1$ are upper triangular and $\tilde{A}_{k,22}$ for $k=1,\ldots,K$ are

upper trapezoidal. The computation of the EPRSF is numerically stable and has a computational complexity $O(Kn^3)$, where $n = \max\{n_i\}$.

The poles of a minimal order periodic system with time-varying dimensions are, as before, the eigenvalues of $\Phi_A(k+K,k)$ and can be computed using the EPRSF of A_k. However, for time-varying dimensions, the poles set depends on the sampling time, and for a given k, it is formed from a so-called *core* set, representing the eigenvalues of $\Phi_{A_{11}}(k+K,k)$, and $n_k - \underline{n}$ null poles. Note that the core poles are independent of k. For a square-system with invertible $\mathcal{D}_k$, the zeros can be computed as before, being the poles of $(\mathcal{A}_k - \mathcal{B}_k\mathcal{D}_k^{-1}\mathcal{C}_k, -\mathcal{B}_k\mathcal{D}_k^{-1}, \mathcal{D}_k^{-1}\mathcal{C}_k, \mathcal{D}_k^{-1})$. In this case, the system zeros for a given sampling time k are formed from a *core* set of n_k zeros and $n_k - \underline{n}$ null zeros.

3.1.2. *Periodic Kalman reachability and observability forms*. The *periodic Kalman reachability form* (PKRF) and the *periodic Kalman observability form* (PKOF) have been introduced by Grasselli (1984) for periodic systems with constant-dimensions and recently extended by Helmke and Verriest (2003) to systems with time-varying state, input and output vector dimensions. It was recently shown by Varga (2003), that this form can be computed using an orthogonal Lyapunov transformation. Specifically, every K-periodic system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is orthogonally Lyapunov equivalent to a K-periodic system $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ in the PKRF

$$\tilde{A}_k = \begin{bmatrix} A_k^r & * \\ 0 & A_k^{\bar{r}} \end{bmatrix}, \qquad \tilde{B}_k = \begin{bmatrix} B_k^r \\ 0 \end{bmatrix}, \qquad \tilde{C}_k = \begin{bmatrix} C_k^r & C_k^{\bar{r}} \end{bmatrix} \tag{11}$$

where $A_k^r \in \mathbb{R}^{r_{k+1} \times r_k}$, $B_k^r \in \mathbb{R}^{r_{k+1} \times m}$, $C_k^r \in \mathbb{R}^{p \times r_k}$ and the subsystem $(\mathcal{A}^r, \mathcal{B}^r, \mathcal{C}^r, \mathcal{D})$ is completely reachable. The TFMs of the corresponding standard lifted-representations of this subsystem and of the original system (1) are the same (Colaneri and Longhi 1995). The unreachable characteristic multipliers of the system (1) are the eigenvalues of $\Phi_{A^{\bar{r}}}(K+1,1)$.

By duality, the PKOF can be computed using an appropriate orthogonal Lyapunov transformation. In this form the transformed system matrices have the form

$$\tilde{A}_k = \begin{bmatrix} A_k^o & 0 \\ * & A_k^{\bar{o}} \end{bmatrix}, \qquad \tilde{B}_k = \begin{bmatrix} B_k^o \\ B_k^{\bar{o}} \end{bmatrix}, \qquad \tilde{C}_k = \begin{bmatrix} C_k^o & 0 \end{bmatrix} \tag{12}$$

where $A_k^o \in \mathbb{R}^{q_{k+1} \times q_k}$, $B_k^o \in \mathbb{R}^{q_{k+1} \times m}$, $C_k^o \in \mathbb{R}^{p \times q_k}$ and the subsystem $(\mathcal{A}^o, \mathcal{B}^o, \mathcal{C}^o, \mathcal{D})$ is completely observable. The TFMs of the standard lifted-representations corresponding to this subsystem and to the original system (1) are the same (Colaneri and Longhi 1995). The unobservable characteristic multipliers of the system (1) are the eigenvalues of $\Phi_{A^{\bar{o}}}(K+1,1)$.

The computation of reachability/observability Kalman decompositions by using orthogonal transformations has been listed among the open computational problems of periodic systems in the recent survey of Varga and Van Dooren (2001). The recently proposed algorithms by Varga (2003) to compute these forms are strongly numerically stable (i.e. the computed condensed forms are exact for an original system with slightly perturbed matrices) and have a worst-case computational complexity of $O(Kn^3)$. The main applications of these algorithms is the computation of minimal realizations of periodic systems.

3.2. *Computation of minimal realizations*

The minimal realization problem at Step 1 of the *Pole-Zero Algorithm* has a very particular structure. To simplify notation, we denote by $(F - zE, G, H, L)$ the stacked lifted system $(F_1^S - zE_1^S, G_1^S, H_1^S, L_1^S)$ (at time $k=1$) and let G_j and H_i be the jth-column of G and the ith-row of H, respectively, and let L_{ij} be the element (i,j) of L. We want to compute a minimal realization of the system $(F - zE, G_j, H_i, L_{ij})$. The matrices G_j and H_i have a very special structure. Since i and j can be uniquely expressed as

$$i = (\ell_i - 1)p + t, \qquad j = (\ell_j - 1)m + s$$

it follows that H_i can be constructed only from $c_{\ell_i,t}$, the row t of C_{ℓ_i}, and G_j can be constructed only from $b_{\ell_j,s}$, the column s of B_{ℓ_j}, as

$$H_i = [\underbrace{0 \cdots 0}_{\sum_{k=1}^{\ell_i-1} n_k} \quad c_{\ell_i,t} \quad \underbrace{0 \cdots 0}_{\sum_{k=\ell_i+1}^{K} n_k}] \tag{13}$$

$$G_j = [\underbrace{0 \cdots 0}_{n_K + \sum_{k=1}^{\ell_j-1} n_k} \quad b_{\ell_j,s}^{\mathrm{T}} \quad \underbrace{0 \cdots 0}_{\sum_{k=\ell_j+1}^{K-1} n_k}]^{\mathrm{T}} \tag{14}$$

Further, L_{ij} is defined as

$$L_{ij} = \begin{cases} d_{\ell_i,ts}, & \text{if } \ell_i = \ell_j \\ 0, & \text{if } \ell_i \neq \ell_j \end{cases}$$

where $d_{\ell_i,ts}$ is the element (t,s) of D_{ℓ_i}. The single-input single-output system $(F - zE, G_i, H_j, L_{ij})$ can be interpreted as a *stacked lifted system* corresponding to a particular single-input single-output periodic system $(\mathcal{A}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}})$, with time-varying dimensions m_k and p_k of the input and output vectors, respectively. For this periodic system, all input vector dimensions are zero excepting $m_s = 1$ and all output vector dimensions are zero excepting $p_t = 1$. Alternatively, we can consider a periodic system with constant input and output vector dimension $(m_k = p_k = 1)$ but with zero $\tilde{B}_k$ for $k \neq s$ and zero $\tilde{C}_k$ for $k \neq t$ and appropriate $\tilde{D}_k$. In both cases, we can now apply minimal realization procedures to $(\mathcal{A}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ to obtain the minimal order periodic system

$(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}})$ with the corresponding stacked minimal realization $(\hat{A} - \lambda \hat{E}, \hat{b}, \hat{c}, \hat{d})$, at Step 1 of the *Pole-Zero Algorithm*. In what follows, two classes of minimal realization approaches are described, each of them having some advantages when applied to the above problem. Both approaches are able to address the computation of minimal realizations for periodic systems with time-varying state, input and output vector dimensions.

3.2.1. *Balancing-related minimal realization*. We shortly present the main steps of a balancing-related minimal realization approach for stable periodic systems (Varga 1999) applied to the periodic system $(\mathcal{A}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ defined in the previous section. The minimal realization procedure involves the computation of two truncation matrices $\mathcal{L}$ and $\mathcal{T}$ such that the matrices of the minimal order system $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}})$ are obtained as

$$\hat{\mathcal{A}} = \sigma \mathcal{L} \mathcal{A} \mathcal{T}, \quad \hat{\mathcal{B}} = \sigma \mathcal{L} \tilde{\mathcal{B}}, \quad \hat{\mathcal{C}} = \tilde{\mathcal{C}} \mathcal{T}, \quad \hat{\mathcal{D}} = \tilde{\mathcal{D}} \tag{15}$$

The computation of truncation matrices relies on the reachability grammian $\mathcal{P}$ and observability grammian $\mathcal{Q}$, which satisfy the periodic Lyapunov equations

$$\left.\begin{aligned} \sigma \mathcal{P} &= \mathcal{A} \mathcal{P} \mathcal{A}^{\mathrm{T}} + \tilde{\mathcal{B}} \tilde{\mathcal{B}}^{\mathrm{T}} \\ \mathcal{Q} &= \mathcal{A}^{\mathrm{T}} \sigma \mathcal{Q} \mathcal{A} + \tilde{\mathcal{C}}^{\mathrm{T}} \tilde{\mathcal{C}} \end{aligned}\right\} \tag{16}$$

Since these grammians are non-negative definite, they can be expressed in Cholesky factorized forms $\mathcal{P} = \mathcal{S}\mathcal{S}^{\mathrm{T}}$ and $\mathcal{Q} = \mathcal{R}^{\mathrm{T}}\mathcal{R}$, where S_k and R_k for $k = 1, \ldots, K$ are $n_k \times n_k$ upper triangular matrices.

For each k, consider the singular value decomposition of $R_k S_k$ in the partitioned form

$$R_k S_k = \begin{bmatrix} U_{k,1} & U_{k,2} \end{bmatrix} \begin{bmatrix} \tilde{\Sigma}_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{k,1} & V_{k,2} \end{bmatrix}^{\mathrm{T}} \tag{17}$$

where $\tilde{\Sigma}_k \in \mathbb{R}^{r_k \times r_k}$, $U_{k,1} \in \mathbb{R}^{n_k \times r_k}$, $V_{k,1} \in \mathbb{R}^{n_k \times r_k}$ and $\tilde{\Sigma}_k > 0$. From the above decomposition define, with $\tilde{\Sigma} = \mathrm{diag}(\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_K)$, the *truncation* matrices

$$\mathcal{L} = \tilde{\Sigma}^{-1/2} \mathcal{U}_1^{\mathrm{T}} \mathcal{R}, \qquad \mathcal{T} = \mathcal{S} \mathcal{V}_1 \tilde{\Sigma}^{-1/2} \tag{18}$$

which are used to determine the reduced system matrices is (15). In the corresponding stacked lifted system $(\hat{A} - \lambda \hat{E}, \hat{b}, \hat{c}, \hat{d})$ resulting at Step 1 of the *Pole-Zero Algorithm*, both $\hat{c}$ and $\hat{b}$ have entirely similar structures with H_i in (13) and G_j in (14), having respectively, $\sum_{k=1}^{\ell_i - 1} r_k$ and $r_K + \sum_{k=1}^{\ell_j - 1} r_k$ zeros in their leading positions, and the appropriate number of zeros in the trailing positions.

The key computation in determining $\mathcal{L}$ and $\mathcal{T}$ is the solution of the two periodic Lyapunov equations in (16) with time-varying dimensions directly for the Cholesky factors of the grammians. A numerically reliable procedure for this computation has been proposed by Varga (1999). Since the computation of truncation matrices (and also of the minimal realization) can be done using only the Cholesky factors (square-roots) of Gramians, the above method is called the *square-root* method. This method leads to a guaranteed enhancement of the overall numerical accuracy of computations.

The resulting minimal system $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}})$ is *balanced*, because the corresponding grammians are equal and diagonal. However, to obtain a minimal realization from a non-minimal one we do not actually need to obtain a minimal realization which is balanced, since this could involve ill-conditioned (i.e. nearly rank deficient) truncation matrices $\mathcal{L}$ and $\mathcal{T}$, if the original system is poorly scaled. Since $\mathcal{L}\mathcal{T} = I$, the pair $(\mathcal{L}, \mathcal{T})$ defines a projector $\mathcal{T}\mathcal{L}$, in analogy to the case of standard systems. For any invertible $\mathcal{W}$, the pair of truncation matrices $(\mathcal{W}^{-1}\mathcal{L}, \mathcal{T}\mathcal{W})$ defines the same projected system, but in a different coordinate form. Thus, to avoid potential accuracy losses, an alternative to balancing is to combine the *square-root* technique with a *balancing-free* approach to compute truncation matrices.

In the *balancing-free square-root* method of Varga (1999), we compute additionally two QR decompositions

$$\mathcal{S} \mathcal{V}_1 = \mathcal{T} \mathcal{X}, \qquad \mathcal{R}^{\mathrm{T}} \mathcal{U}_1 = \mathcal{Z} \mathcal{Y} \tag{19}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are non-singular matrices and $\mathcal{T}$ and $\mathcal{Z}$ are matrices with orthonormal columns. With the already computed $\mathcal{T}$ we define the corresponding $\mathcal{L}$ as

$$\mathcal{L} = (\mathcal{Z}^{\mathrm{T}} \mathcal{T})^{-1} \mathcal{Z}^{\mathrm{T}} \tag{20}$$

The minimal realization $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \mathcal{D})$ is obtained as in (15). For the purpose of the *Pole-Zero Algorithm*, this approach is that one to be preferred.

The computational complexity of the approach of Varga (1999) to determine a minimal realization is $O(Kn^3)$. The most time-consuming operation in this process is the solution of the two periodic Lyapunov equations satisfied by the grammians. When employing the procedure of Varga (1999) for this purpose, the first step of the solution method is the reduction of the periodic matrix A_k to an EPRSF. Then, the Cholesky factors of the grammians are computed directly by solving reduced periodic Lyapunov equations (i.e. with A_k in EPRSF). Since for each element of the $pK \times mK$ TFM the minimal realization problem involves the same periodic state matrix, the reduction to EPRSF has to be done only once to put the original periodic system in a coordinate form with the state matrix in EPRSF. This is achieved by applying a Lyapunov similarity transformation as in (8) with the appropriate periodic orthogonal transformation matrix $\mathcal{Z}$.

The minimal realization method based on balancing technique is not restricted to asymptotically stable periodic systems. For an unstable system, a simple scaling can be used to enforce the stability of the starting representation. For instance, it is possible to replace

only A_1 by αA_1, where $0<\alpha<1$ is chosen such that $\alpha\Phi_A(K+1,1)$ has eigenvalues in the open unit disc. For the α-scaled system, we can apply either the *square-root* or *balancing-free square-root approach* to determine a minimal system. Finally, the computed $\hat{A}_1$ needs to be rescaled to $\hat{A}_1/\alpha$.

The balancing-related approach relies on the computation of the EPRSF of A_k, and involves the solution of two non-negative definite periodic Lyapunov equations. This algorithm is numerically reliable, since each computational step relies on strongly backward stable algorithms. The main advantage of this algorithm is that the K rank decisions necessary to obtain the state-vector dimensions of a minimal realization are performed only once at the end of the algorithm. Thus, this approach is very reliable in determining the orders of the minimal realizations.

3.2.2. *Kalman forms based minimal realization.* The orthogonal reduction of the periodic triple $(\mathcal{A},\tilde{\mathcal{B}},\tilde{\mathcal{C}})$ to the periodic Kalman structural forms, PKRF and PKOF, can be used to devise a strongly numerically stable periodic minimal realization algorithm with a lower computational cost than the balancing-related approach. The procedure to compute minimal realizations is conceptually straightforward, and involves the successive elimination of the unobservable and unreachable parts of the system. A two steps procedure is formalized below:

Minimal realization procedure:

1. Compute the PKOF of the periodic triple $(\mathcal{A},\tilde{\mathcal{B}},\tilde{\mathcal{C}})$ to obtain a completely observable realization $(\mathcal{A}^o,\mathcal{B}^o,\mathcal{C}^o)$.
2. Compute the PKRF of the observable periodic triple $(\mathcal{A}^o,\mathcal{B}^o,\mathcal{C}^o)$ to obtain the minimal realization as the completely reachable part $(\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{\mathcal{C}})$.

This algorithm is strongly backward stable since it can be shown that the computed minimal realization is exact for an original system with slightly perturbed matrices (Varga 2003). For a system with constant state dimension n, the above procedure performs in the worst-case (i.e. the system is reachable and observable) about $\frac{10}{3}Kn^3$ operations. Thus, the actual number of necessary operations to determine a minimal realization for a single entry of the TFM is usually much less than for the balancing-related approach. Additional gain can be achieved if the first step is performed only once for each of Kp rows of H. This can be achieved by performing the first step on the triple $(\mathcal{A},\mathcal{B},\tilde{\mathcal{C}})$, where $\tilde{\mathcal{C}}$ is the periodic matrix constructed for H_j, the jth column of H. Comparing computational efforts of the two approaches, it is expected that for system with a large number of entries or large periods, the difference between the two procedures to become less significant.

Remark: At the end of the minimal realization procedure, the resulting system $(\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{\mathcal{C}},\hat{\mathcal{D}})$ is in a particular PKRF. The periodic matrix $\hat{A}_k$ is in an *extended periodic Hessenberg form* (EPHF), where all $\hat{A}_k$ are upper trapezoidal or upper triangular excepting, say $\hat{A}_s$, whose leading square block is upper Hessenberg. The index s corresponds to the only non-empty column matrix $\hat{B}_s$ resulting from G_j of the form (14), where the column matrix $\hat{B}_s$ has all elements zero excepting the first one. Furthermore, the only non-empty row matrix $\hat{C}_t$ results from H_i of the form (13). The only matrix $\hat{D}_k$ with non-empty rows and columns corresponds to the case when $k=s=t$.

3.3. *Computation of poles and zeros*

To compute the poles of the periodic system $(\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{\mathcal{C}},\hat{\mathcal{D}})$, the eigenvalues of the *monodromy* matrix $\Phi_{\hat{A}}(K+1,1)$ must be determined. This computation can be done without forming this matrix product explicitly, by reducing the K-periodic matrix $\hat{A}_k$ to the PRSF in the case of constant dimensions, or to the EPRSF in the case of time-varying dimensions. Note that when employing the periodic Kalman forms based minimal realization approach, the resulting periodic pair $(\hat{\mathcal{A}},\hat{\mathcal{B}})$ is in the PKRF. In this condensed form, the periodic matrix $\hat{A}_k$ is already in an EPHF, which represents the starting form to compute the EPRSF (Varga 1999). Therefore, this first reduction step is not anymore necessary to be performed to compute the system poles.

The zeros of the descriptor system $(\hat{A}-z\hat{E},\hat{b},\hat{c},\hat{d})$ representing the stacked lifted representation of $(\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{\mathcal{C}},\hat{\mathcal{D}})$ are defined as the Smith zeros of the structured system pencil

$$\hat{S}(z)=\begin{bmatrix}\hat{A}-z\hat{E} & \hat{b}\\ \hat{c} & \hat{d}\end{bmatrix} \tag{21}$$

To compute system zeros, it is possible to exploit the structure of this pencil by eliminating most of its simple eigenvalues at infinity. We present here only the main idea of the procedure to compute the finite zeros. A more general algorithm which allows to compute the infinite zeros and the Kronecker structure has been recently developed by Varga and Van Dooren (2002).

Consider $\check{S}(z)$, a system pencil with permuted block rows and columns

$$\check{S}(z)=\begin{bmatrix}\hat{S}_1 & -\hat{T}_1 & O & \cdots & O\\ O & \hat{S}_2 & -\hat{T}_2 & \cdots & O\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ O & & & \hat{S}_{K-1} & -\hat{T}_{K-1}\\ -z\hat{T}_K & O & \cdots & O & \hat{S}_k\end{bmatrix} \tag{22}$$

with

$$\hat{S}_i = \begin{bmatrix} \hat{A}_i & \hat{B}_i \\ \hat{C}_i & \hat{D}_i \end{bmatrix}, \qquad \hat{T}_i = \begin{bmatrix} I_{r_{i+1}} & O \\ O & O \end{bmatrix}$$

for $i = 1, \ldots, K$. Here r_i, $i = 1, \ldots, K$ are the time-varying dimensions of the state vector of the minimal order periodic system $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}})$.

Consider the orthogonal transformation matrix U_1 to compress the matrix

$$\begin{bmatrix} -\hat{T}_1 \\ \hat{S}_2 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} R_1 \\ O \end{bmatrix}$$

where R_1 is a full row rank matrix. Applying U_1^{T} to the first two blocks rows of $\breve{S}(z)$ we obtain for the non-zero blocks

$$U_1^{\mathrm{T}} \begin{bmatrix} \hat{S}_1 & -\hat{T}_1 & O \\ O & \hat{S}_2 & -\hat{T}_2 \end{bmatrix} = \begin{bmatrix} * & R_1 & * \\ \bar{S}_2 & O & -\bar{T}_2 \end{bmatrix}$$

which defines the new matrices $\bar{S}_2$ and $\bar{T}_2$. Then, construct the transformations U_i for $i = 2, \ldots, K-1$ such that

$$U_i^{\mathrm{T}} \begin{bmatrix} \bar{S}_i & -\bar{T}_i & O \\ O & \hat{S}_{i+1} & -\hat{T}_{i+1} \end{bmatrix} = \begin{bmatrix} * & R_i & * \\ \bar{S}_{i+1} & O & -\bar{T}_{i+1} \end{bmatrix}$$

where R_i is a full row rank matrix. Applying the transformations U_i successively to the ith and $(i+1)$th block rows of the pencil $\breve{S}(z)$, we get the reduced pencil

$$\left[\begin{array}{c|cccc} * & R_1 & * & \cdots & * \\ * & O & R_2 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & O & O & \cdots & R_{K-1} \\ \hline \bar{S}_K - z\bar{T}_K & O & O & \cdots & O \end{array}\right]$$

which is orthogonally similar to the original system pencil $\hat{S}(z)$. Since the matrices R_i have full row rank, the subpencil $\bar{S}_K - z\bar{T}_K$ contains all finite zeros of the original pencil. This pencil has column and row dimensions at most $r_1 + 1$. To compute the finite zeros of the periodic system, we can now apply a general algorithm, like that of Varga (1996), to compute the finite eigenvalues of this low-order subpencil. This method requires a system with constant state dimension $r = r_i$, about $2 \times (13.3\,K + 27.5)r^3$ flops (floating point operations) in the worst case.

Remark: When using the periodic Kalman forms based minimal realization approach, it is possible to exploit the particular structure of the resulting matrices (see the remark in the previous subsection). For example, in the case $s = t$ and $\hat{D}_s \neq 0$, the zeros are the eigenvalues of the product $\hat{A}_K \cdots \hat{A}_{s+1} \times (\hat{A}_s - \hat{B}_s \hat{D}_s^{-1} \hat{C}_s) \hat{A}_{s-1} \cdots \hat{A}_1$ and can be computed via the EPRSF. Computational effort saving arises by observing that the factors of this product are already in an EPHF. In the case $\hat{D}_s = 0$, the finite zeros can be isolated by applying a special non-orthogonal similarity transformation to the pencil (22), which corresponds to a particular periodic output injection $\hat{L}_k$ acting only at time moment $k = s$. Let j be the least index such that $\hat{c}_{s;1,j} \neq 0$. By choosing $\hat{L}_s$ of the form

$$\hat{L}_s^{\mathrm{T}} = [\underbrace{0 \cdots 0}_{j} - \hat{a}_{s;j+1,j} / \hat{c}_{s;1,j}\, 0 \cdots 0]$$

a null element is introduced in position $(j+1, j)$ of $\hat{A}_s + \hat{L}_s \hat{C}_s$. It can be shown that the product $\hat{A}_K \cdots \hat{A}_{s+1} \times (\hat{A}_s + \hat{L}_s \hat{C}_s) \hat{A}_{s-1} \cdots \hat{A}_1$ becomes block upper triangular, with the trailing $(r_1 - j) \times (r_1 - j)$ block containing the $r_1 - j$ finite system zeros. Thus, the finite zeros can be computed via the EPRSF of the $(n_{k+1} - j) \times (n_k - j)$ submatrices in the trailing positions of the above factors. Once again the factors of this product are in an EPHF. Similar simplifications can be achieved also in the case $s \neq t$.

3.4. *Computation of gain*

To compute the gain, the main computation is to solve the linear system $Hw = g$, where $H = z_0 \hat{E} - \hat{A}$ and $g = \hat{b}$. This is a potentially large order structured linear system with

$$H = \begin{bmatrix} H_{11} & & & H_{1,K} \\ H_{21} & H_{22} & & \\ & \ddots & \ddots & \\ & & H_{K,K-1} & H_{K,K} \end{bmatrix}$$

$$:= \begin{bmatrix} z_0 I_{r_1} & & & -\hat{A}_K \\ -\hat{A}_1 & I_{r_2} & & \\ & \ddots & \ddots & \\ & & -\hat{A}_{K-1} & I_{r_k} \end{bmatrix}, \qquad g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_K \end{bmatrix}$$

where only the non-zero block elements of H are shown, and g is partitioned according to the block row structure of H. The solution vector w is obtained in a partitioned form similar to that of g

$$w = \begin{bmatrix} w_1^{\mathrm{T}} & w_2^{\mathrm{T}} & \cdots & w_K^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

The standard Gaussian elemination method (Golub and Van Loan 1989) to solve the linear equation $Hw = g$ has two main steps. First, the LU factorization of H is computed by using partial pivoting, to obtain $PH = LU$, where P is a permutation matrix, L is a unit lower triangular matrix and U is an upper triangular matrix. Then, by using forward and backward substitutions, the solution w is computed as $w = U^{-1}L^{-1}Pg$. For the particular structure of H above, it can be easily observed that the resulting L is block-bidiagonal, and U has

non-zero blocks only on the main diagonal, first supra-diagonal and in the last block column. For the efficient solution of the equation $Hw = g$, it is advantageous to combine the LU factorization step with the solution steps by applying the elementary row transformations also to the right-hand side g, such that in parallel with the computation of non-zero blocks of U we also compute $L^{-1}Pg$. The following algorithm can be used for this purpose.

Algorithm:

if $K = 1$, **then** solve $(z_0 I_{r_1} - \hat{A}_1)w = g$ and **exit**
Comment. Compute the block-LU factorization $PH = LU$ and $g \leftarrow L^{-1}Pg$.
for $i = 1, \ldots, K-1$
Compute the LU factorization

$$P_i \begin{bmatrix} H_{ii} \\ H_{i+1,i} \end{bmatrix} = L_i \begin{bmatrix} U_{ii} \\ O \end{bmatrix}$$

Compute

$$\left[\begin{array}{c|c|c} U_{i,i+1} & U_{i,K} & g_i \\ H_{i+1,i+1} & H_{i+1,K} & g_{i+1} \end{array}\right] := L_i^{-1} P_i \left[\begin{array}{c|c|c} O & H_{i,K} & g_i \\ H_{i+1,i+1} & H_{i+1,K} & g_{i+1} \end{array}\right]$$

end
Comment. Compute $w = U^{-1}g$ by backward substitution.
Solve $U_{KK}w_K = g_K$ and
$U_{K-1,K-1}w_{K-1} = g_{K-1} - U_{K-1,K}w_K$.
for $i = K-2, \ldots, 1$
Solve $U_{ii}w_i = g_i - U_{i,i+1}w_{i+1} - U_{i,K}w_K$
end

The main computations in this algorithm are the $K-1$ successive LU decompositions of $(r_i + r_{i+1}) \times r_i$ matrices and the application of r_i elementary transformations to $(r_i + r_{i+1}) \times (r_{i+1} + r_K + 1)$ matrices. For a constant system with dimension $r = r_i$, this algorithm performs about $5r^3/6$ flops to compute the LU-decomposition of a $2r \times r$ matrix and additionally, $2r^3$ flops to apply r elementary transformations (Golub and Van Loan 1989). Thus, the soluion of the equation $Hw = g$ with the above algorithm can be computed with about $\max\{3(K-1)r^3, r^3/3\}$ flops.

4. Examples

Example 1: Consider the 2-periodic single-input single-output system described by the following matrices

$$A_1 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1 = 1, \quad D_1 = 0$$
$$A_2 = \begin{bmatrix} 0 & 0.5 \end{bmatrix}, \quad B_2 = 1, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_2 = 0$$

where the state-space dimensions are $n_1 = 1$, $n_2 = 2$. This system is in a minimal balanced state-space representation and is asymptotically stable. The corresponding TFM of the *standard lifted system* is a 2×2 rational matrix. To compute the $(1,1)$ element of $W(z)$ we form the corresponding periodic system

$$A_1 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{C}_1 = 1, \quad \tilde{D}_1 = 0$$
$$A_2 = \begin{bmatrix} 0 & 0.5 \end{bmatrix}, \quad \tilde{B}_2 = 1, \quad \tilde{C}_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \tilde{D}_2 = 0$$

This system is not minimal and the minimal realization has all state dimensions zero. Thus, the corresponding entry is $w_{11}(z) = 0$.

To compute $w_{12}(z)$, we form the corresponding periodic system

$$A_1 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{C}_1 = 1, \quad \tilde{D}_1 = 0$$
$$A_2 = \begin{bmatrix} 0 & 0.5 \end{bmatrix}, \quad \tilde{B}_2 = 1, \quad \tilde{C}_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \tilde{D}_2 = 0$$

which has a minimal realization with constant state dimensions $r_1 = r_2 = 1$

$$\hat{A}_1 = 0.5, \quad \hat{B}_1 = 0, \quad \hat{C}_1 = 1 \quad \hat{D}_1 = 0$$
$$\hat{A}_2 = 0.5 \quad \hat{B}_2 = 1, \quad \hat{C}_2 = 0, \quad \hat{D}_2 = 0$$

This system has a pole at $\rho = 0.25$, but has no finite zeros. The resulting gain is 1, thus $w_{12}(z) = 1/(z - 0.25)$.

To compute $w_{21}(z)$, we form the corresponding periodic system

$$A_1 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{C}_1 = 0, \quad \tilde{D}_1 = 0$$
$$A_2 = \begin{bmatrix} 0 & 0.5 \end{bmatrix}, \quad \tilde{B}_2 = 0, \quad \tilde{C}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \tilde{D}_2 = 0$$

This system has a minimal realization with state dimensions $r_1 = 0$ and $r_2 = 1$, thus $\hat{A}_1$, $\hat{C}_1$, $\hat{A}_2$ and $\hat{B}_2$ are 1×0, 1×0, 0×1 and 0×1 (empty) matrices, respectively, and the rest of matrices are

$$\hat{B}_1 = 1, \quad \hat{D}_1 = 0$$
$$\hat{C}_2 = 1, \quad \hat{D}_2 = 0$$

This system has no poles and no zeros, but it has a non-zero gain equal to 1. Thus, $w_{21}(z) = 1$. The element $w_{22}(z)$ is zero. The resulting TFM of the *standard lifted system* at sampling time $k = 1$ is

$$W_1(z) = \begin{bmatrix} 0 & \dfrac{1}{z - 0.25} \\ 1 & 0 \end{bmatrix}$$

Example 2: Consider the 3-periodic single-input single-output system

$$A_1=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad B_1=\begin{bmatrix}3\\0\end{bmatrix},\quad C_1=\begin{bmatrix}0&1\end{bmatrix},\quad D_1=1$$
$$A_2=\begin{bmatrix}1&2\\0&0\end{bmatrix},\quad B_2=\begin{bmatrix}0\\1\end{bmatrix},\quad C_2=\begin{bmatrix}2&4\end{bmatrix},\quad D_2=3$$
$$A_3=\begin{bmatrix}0&0\\1&4\end{bmatrix},\quad B_3=\begin{bmatrix}0\\1\end{bmatrix},\quad C_3=\begin{bmatrix}3&1\end{bmatrix},\quad D_3=1$$

with state dimensions $n_1=n_2=n_3=2$. This system is not minimal and not asymptotically stable.

The corresponding TFM of the *standard lifted system* is a 3×3 rational matrix. We only show how to compute element $w_{11}(z)$. The corresponding periodic system

$$A_1=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad \tilde{B}_1=\begin{bmatrix}3\\0\end{bmatrix},\quad \tilde{C}_1=\begin{bmatrix}0&1\end{bmatrix},\quad \tilde{D}_1=1$$
$$A_2=\begin{bmatrix}1&2\\0&0\end{bmatrix},\quad \tilde{B}_2=\begin{bmatrix}0\\0\end{bmatrix},\quad \tilde{C}_2=\begin{bmatrix}0&0\end{bmatrix},\quad \tilde{D}_2=0$$
$$A_3=\begin{bmatrix}0&0\\1&4\end{bmatrix},\quad \tilde{B}_3=\begin{bmatrix}0\\0\end{bmatrix},\quad \tilde{C}_3=\begin{bmatrix}0&0\end{bmatrix},\quad \tilde{D}_3=0$$

has a minimal realization with constant state dimensions $r_1=r_2=r_3=1$ given by

$$\hat{A}_1=-1,\quad \hat{B}_1=3,\quad \hat{C}_1=-1,\quad \hat{D}_1=1$$
$$\hat{A}_2=1,\quad \hat{B}_2=0,\quad \hat{C}_2=0,\quad \hat{D}_2=0$$
$$\hat{A}_3=-1,\quad \hat{B}_3=0,\quad \hat{C}_3=0,\quad \hat{D}_3=0$$

This system has a pole at $\rho=1$ and a zero at $\mu=-2$. The resulting gain is 1, and thus $w_{11}(z)=(z+2)/(z-1)$. The computed TFM of the periodic system at sampling time $k=1$ is

$$W(z)=\frac{1}{z-1}\begin{bmatrix}z+2&4&1\\6z&3z+5&2\\9z&z+11&z+2\end{bmatrix}$$

Example 3: Consider a discrete-time periodic system originating from a continuous-time periodic model of a spacecraft pointing and attitude system. This system has one input and two outputs, and is described in Pittelkau (1993). The continuous-time linearized state space model of the spacecraft system is described by the matrices

$$A=\begin{bmatrix}0&0&0.053\,180\,64&0\\0&0&0&0.053\,180\,64\\-0.001\,352\,134&0&0&-0.070\,992\,73\\0&-0.000\,755\,718\,2&0.037\,815\,55&0\end{bmatrix}$$

$$B(t)=\begin{bmatrix}0\\0\\0.138\,973\,5\times 10^{-6}\sin(\omega_0 t)\\-0.370\,133\,6\times 10^{-7}\cos(\omega_0 t)\end{bmatrix}$$

$$C=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix},\quad D=\begin{bmatrix}0\\0\end{bmatrix}$$

where $\omega_0=0.001\,034\,48$ rad/s is the orbital frequency. Notice that A is a constant matrix with all its eigenvalues on the imaginary axis. The matrix $B(t)$ is however a time-dependent periodic matrix with the period $2\pi/\omega_0$. The discretized system for different sampling periods K has been used by Varga and Pieters (1998) to design periodic output feedback controllers for this system. For a given K, the corresponding sampling period is $T=2\pi/(\omega_0 K)$. The matrices of the discrete-time periodic system can be computed explicitly as $A_k=\exp(AT)$, $B_k=\int_{(k-1)T}^{kT}\mathrm{e}^{[AkT-\tau]}B(\tau)\,\mathrm{d}\tau$. To show the applicability of our algorithm to periodic systems with large periods, we computed the TFM for $K=120$. Since the corresponding TFM is an 240×120 rational matrix, we will illustrate only the computation of element $\omega_{100,100}(z)$.

For reference purposes we give the matrices of the discretized periodic model which results for $K=120$ and $T=50.614\,68$ s

$$A_k=\begin{bmatrix}0.950\,686\,0&0.042\,986\,6&0.482\,732\,0&-2.556\,438\,3\\-0.040\,968\,4&0.972\,162\,8&1.361\,732\,8&0.508\,145\,4\\-0.012\,273\,6&0.036\,328\,0&-0.867\,139\,4&-0.601\,429\,5\\-0.034\,622\,5&-0.007\,220\,9&0.320\,362\,2&-0.845\,662\,6\end{bmatrix}$$

$$B_k=10^{-5}\times\begin{bmatrix}0.222\,092\,5\\-0.130\,053\,6\\0.187\,721\,7\\-0.027\,116\,7\end{bmatrix}\cos\left(2\pi\frac{k-1}{K}\right)$$
$$+10^{-5}\times\begin{bmatrix}0.503\,562\,0\\0.424\,108\,7\\0.121\,829\,0\\0.358\,382\,6\end{bmatrix}\sin\left(2\pi\frac{k-1}{K}\right)$$

The computed poles, zeros and the gain of the transfer function $\omega_{100,100}(z)$ are, respectively

$$\rho_1=0.7626+0.6469i,\quad \mu_1=0.3029+0.6419i$$
$$\rho_2=0.7626-0.6469i,\quad \mu_2=0.3029-0.6419i$$
$$\rho_3=0.9942+0.1077i,\quad \mu_3=0.9685$$
$$\rho_4=0.9942-0.1077i$$
$$\gamma=2.3273\times 10^{-6}$$

Thus, the resulting transfer function is

$$w_{100,100}(z)=\frac{2.3273\times 10^{-6}(z^3-1.574z^2+1.09z-0.4879)}{z^4-3.514z^3+5.033z^2-3.514z+1}$$

Note that the order of the *stacked lifted system* is 480. Although the direct application of algorithm of Varga (1989) to this system is still feasible, it is certainly too expensive to solve this problem.

In table 1, computational times are given to determine a single element of the TFM for different values of K. The computations have been done on a 866 MHz PC running MATLAB 6.5 under Windows ME. For minimal

realizations, the periodic Kalman forms based procedure has been used.

K	20	40	80	120	240
Time (s)	0.06	0.17	0.22	0.44	0.88

Table. 1 Computational times for a single element.

It is easy to see that the computational time varies almost linearly with K, and this confirms our claim for a computational complexity of $O(Kn^3)$ of the proposed approach. In contrast, when applying the algorithm of Varga (1989) to the *stacked lifted system*, the times for $K=120$ and $K=240$ are 1.98 and 15.05 s, respectively, which clearly indicates a computational complexity of $O(K^3n^3)$.

5. Conclusion

A reliable numerical algorithm has been proposed to compute the TFM corresponding to lifted representations of periodic systems. The proposed algorithm is generally applicable to periodic systems with time-varying state dimensions. The individual entries of the TFM are obtained in a cancelled, minimal *zeros-poles-gain* form, regardless of whether the original system is minimal or not.

The proposed algorithm relies on the extensive use of orthogonal transformations. A main step in the proposed algorithm is the computation of minimal periodic realizations. This can be done by either performing exact model reduction based on accuracy enhancing balancing-free square-root techniques or by applying a recently developed algorithm based on reachability/observability Kalman structural forms. Note that the latter approach is strongly numerical stable relying exclusively on structure preserving orthogonal similarity transformations. For the subsequent computation of poles, zeros and gains, numerically stable algorithms can be employed as well. The overall procedure is oriented towards an efficient software implementation using available high performance numerical linear algebra software. A prototype implementation has been performed for MATLAB and relies, among others, on the generalized zeros computation tools available in the Descriptor Toolbox (Varga 2000 b).

The computational complexity of the proposed algorithm for one entry of the TFM is $O(Kn^3)$ and is much less than the computational complexity $O(K^3n^3)$ resulting when applying existing procedures (Varga 1989) directly to the *stacked lifted representation* of the periodic system. However, the worst-case overall computational complexity for a system with m-inputs and p-outputs involving the computation of $pK \times mK$ elements can be still significant for large values of the period K. Therefore, this problem is perfectly suited to be solved on parallel machines, since all $pK \times mK$ elements can be computed independently in parallel.

The proposed approach can be extended to compute the TFM of periodic descriptor systems of the form

$$E_k x(k+1) = A_k x(k) + B_k u(k)$$
$$y(k) = C_k x(k) + D_k u(k)$$

where the descriptor matrix E_k can be, in the most general case, rectangular. The main necessary computational ingredient (to be developed) for such a method is an efficient algorithm to compute minimal realizations of periodic descriptor systems. Note that to compute finite poles, finite zeros and gains, the algorithms proposed in this paper can be readily extended to the descriptor case.

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