Computation of minimal periodic realizations of transfer-function matrices

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Abstract

We present a numerical approach to compute a minimal periodic state-space realization of a transfer-function matrix corresponding to a lifted state-space representation. The proposed method determines a realization with time-varying state dimensions by using exclusively orthogonal transformations. The new method is numerically reliable, computationally efficient and thus well suited for robust software implementations.

1 Introduction

We consider the development of an efficient and reliable numerical algorithm for the following *periodic realization problem* (PRP): Given a $Np \times Nm$ *transfer-function matrix* (TFM) $W(z)$, determine a minimal periodic realization (i.e., completely reachable and completely observable) of the form

$$\begin{array}{rcl} x(k+1) & = & A_k x(k) + B_k u(k) \\ y(k) & = & C_k x(k) + D_k u(k) \end{array} \quad (1)$$

such that the TFM of the standard lifted representation of (1) (see next section) is equal to $W(z)$. In (1), $A_k \in \mathbb{R}^{n_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{n_{k+1}\times m}$, $C_k \in \mathbb{R}^{p\times n_k}$, $D_k \in \mathbb{R}^{p\times m}$ are periodic matrices with period $N \geq 1$. Note that generally, the minimal periodic realization of a given $W(z)$ is a periodic system with time-varying state dimensions [2].

A computational algorithm to solve the above PRP is useful in many applications. For example, using lifted representations of periodic systems (as those introduced in [9, 10, 3, 6]), it is relatively easy to compute left or right inverses of a periodic system by manipulating the associated system pencil matrix (see for example [12]). However, from the resulting representation of the inverse it is impossible in general to directly recover the underlying periodic representation. A minimal realization algorithm for periodic systems can be useful by allowing the following computational detour: compute first the corresponding TFM or a minimal state-space representation of the inverse and then compute its minimal periodic realization.

Another possible application is in the identification of periodic systems. For example, one of proposed subspace identification algorithms (see [15]) is only applicable provided the underlying periodic system is uniformly reachable and observable (i.e., the periodic system has constant state dimension). This condition is however not always fulfilled, a notable exception being the class of multirate systems modelled as periodic systems. In this case, a computational detour is to identify first the TFM or state-space realization of the lifted system by using appropriate subspace identification methods, and then convert this model into a periodic minimal realization.

Finally, the TFM to state-space conversion is a useful transformation which must belong to any software toolbox devoted to the manipulation of periodic system descriptions. Note that for the reverse transformation (the state-space to TFM conversion), a numerically reliable algorithm has been recently developed in [13].

A realization algorithm for periodic systems has been proposed by Lin and King [8]. The resulting periodic realization is however not minimal, because the state dimension is forcedly chosen constant. The determination of this state dimension requires checking of $N-1$ rank conditions. To compute a minimal realization, we need to combine this algorithm with a minimal realization algorithm, as for instance, that proposed in [11]. Besides the higher computational costs of the overall approach, both main computational steps use non-orthogonal transformations. Thus in general, this combination approach is numerically not satisfactory.

In this paper we propose a computational procedure which improves the algorithm of [8] in two directions. Firstly, the new procedure computes directly a periodic minimal realization with time-varying state dimensions starting from a minimal realization of $W(z)$ as a standard state-space system. Secondly, the procedure relies exclusively on performing orthogonal transformations, thus is completely satisfactory from numerical point of view. Therefore, the new method is well suited for robust software implementations.

2 Periodic realization problem

First, we introduce some notations and recall the definitions of reachability, observability and minimality of periodic systems (see [4, 2]). The transition matrix of the system (1) is defined by the $n_j \times n_i$ matrix $\Phi_A(j,i) = A_{j-1}A_{j-2}\cdots A_i$, where $\Phi_A(i,i) := I_{n_i}$. The state transition matrix over one period $\Phi_A(j+N,j) \in \mathrm{R}^{n_j\times n_j}$ is called the *monodromy matrix* of system (1) at time j and its eigenvalues are called *characteristic multipliers* at time j.

Definition 1. The periodic system (1) is *reachable at time* i if

$$\operatorname{rank}\mathcal{C}_i = n_i, \tag{2}$$

where $\mathcal{C}_i$ is the infinite columns matrix

$$\mathcal{C}_i = [\, B_{i-1} \;\; A_{i-1}B_{i-2} \;\cdots\; \Phi_A(i,j+1)B_j \;\cdots]. \tag{3}$$

The periodic system (1) is *completely reachable* if (2) holds for $i = 1,\ldots,N$.

Definition 2. The periodic system (1) is *observable at time* i if

$$\operatorname{rank}\mathcal{O}_i = n_i, \tag{4}$$

where $\mathcal{O}_i$ is the infinite rows matrix

$$\mathcal{O}_i = \begin{bmatrix} C_i \\ C_{i+1}A_i \\ \vdots \\ C_j\Phi_A(j,i) \\ \vdots \end{bmatrix}. \tag{5}$$

The periodic system (1) is *completely observable* if (4) holds for $i = 1,\ldots,N$.

Definition 3. The periodic system (1) is *minimal* if it is completely reachable and completely observable.

To define the TFM of the periodic system (1), we consider the time-invariant representation corresponding to the associated lifted system introduced in [9] which uses the input-output behavior of the system over time intervals of length N, rather than 1. For a given sampling time k, the corresponding Nm-dimensional input, Np-dimensional output, and n_k-dimensional state vectors are defined as

$$\begin{aligned} \widetilde{u}_k(h) &= [u^T(k+hN)\cdots u^T(k+hN+N-1)]^T, \\ \widetilde{y}_k(h) &= [y^T(k+hN)\cdots y^T(k+hN+N-1)]^T, \\ \widetilde{x}_k(h) &= x(k+hN) \end{aligned} \tag{6}$$

The lifted system has the form

$$\begin{aligned} \widetilde{x}_k(h+1) &= F_k\widetilde{x}_k(h) + G_k\widetilde{u}_k(h) \\ \widetilde{y}_k(h) &= H_k\widetilde{x}_k(h) + L_k\widetilde{u}_k(h) \end{aligned} \tag{7}$$

where

$$\begin{aligned} F_k &= \Phi_A(k+N,k) \\ G_k &= [\, \Phi_A(k+N,k+1)B_k \;\cdots B_{k+N-1}\,] \\ H_k &= \begin{bmatrix} C_k \\ \vdots \\ C_{k+N-1}\Phi_A(k+N-1,k) \end{bmatrix} \\ L_k &= \begin{bmatrix} D_k & 0 & \cdots & 0 \\ L_{k,2,1} & D_{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{k,N,1} & L_{k,N,2} & \cdots & D_{k+N-1} \end{bmatrix} \end{aligned}$$

with $L_{k,i,j} = C_{k+i-1}\Phi_A(k+i-1,k+j)B_{k+j-1}$, for $i = 2,...,N$, $j = 1,2,\ldots N-1$, and $i > j$.

The system (7) is called the *standard lifted system* at time k of the given N-periodic system (1). The lifted system (7) shares the same structural properties as the original periodic system (1). In particular, the system (7) is reachable (observable) if and only if the system (1) is reachable (observable) at time k.

The associated TFM $W_k(z)$ is

$$W_k(z) = H_k(zI_{n_k} - F_k)^{-1}G_k + L_k \tag{8}$$

and depends on the sampling time k. Thus, a given TFM $W(z)$ can be realized in N instances as a periodic system.

In what follows, we assume that the given $Np \times Nm$ TFM $W(z)$ corresponds to the time moment $k = 1$ for which reason we will drop the index k. Realizations at time moments $k > 1$ can be easily obtained by cyclic permutations of the matrices determined for $k = 1$. We have the following result showing the existence of periodic realizations [7]:

Theorem 2.1 *The $Np \times Nm$ TFM $W(z)$ has a periodic realization of the form (1) iff the $W(\infty)$ matrix with $p \times m$ diagonal blocks is lower block triangular.*

Remark. Theorem 2.1 can be relaxed by allowing for more general TFMs for which $W(\infty)$ has a block structure which can be brought into a lower block triangular form by means of block row and block column permutations. This corresponds to define suitable permutations of block inputs and block outputs in (6) for the lifted system. In this way, we can determine periodic realizations of several TFMs (see Section 5), which according to Theorem 2.1, can not be realized as periodic systems.

For our developments, we assume in what follows that $W(z)$ fulfills the condition of Theorem 2.1 and has a minimal realization of order $\overline{n}$, as a standard system $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ satisfying

$$W(z) = \overline{C}(zI_{\overline{n}} - \overline{A})^{-1}\overline{B} + \overline{D}. \tag{9}$$

To solve the PRP we have to compute the N-periodic system matrices A_k, B_k, C_k, D_k which satisfy the conditions

$$\overline{A} = F, \quad \overline{B} = G, \quad \overline{C} = H, \quad \overline{D} = L \tag{10}$$

Moreover, the periodic realization (A_k, B_k, C_k, D_k) is required to be minimal, that is, completely reachable and completely observable.

3 Periodic realization algorithm

If we partition the matrices $\overline{B}$, $\overline{C}$, and $\overline{D}$ to correspond to the N block rows and N block columns of $W(z)$, we have

$$\overline{B} = \begin{bmatrix} \overline{B}_1 & \overline{B}_2 & \cdots & \overline{B}_N \end{bmatrix},$$

$$\overline{C} = \begin{bmatrix} \overline{C}_1 \\ \overline{C}_2 \\ \vdots \\ \overline{C}_N \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} \overline{D}_{11} & 0 & \cdots & 0 \\ \overline{D}_{21} & \overline{D}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{D}_{N1} & \overline{D}_{N2} & \cdots & \overline{D}_{NN} \end{bmatrix}$$

From (10), we get the following equations to be satisfied by the periodic system matrices A_k, B_k, C_k, D_k:

$$\begin{array}{ll} \overline{A} & = A_N A_{N-1} \cdots A_1 \\ \overline{B}_i & = A_N \cdots A_{i+1} B_i, \quad i = 1, \ldots, N \\ \overline{C}_i & = C_i A_{i-1} \cdots A_1, \quad i = 1, \ldots, N \\ \overline{D}_{ii} & = D_i, \quad i = 1, \ldots, N \\ \overline{D}_{ij} & = C_i A_{i-1} \cdots A_{j+1} B_j, \quad i > j,\ i, j = 2, \ldots, N-1 \end{array}$$

We solve the above equations by generating recursively the system matrices. We have immediately, that

$$B_N = \overline{B}_N, \ \ C_1 = \overline{C}_1, \ \ D_i = \overline{D}_{ii}, \ \ i = 1, \ldots, N \tag{11}$$

and thus

$$n_1 = \overline{n} \tag{12}$$

Analogously to [8], we consider for $i = 1, \ldots, N-1$ the matrices

$$K_i = \left[\begin{array}{c|ccc} \overline{A} & \overline{B}_1 & \cdots & \overline{B}_i \\ \hline \overline{C}_N & \overline{D}_{N,1} & \cdots & \overline{D}_{N,i} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{C}_{i+1} & \overline{D}_{i+1,1} & \cdots & \overline{D}_{i+1,i} \end{array}\right]$$

which can be factored in the form

$$K_i = \left[\begin{array}{c} A_N \cdots A_{i+1} \\ \hline C_N A_{N-1} \cdots A_{i+1} \\ \vdots \\ C_{i+1} \end{array}\right] \left[\begin{array}{c|ccc} A_i \cdots A_1 & A_i \cdots A_2 B_1 & \cdots & B_i \end{array}\right]$$

Let

$$n_{i+1} := \operatorname{rank} K_i \tag{13}$$

and consider the full rank orthogonal factorization

$$K_i = U_i R_i, \tag{14}$$

with U_i having n_{i+1} orthonormal columns (i.e., $U_i^T U_i = I_{n_{i+1}}$), and R_i having full row rank n_{i+1}. If we partition U_i and R_i according to the row and column structure of K_i, respectively, we have

$$U_i = \left[\begin{array}{c} U_{i0} \\ U_{iN} \\ \vdots \\ U_{i,i+2} \\ \hline U_{i,i+1} \end{array}\right] := \left[\begin{array}{c} \overline{U}_i \\ \hline U_{i,i+1} \end{array}\right]$$

$$R_i = \left[\begin{array}{cccc|c} R_{i0} & R_{i1} & \cdots & R_{i,i-1} & R_{ii} \end{array}\right]$$

Thus, we can immediately take

$$B_i = R_{ii}, \quad C_{i+1} = U_{i,i+1} \tag{15}$$

If $i = 1$, we can choose

$$A_1 = R_{10} \tag{16}$$

while for $i = N$ we can choose

$$A_N = U_{N-1,0} \tag{17}$$

For $i = 2, \ldots, N-1$, it is easy to see that A_i satisfies

$$U_i A_i = \overline{U}_{i-1}$$

Provided $\mathcal{R}(\overline{U}_{i-1}) \subseteq \mathcal{R}(U_i)$, and recalling that U_i has full column rank and orthonormal columns, we get the unique solution

$$A_i = U_i^T \overline{U}_{i-1} \tag{18}$$

To show that indeed $\mathcal{R}(\overline{U}_{i-1}) \subseteq \mathcal{R}(U_i)$, we observe that K_i and K_{i-1} can be partitioned as

$$K_i = \left[\begin{array}{c|c} X_i & \begin{array}{c} \overline{B}_i \\ \overline{D}_{N,i} \\ \vdots \\ \overline{D}_{i+1,i} \end{array} \end{array}\right]$$

$$K_{i-1} = \left[\begin{array}{cccc} & X_i & & \\ \hline \overline{C}_i & \overline{D}_{i,1} & \cdots & \overline{D}_{i,i-1} \end{array}\right]$$

Taking into account the partitioning of U_{i-1} in the form

$$U_{i-1} = \left[\begin{array}{c} \overline{U}_{i-1} \\ \hline U_{i-1,i} \end{array}\right]$$

it follows that $\mathcal{R}(\overline{U}_{i-1}) = \mathcal{R}(X_i) \subseteq \mathcal{R}(K_i) = \mathcal{R}(U_i)$.

We can easily prove our main result:

Theorem 3.1 *The periodic realization computed by using the formulas (11)-(18) is minimal.*

Proof. We need to show that the computed state dimensions (12) and (13) satisfy

$$n_i = \operatorname{rank} \mathcal{C}_i = \operatorname{rank} \mathcal{O}_i, \quad i = 1, \ldots, N$$

First we show that

$$\operatorname{rank} \mathcal{C}_1 = n_1$$

where

$$\mathcal{C}_1 = [\, B_N \;\; A_N B_{N-1} \;\; \cdots \;\; A_N \cdots A_2 B_1 \;\; A_N \cdots A_1 B_N \cdots \,]$$

Since the pair $(\overline{A}, \overline{B})$ is controllable, we have that

$$n_1 = \overline{n} = \operatorname{rank} [\, \overline{B} \;\; \overline{AB} \;\; \cdots \;\; \overline{A}^{n-1} \overline{B} \,] = \operatorname{rank} \mathcal{C}_1$$

Similarly, from the observability of the pair $(\overline{A}, \overline{C})$, we can show that

$$n_1 = \operatorname{rank} \mathcal{O}_1$$

For $i \geq 1$ we have successively

$$\begin{aligned} \operatorname{rank} \mathcal{C}_{i+1} &= \operatorname{rank} \big[B_i \big| A_i B_{i-1} \big| \cdots \big| A_i \cdots A_2 B_1 \big| A_i \cdots A_1 B_N \big| \cdots \big] \\ &= \operatorname{rank} \big[B_i \big| A_i B_{i-1} \big| \cdots \big| A_i \cdots A_2 B_1 \big| A_i \cdots A_1 \mathcal{C}_1 \big] \\ &= n_{i+1} \end{aligned}$$

where the last equality results from the full rank factorization (14) and the full row rank of $\mathcal{C}_1$. Similarly we can prove the observability properties. □

There are two main improvements offered by our algorithm with respect to the algorithm of [8]. The first improvement is that we determine directly a periodic realization with minimal order state dimensions. The resulting periodic realization is minimal and has, in general, time-varying state dimensions. In contrast, the algorithm of [8] determines generally a non-minimal realization with unreachable and/or unobservable characteristic multipliers in the origin.

The second improvement is the overall numerically reliability of our algorithm and the straightforward computation of the state matrices of the periodic realization. The computation of the minimal periodic realization involves performing $N-1$ rank revealing QR-factorizations (e.g., QR-factorizations with column pivoting or singular value decompositions [5]) and all matrices are generated in terms of the resulting computed quantities in these factorizations. By using the orthogonal full rank factorizations (14), we can explicitly solve the equation $U_i A_i = \overline{U}_{i-1}$ satisfied by A_i. Thus all system matrices can be determined by using exclusively orthogonal transformations. In contrast, the algorithm of [8] determine A_i by solving two matrix equations involving both U_i and R_i (see Lemma 2.1 of [8]). Thus, our algorithm can be considered completely satisfactory from numerical point of view along the lines of requirements formulated in [14].

We can roughly estimate the computational effort required by our algorithm assuming constant dimension $n_i = n$, $i = 1, \ldots, N$ for the resulting periodic realization. We assume that we use QR-decompositions with column pivoting based on Householder transformations to compute the full rank orthogonal factorizations (see [5] for details). Note that for a generic $m \times n$ matrix X of rank r, the computation of the $r \times n$ matrix R in the full rank QR factorization $X = QR$ requires about $r(mn + \frac{2r^2}{3} - (m+n)r^2)$ *flops* (1 *flop* = 1 multiplication + 1 addition), while the economic accumulation of the r Householder transformations in Q requires $2r^2(m - \frac{2r}{3})$ *flops*. Thus in total we need about $r(2mn - \frac{r^2}{3} + (m-n)\frac{r}{2})$ *flops*. In the minimal realization algorithm we need to compute $N-1$ full rank factorizations of the K_i matrices of dimensions $(\overline{n} + (N-i)p) \times (\overline{n} + im)$, for $i = 1, \ldots, N-1$. This involves approximately

$$2n(N{-}1) \left(\overline{n}^2 - \frac{n^2}{3} + \frac{\overline{n}(m+p) + n(p-m)}{2} N + \frac{pm}{6} N^2 \right)$$

flops. By assuming $n \approx \overline{n}$ and $p = m$, we obtain a simpler expression for the approximate number of required *flops*

$$n(N{-}1) \left(\frac{4n^2}{3} + 2npN + \frac{p^2}{3} N^2 \right)$$

To the above figures we have to add the number of *flops* necessary to compute the state-space realization of $W(z)$.

4 Example

We consider the example used in [8] to show the main computational steps. For the sake of clarity, we will use non-orthogonal computations to compute the full rank factorizations. Let

$$W(z) = \frac{1}{z-1} \begin{bmatrix} z+2 & 4 & 1 \\ 6z & 3z+5 & 2 \\ 9z & z+11 & z+2 \end{bmatrix}$$

and consider the same state realization as in [8]

$$\left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array} \right] = \left[\begin{array}{c|ccc} 1 & 3 & 4 & 1 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & 9 & 1 & 1 \end{array} \right]$$

We have immediately

$$B_3 = 1, \quad C_1 = 1, \quad D_1 = 1, \quad D_2 = 3, \quad D_3 = 1.$$

From

$$K_1 = \left[\begin{array}{c|c} 1 & 3 \\ \hline 3 & 9 \\ 2 & 6 \end{array}\right] = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}$$

we obtain

$$A_1 = 1, \quad B_1 = 3, \quad C_2 = 2.$$

Further, from

$$K_2 = \left[\begin{array}{c|cc} 1 & 3 & 4 \\ \hline 3 & 9 & 1 \end{array}\right] = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we obtain

$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 3 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 4 \end{bmatrix}$$

A_2 results from

$$\begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} A_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

as

$$A_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The resulting periodic realization has state dimensions $n_1 = 1$, $n_2 = 1$ and $n_3 = 2$ and is minimal. In contrast, the realization obtained in [8] has constant order $n = 2$ and is not minimal.

5 Periodic realizations of some TFMs

In this section we summarize (without proofs) some known (and less known) results concerning the realizability of certain TFMs which can be useful in building periodic realizations of particular TFMs. Note that while some of these results formally duplicates similar ones for standard discrete-time state-space systems, other results show that when manipulating lifted representations, a certain care must be exercised to avoid nonfeasible/noncausal realization problems (see [1] for an extensive account on lifted representations).

In the light of Theorem 2.1, the realizability of a given $W(z)$ is guaranteed if $W(z)$ is proper and $W(\infty)$ is block lower triangular. The following facts relies on preserving this property for several TFM constructs.

Proposition 5.1 *Assume $W(z)$ has an N-periodic realization (A_k, B_k, C_k, D_k) with D_k nonsingular for $k = 1, \ldots, N$. Then, $W^{-1}(z)$ has a periodic realization given by $(A_k - B_k D_k^{-1} C_k, -B_k D_k^{-1}, D_k^{-1} C_k, D_k^{-1})$.*

Proposition 5.2 *Assume $W_i(z)$ has the N-periodic realization $(A_k^{(i)}, B_k^{(i)}, C_k^{(i)}, D_k^{(i)})$ for $i = 1, 2$. Then, $W_1(z)W_2(z)$ has a periodic realization given by $(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k)$, where*

$$\left[\begin{array}{c|c} \widetilde{A}_k & \widetilde{B}_k \\ \hline \widetilde{C}_k & \widetilde{D}_k \end{array}\right] = \left[\begin{array}{cc|c} A_k^{(1)} & B_k^{(1)} C_k^{(2)} & B_k^{(1)} D_k^{(2)} \\ 0 & A_k^{(2)} & B_k^{(2)} \\ \hline C_k^{(1)} & D_k^{(1)} C_k^{(2)} & D_k^{(1)} D_k^{(2)} \end{array}\right]$$

Proposition 5.3 *Assume $W_i(z)$ has the N-periodic realization $(A_k^{(i)}, B_k^{(i)}, C_k^{(i)}, D_k^{(i)})$ for $i = 1, 2$. Then, $W_1(z) + W_2(z)$ has a periodic realization given by $(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k)$, where*

$$\left[\begin{array}{c|c} \widetilde{A}_k & \widetilde{B}_k \\ \hline \widetilde{C}_k & \widetilde{D}_k \end{array}\right] = \left[\begin{array}{cc|c} A_k^{(1)} & 0 & B_k^{(1)} \\ 0 & A_k^{(2)} & B_k^{(2)} \\ \hline C_k^{(1)} & C_k^{(2)} & D_k^{(1)} + D_k^{(2)} \end{array}\right]$$

The following fact is an immediate consequence of Propositions 5.1, 5.2 and 5.3.

Proposition 5.4 *Assume $W_i(z)$ has the N-periodic realization $(A_k^{(i)}, B_k^{(i)}, C_k^{(i)}, D_k^{(i)})$ for $i = 1, 2$, such that $I - D_k^{(1)} D_k^{(2)}$ are invertible for $k = 1, \ldots, N$. Then, $W_1(z)(I + W_2(z)W_1(z))^{-1}$ has a periodic realization given by $(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k)$, where*

$$\widetilde{A}_k = \begin{bmatrix} A_k^{(1)} - B_k^{(1)} D_k^{(2)} R_k^{-1} C_k^{(1)} & -B_k^{(1)} \widetilde{R}_k^{-1} C_k^{(2)} \\ B_k^{(2)} R_k^{-1} C_k^{(1)} & A_k^{(2)} - B_k^{(2)} D_k^{(1)} \widetilde{R}_k^{-1} C_k^{(2)} \end{bmatrix}$$
$$\widetilde{B}_k = \begin{bmatrix} B_k^{(1)} \widetilde{R}_k^{-1} \\ B_k^{(2)} D_k^{(1)} \widetilde{R}_k^{-1} \end{bmatrix}$$
$$\widetilde{C}_k = \begin{bmatrix} R_k^{-1} C_k^{(1)} & -R_k^{-1} D_k^{(1)} C_k^{(2)} \end{bmatrix}$$
$$\widetilde{D}_k = D_k^{(1)} \widetilde{R}_k^{-1}$$

where $R_k = I - D_k^{(1)} D_k^{(2)}$ and $\widetilde{R}_k = I - D_k^{(2)} D_k^{(1)}$.

Some constructs using realizable TFMs have no causal periodic realizations (although can be realized as descriptor-type periodic systems). However, in the light of the **Remark** in Section 2, we can still build several useful system realizations which correspond to permuted input and outputs blocks in (6). The following two facts (given without proofs) are just a representative selection of some possible results.

Proposition 5.5 *Assume $W_i(z)$ has the N-periodic realization $(A_k^{(i)}, B_k^{(i)}, C_k^{(i)}, D_k^{(i)})$ for $i = 1, 2$. Then the periodic system realization given by $(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k)$, where*

$$\left[\begin{array}{c|c} \widetilde{A}_k & \widetilde{B}_k \\ \hline \widetilde{C}_k & \widetilde{D}_k \end{array}\right] = \left[\begin{array}{cc|cc} A_k^{(1)} & 0 & B_k^{(1)} & 0 \\ 0 & A_k^{(2)} & 0 & B_k^{(2)} \\ \hline C_k^{(1)} & 0 & D_k^{(1)} & 0 \\ 0 & C_k^{(2)} & 0 & D_k^{(2)} \end{array}\right]$$

corresponds to the TFM $P\begin{bmatrix} W_1(z) & 0 \\ 0 & W_2(z) \end{bmatrix}P$*, with* P *the following permutation matrix*

$$P = \left[\begin{array}{cccc|ccccc} I & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I \end{array}\right]$$

Similar results are valid for periodic systems corresponding to permuted column or row concatenations of TFMs.

A dual periodic system can be defined using the following fact.

Proposition 5.6 *Let* $W(z)$ *have the* N*-periodic realization* (A_k, B_k, C_k, D_k)*. Then, the periodic realization* $(A^T_{N-k+1}, C^T_{p(k)}, B^T_{N-k+1}, D^T_{p(k)})$ *with* $p(k) = mod(N-k+2, N)$ *corresponds to the TFM* $PW^T(z)P$*, where* P *is the permutation matrix*

$$P = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & I \\ \vdots & \vdots & & \vdots \\ 0 & I & \cdots & 0 \end{bmatrix}$$

Interestingly, although the transposed TFM $W^T(z)$ has in general no standard periodic realization, the conjugated TFM $W^\sim(z) := W^T(1/z)$ does.

Proposition 5.7 *Assume* $W(z)$ *has an* N*-periodic realization* (A_k, B_k, C_k, D_k) *with constant state dimension and with* A_k *nonsingular for* $k = 1, \ldots, N$*. Then, the conjugated TFM* $W^\sim(z)$ *has a periodic realization given by* $(A_k^{-T}, -A_k^{-T}C_k^T, B_k^T A_k^{-T}, D_k^T - B_k^T A_k^{-T} C_k^T)$*.*

6 Conclusion

We proposed a numerically sound and computationally efficient approach to compute minimal periodic realizations of transfer-funtion matrices. The resulting periodic representations have in general time-varying dimensions. The proposed approach relies exclusively on numerically stable algorithms, the key computations being $N-1$ rank revealing orthogonal decompositions. The proposed approach is straightforward to implement as robust numerical software. Numerical examples computed with a MATLAB-based implementation show the applicability of this method to high order periodic systems.

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