



Brief Paper

Accuracy-enhancing methods for balancing-related frequency-weighted model and controller reduction[☆]

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Abstract

We consider the solution of the balancing-related frequency-weighted model and controller reduction problems using accuracy enhanced numerical algorithms. We propose first new stability-enforcing choices of the frequency-weighted grammians which can guarantee the stability of reduced models for two-sided frequency weights. Then we show that for the frequency-weighted controller reduction problems with standard stability and performance-enforcing frequency weights the computation of the frequency-weighted grammians can be done by solving reduced order Lyapunov equations. For both frequency-weighted model and controller reduction problems we indicate how to compute the grammians directly in terms of their Cholesky factors. This allows the extension of the square-root and balancing-free accuracy-enhancing techniques to the frequency-weighted case.

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1. Introduction

Consider the n th order original continuous-time state-space model $\mathbf{G} := (A, B, C, D)$ with the *transfer-function matrix* (TFM)

$$G(s) = C(sI - A)^{-1}B + D$$

and let $\mathbf{G}_r := (A_r, B_r, C_r, D_r)$ be an r th order approximation of the original model ($r < n$), with the TFM

$$G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r.$$

The methods for *frequency-weighted model reduction* (FWMR) try to minimize a weighted approximation error of the form

$$\|W_o(G - G_r)W_i\|_\infty, \tag{1}$$

where W_o and W_i are suitably chosen output and input weighting TFMs, respectively.

Notation: Throughout the paper, we will use the bold-notation \mathbf{G} to denote a state-space system having the TFM $G(s)$ or G . This notation is also used to denote state-space realizations of systems corresponding to operations with TFMs, as for example, $\mathbf{G}_1\mathbf{G}_2$ to denote the series coupling of two systems having the TFM $G_1(s)G_2(s)$ or $\mathbf{G}_1 + \mathbf{G}_2$ to denote the (additive) parallel coupling of two systems with TFM $G_1(s) + G_2(s)$.

Controller reduction problems are frequently formulated as FWMR problems (Anderson & Liu, 1989). Let $\mathbf{K} = (A_c, B_c, C_c, D_c)$ be a stabilizing controller of order n_c for the system \mathbf{G} . We want to find \mathbf{K}_r , an r_c th order approximation of \mathbf{K} having the same number of unstable poles as \mathbf{K} , such that a weighted error of the form

$$\|W_o(K - K_r)W_i\|_\infty \tag{2}$$

is minimized. To enforce closed-loop stability one-sided weights of the form

$$W_o = (I + GK)^{-1}G, \quad W_i = I \tag{3}$$

$$W_o = I, \quad W_i = G(I + KG)^{-1} \tag{4}$$

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can be used, while performance-preserving considerations lead to two-sided weights

$$W_o = (I + GK)^{-1}G, \quad W_i = (I + GK)^{-1}. \quad (5)$$

To solve the approximation problem of minimizing the weighted error norm (1), a possible suboptimal but easy to apply approach has been proposed by Enns (1984). This approach is summarized below:

FWMR procedure

1. Compute an additive stable–unstable spectral decomposition of \mathbf{G} as $\mathbf{G} = \mathbf{G}_s + \mathbf{G}_u$, where \mathbf{G}_s , of order n_s , contains the stable poles of \mathbf{G} and \mathbf{G}_u , of order $n - n_s$, contains the unstable poles of \mathbf{G} .
2. Compute the controllability grammian of $\mathbf{G}_s \mathbf{W}_i$ and the observability grammian of $\mathbf{W}_o \mathbf{G}_s$ and define, according to Enns (1984) (see Section 2), appropriate n_s order frequency-weighted controllability and observability grammians P_E and Q_E , respectively.
3. Using P_E and Q_E in place of standard grammians of \mathbf{G}_s , determine a reduced order approximation \mathbf{G}_{sr} by applying the *balanced truncation* (BT) approach of Moore (1981).
4. Compute $\mathbf{G}_r = \mathbf{G}_{sr} + \mathbf{G}_u$.

In this paper we discuss several enhancements of Enns' approach. A first possibility which we only mention in passing, is to use the *singular perturbation approximation* (SPA) method (Liu & Anderson, 1989) instead of BT at step 3 of the FWMR procedure. As revealed by several examples given in (Varga & Anderson, 2001) (see also Section 5), the SPA used in conjunction with frequency-weighted balancing techniques exhibits superior performance over the more traditionally used BT method, both in preserving stability as well as in providing better approximations.

Further enhancing of Enns' method tries to remedy the main difficulty with this method, namely, the lack of guarantee of stability of the reduced model in the case of two-sided weighting. In Section 2 we propose new parameterized selection schemes of the grammians which combine the advantages of Enns' method with those of the stability guaranteeing techniques proposed by Lin and Chiu (1992) and Wang, Sreeram, and Liu (1999). The numerical example in Section 5 illustrates the additional flexibility provided by the new selection schemes.

In Section 3 we discuss enhancements of Enns' approach when applied to the controller reduction problem (2) for the special weights defined in (3)–(5). The main result shows that for an *arbitrary* stabilizing controller, the computation of grammians can be efficiently performed by solving Lyapunov equations of order at most $n + n_c$ (instead the expected order of $n + 2n_c$). This result and its corollary extend similar results of Schelfhout and De Moor (1996), established for *stable* controllers and strictly proper plants,

and of Liu, Anderson, and Ly (1990), stated for *stable* state-feedback and full-order estimator-based controllers, respectively.

Reliable numerical methods for unweighted approximations are well understood and accompanying robust numerical software based on *balancing-free square-root* accuracy-enhancing techniques (Varga, 1991a,b) is freely available (Varga, 2001). In Section 4 we extend these techniques to the balancing-related frequency-weighted approximation method. Specifically, we develop formulas to compute the Cholesky (square-root) factors of the frequency-weighted grammians for all proposed choices in Sections 2 and 3.

2. Balancing-related frequency-weighted model reduction

2.1. Review of known techniques

Balancing-related model reduction can be interpreted as performing a similarity transformation Z on a *stable* original system $\mathbf{G} = (A, B, C, D)$ yielding

$$\left[\begin{array}{c|c} Z^{-1}AZ & Z^{-1}B \\ \hline CZ & D \end{array} \right] := \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array} \right] \quad (6)$$

and then defining the reduced model of order r as $\mathbf{G}_r = (A_{11}, B_1, C_1, D)$. When partitioning compatibly Z and Z^{-1} in the form

$$Z := [T \ U], \quad Z^{-1} := \begin{bmatrix} L \\ V \end{bmatrix}, \quad (7)$$

then $\Pi = TL$ is a projector on T along L , and $LT = I_r$. Thus the reduced system is given by

$$\mathbf{G}_r = (LAT, LB, CT, D). \quad (8)$$

The matrices L and T , called *truncation matrices*, are the only ones necessary to determine the matrices of the reduced model \mathbf{G}_r .

Partitioned representations as in (6) can also be used to construct a SPA \mathbf{G}_r of \mathbf{G} by employing the SPA formulas

$$\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array} \right] = \left[\begin{array}{cc|c} A_{11} - A_{12}A_{22}^{-1}A_{21} & B_1 - A_{12}A_{22}^{-1}B_2 \\ \hline C_1 - C_2A_{22}^{-1}A_{21} & D - C_2A_{22}^{-1}B_2 \end{array} \right]. \quad (9)$$

Note that the SPA method preserves the DC-gain of an original stable system.

For balancing-related model reduction methods, the computation of suitable truncation matrices relies on two positive semidefinite symmetric matrices P and Q , called generically the controllability and observability grammians, respectively. For the unweighted BT and SPA methods applied to a system \mathbf{G} (Moore, 1981; Liu & Anderson, 1989),

P and Q satisfy a pair of Lyapunov equations:

$$\begin{aligned} AP + PA^T + BB^T &= 0, \\ A^T Q + QA + C^T C &= 0. \end{aligned} \tag{10}$$

Since the grammians of a stable system are positive semidefinite matrices, the stability of the reduced models follows from the guaranteed positive semidefiniteness of their grammians.

The grammians can be always determined in Cholesky factorized forms $P = SS^T$ and $Q = R^T R$, where S and R are upper-triangular matrices. The computation of the truncation matrices L and T can be done from the *singular value decomposition* (SVD)

$$RS = [U_1 \quad U_2] \text{diag}(\Sigma_1, \Sigma_2) [V_1 \quad V_2]^T,$$

where $\Sigma_1 = \text{diag}(\sigma_1, \dots, \sigma_r)$, $\Sigma_2 = \text{diag}(\sigma_{r+1}, \dots, \sigma_n)$, and $\sigma_1 \geq \dots \geq \sigma_r > \sigma_{r+1} \geq \dots \geq \sigma_n \geq 0$. Note that $\sigma_1, \dots, \sigma_n$ are the *Hankel-singular values* of the system \mathbf{G} . To compute L and T , the formulas

$$L = \Sigma_1^{-1/2} U_1^T R, \quad T = S V_1 \Sigma_1^{-1/2} \tag{11}$$

proposed by Tombs and Postlethwaite (1987) can be used. This is the so-called *square-root* method to compute the truncation matrices, and its name suggests that all computations are done in terms of square-root quantities (i.e., the Cholesky factors).

The *frequency-weighted balanced truncation* (FWBT) and *frequency-weighted singular perturbation approximation* (FWSPA) methods extend the unweighted BT and SPA methods to the frequency-weighted case and rely on so-called *frequency-weighted grammians*. The first FWBT approach has been proposed by Enns (1984). From now, let assume G and the two weights W_o and W_i are all stable TFMs, and let $\mathbf{W}_o = (A_o, B_o, C_o, D_o)$ and $\mathbf{W}_i = (A_i, B_i, C_i, D_i)$ be minimal state-space realizations of the weighting matrices. Consider the following realizations of $\mathbf{G}\mathbf{W}_i$ and $\mathbf{W}_o\mathbf{G}$:

$$\mathbf{G}\mathbf{W}_i = \begin{bmatrix} \bar{A}_i & \bar{B}_i \\ \bar{C}_i & \bar{D}_i \end{bmatrix} = : \begin{bmatrix} A & BC_i & BD_i \\ 0 & A_i & B_i \\ C & DC_i & DD_i \end{bmatrix}, \tag{12}$$

$$\mathbf{W}_o\mathbf{G} = \begin{bmatrix} \bar{A}_o & \bar{B}_o \\ \bar{C}_o & \bar{D}_o \end{bmatrix} = : \begin{bmatrix} A_o & B_o C & B_o D \\ 0 & A & B \\ C_o & D_o C & D_o D \end{bmatrix}. \tag{13}$$

Let

$$\bar{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \tag{14}$$

be the controllability grammian of $\mathbf{G}\mathbf{W}_i$ and the observability grammian of $\mathbf{W}_o\mathbf{G}$, respectively, partitioned such that P_{11} and Q_{22} are $n \times n$ matrices. The approach proposed by Enns defines

$$P_E = P_{11}, \quad Q_E = Q_{22} \tag{15}$$

as the frequency-weighted controllability and observability grammians, respectively. Although this method has been successfully employed in many applications, its main weakness is the lack of guaranteed stability of the reduced model in the case of two-sided weighting ($W_o \neq I$ and $W_i \neq I$) (Anderson & Liu, 1989).

An alternative approach was proposed by Lin and Chiu (1992) which, under certain assumptions, guarantees stability in the case of two-sided weighting. Provided P_{22} and Q_{11} are nonsingular (condition ensured if \mathbf{W}_i and \mathbf{W}_o are minimal realizations), the frequency-weighted grammians are chosen as

$$\begin{aligned} P_L &= P_{11} - P_{12} P_{22}^{-1} P_{12}^T, \\ Q_L &= Q_{22} - Q_{12}^T Q_{11}^{-1} Q_{12}. \end{aligned} \tag{16}$$

Since P_L and Q_L satisfy this time Lyapunov equations of the form (10) (however with different B and C matrices), the stability of the reduced model is automatically guaranteed (Pernebo & Silverman, 1982). The main weakness of this approach is the requirement that no pole-zero cancellations occur when forming $\mathbf{G}\mathbf{W}_i$ or $\mathbf{W}_o\mathbf{G}$. For example, this restriction prevents the applicability of this method to solve controller reduction problems involving weights as in (3), (4), or (5), where pole-zero cancellation takes always place (see also Section 3).

Another modification of the method of Enns which guarantees stability in the case of two-sided weighting has been proposed by Wang et al. (1999) for continuous-time systems. Two frequency-weighted grammians P_W and Q_W are determined as the solutions of the pair of Lyapunov equations

$$\begin{aligned} AP_W + P_W A^T + \tilde{B} \tilde{B}^T &= 0, \\ Q_W A + A^T Q_W + \tilde{C}^T \tilde{C} &= 0, \end{aligned} \tag{17}$$

where \tilde{B} and \tilde{C} are fictitious input and output matrices. These matrices are computed from the orthogonal eigendecompositions of the symmetric matrices $X = -AP_E - P_E A^T$ and $Y = -A^T Q_E - Q_E A$:

$$X = U \Theta U^T, \quad Y = V \Gamma V^T, \tag{18}$$

where Θ and Γ are real diagonal matrices. Specifically, \tilde{B} and \tilde{C} are determined as

$$\tilde{B} = U |\Theta|^{1/2}, \quad \tilde{C} = |\Gamma|^{1/2} V^T,$$

where $|\cdot|$ denotes a matrix formed from the absolute values of its elements. It is easy to see that with this choice of grammians we have $P_W - P_E = : \Delta P_W \geq 0$ and $Q_W - Q_E = : \Delta Q_W \geq 0$. Thus, the system $(A, \tilde{B}, \tilde{C})$ is minimal provided the original system is minimal.

2.2. New developments

It is possible to construct frequency-weighted grammians which lead to a combination of the approach of Lin and Chiu

(1992) with Enns' method. In such a *combination method* we can use as frequency-weighted grammians

$$P_{\text{EL}} = P_{11} - \alpha_c^2 P_{12} P_{22}^{-1} P_{12}^T, \quad (19)$$

$$Q_{\text{EL}} = Q_{22} - \alpha_o^2 Q_{12}^T Q_{11}^{-1} Q_{12},$$

where for $\alpha_c = \alpha_o = 0$ we have the choice for Enns' method, while for $\alpha_c = \alpha_o = 1$ we have the choice for the method of Lin and Chiu (1992) with stability guarantee. Because stability is guaranteed for $\alpha_c = \alpha_o = 1$, it is to be expected this to be also true for nearby subunitary values of α_c and α_o (on the basis of a continuous variation of spectrum with α_c and α_o). Thus, stability will be guaranteed in a whole neighborhood of $\alpha_c = \alpha_o = 1$ regardless of whether pole-zero cancellations occur or not. This feature can be seen as a simultaneous enhancement of both methods.

Concerning enhancements of the approach of Wang et al. (1999), it appears that in some cases the distance to Enns' choice (i.e., the sizes of ΔP_{W} and ΔQ_{W}) can be reduced by another choice of \tilde{B} and \tilde{C} . Consider $\Theta = \text{diag}(\Theta_1, \Theta_2)$ and $\Gamma = \text{diag}(\Gamma_1, \Gamma_2)$ in the decompositions of X and Y in (18) partitioned such that $\Theta_1 > 0$ and $\Theta_2 \leq 0$, $\Gamma_1 > 0$ and $\Gamma_2 \leq 0$. Partition $U = [U_1 \ U_2]$ and $V = [V_1 \ V_2]$ in accordance with the partitioning of Θ and Γ , respectively, and define

$$\tilde{B} = U_1 \Theta_1^{1/2}, \quad \tilde{C} = \Gamma_1^{1/2} V_1^T. \quad (20)$$

With this choice, an a priori weighted approximation error bound as in (Wang et al., 1999) holds, provided some rank conditions are additionally fulfilled. This modification of the method of Wang et al. (1999) appears to be useful in some applications (see example in Section 5).

It is possible now to consider a *modified combination method*, where we include all above modifications in a single parameterized approach, which guarantees the stability of reduced models for two-sided weighting. This can be done by defining new fictitious input and output matrices \hat{B} and \hat{C} , respectively, such that the frequency-weighted grammians P_{V} and Q_{V} satisfy

$$AP_{\text{V}} + P_{\text{V}}A^T + \hat{B}\hat{B}^T = 0, \quad (21)$$

$$Q_{\text{V}}A + A^TQ_{\text{V}} + \hat{C}^T\hat{C} = 0,$$

\hat{B} and \hat{C} are defined just like \tilde{B} and \tilde{C} in (20), using the decompositions (18) but with $X = -AP_{\text{EL}} - P_{\text{EL}}A^T$ and $Y = -A^TQ_{\text{EL}} - Q_{\text{EL}}A$. Here, P_{EL} and Q_{EL} are the frequency-weighted grammians in (19) used in the combination method.

The new selection schemes provide a wide range of possibilities for combinations in choosing grammians, thus enlarging significantly the applicability of balancing-related approach to FWMR. For example, any combination of grammians ($P_{\text{EL}}, Q_{\text{V}}$), ($P_{\text{V}}, Q_{\text{EL}}$), or ($P_{\text{V}}, Q_{\text{V}}$), for all subunitary values of parameters α_c and α_o guarantees the stability of approximations for two-sided weighting. Note that the

increase of computational cost associated with the use of the new selection schemes is practically negligible.

3. Frequency-weighted controller reduction

In this section we consider the application of balancing-related FWMR techniques to solve the controller reduction problem (2) for the special weights defined in (3), (4), or (5). Assume that the open-loop system $\mathbf{G} = (A, B, C, D)$ has order n and the stabilizing controller $\mathbf{K} = (A_c, B_c, C_c, D_c)$ has order n_c . In the case of a stable controller, the computation of the frequency-weighted grammians P_{E} and Q_{E} apparently involves the solution of Lyapunov equations of order $n + 2n_c$. Controller synthesis methods based on the LQG- or H_{∞} -design methodologies lead typically to a controller order $n_c = n$, so that, for the reduction of such controllers the solution of Lyapunov equations of order $3n$ are apparently necessary. In what follows we show that it is always possible to solve Lyapunov equations of order at most $n + n_c$ to compute the frequency-weighted controllability and observability grammians in controller reduction problems with the above special weights.

Since in general the controller can be unstable, only the stable part of the controller is reduced and a copy of the unstable part is kept in the reduced controller. Therefore, we assume a state-space representation of the controller with A_c already reduced to a block-diagonal form

$$\mathbf{K} = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array} \right] = \left[\begin{array}{cc|c} A_{c1} & 0 & B_{c1} \\ 0 & A_{c2} & B_{c2} \\ \hline C_{c1} & C_{c2} & D_c \end{array} \right], \quad (22)$$

where $\Lambda(A_{c1}) \subset C^+$ and $\Lambda(A_{c2}) \subset C^-$. Here C^- denotes the open left half complex plane, while C^+ denotes the complement of C^- in C . The above form corresponds to an additive decomposition of the controller TFM as $K = K_u + K_s$, where $\mathbf{K}_u = (A_{c1}, B_{c1}, C_{c1}, 0)$ contains the unstable poles of \mathbf{K} and $\mathbf{K}_s = (A_{c2}, B_{c2}, C_{c2}, D_c)$, of order n^- , contains the stable poles of \mathbf{K} .

For our developments, we build the state matrix of the realizations of the weights in (3), (4), or (5) in the form

$$A_{\text{w}} = \left[\begin{array}{cc} A - BD_cR^{-1}C & B\tilde{R}^{-1}C_c \\ -B_cR^{-1}C & A_c - B_cR^{-1}DC_c \end{array} \right],$$

where $R = I + DD_c$ and $\tilde{R} = I + D_cD$. Since the controller is stabilizing, A_{w} has all its eigenvalues in C^- .

The following theorem extends the result of Schelfhout and De Moor (1996) obtained for a stable controller stabilizing a strictly proper plant ($D = 0$).

Theorem 1. *For a given n th order continuous-time system $\mathbf{G} = (A, B, C, D)$ assume that $\mathbf{K} = (A_c, B_c, C_c, D_c)$ is an n_c th order stabilizing controller with $I + DD_c$ nonsingular and having a state-space realization of the form (22). Then, the*

frequency-weighted controllability and observability grammians, P_E and Q_E , respectively, for Enns' method applied to frequency-weighting controller reduction problems with weights defined in (3), (4), or (5) can be computed by solving Lyapunov equations of order at most $n + n_c$ as follows:

(1) For $W_o = (I + GK)^{-1}G$ and $W_i = I$, P_E satisfies

$$A_{c2}P_E + P_E A_{c2}^T + B_{c2}B_{c2}^T = 0 \quad (23)$$

and Q_E is the $n^- \times n^-$ trailing block of Q_o satisfying

$$A_w^T Q_o + Q_o A_w + C_o^T C_o = 0 \quad (24)$$

with $C_o = [-R^{-1}C \quad -R^{-1}DC_c]$.

(2) For $W_o = I$ and $W_i = G(I + GK)^{-1}$, P_E is the $n^- \times n^-$ trailing block of P_1 satisfying

$$A_w P_i + P_i A_w^T + B_i B_i^T = 0 \quad (25)$$

with

$$B_i = \begin{bmatrix} -B\tilde{R}^{-1} \\ B_c D\tilde{R}^{-1} \end{bmatrix}$$

and Q_E satisfies

$$A_{c2}^T Q_E + Q_E A_{c2} + C_{c2}^T C_{c2} = 0. \quad (26)$$

(3) For $W_o = (I + GK)^{-1}G$ and $W_i = (I + GK)^{-1}$, P_E is the $n^- \times n^-$ trailing block of P_1 satisfying (25) with

$$B_i = \begin{bmatrix} BD_c R^{-1} \\ B_c R^{-1} \end{bmatrix}$$

and Q_E is the $n^- \times n^-$ trailing block of Q_o satisfying (24).

Proof. For details see (Varga & Anderson, 2002).

By solving lower order Lyapunov equations, the achieved reduction of the computational effort can be significant for large order systems. For example, if $n_c = n$, by solving $2n$ order Lyapunov equations instead of $3n$ order ones, the computation of one of grammians can be done $(3/2)^3 \approx 3.4$ times faster.

If we try to apply the method of Lin and Chiu (1992), we get $P_L = 0$, and/or $Q_L = 0$, because pole-zero cancellations take place when forming $W_o K_s$ and/or $K_s W_i$. It follows that this method is not applicable for this class of controller reduction problems. Since the grammians for the proposed combination method are $P_{EL} = (1 - \alpha_c^2)P_E$ and $Q_{EL} = (1 - \alpha_o^2)Q_E$, this choice of grammians for subunitary α_c and α_o is practically the same as for Enns' method. To guarantee the stability of the reduced stable part of the controller, the choice P_V or Q_V corresponding to the modified combination method is appropriate.

Simplifications arise also in the case of a state-feedback and full-order observer based controller of the form

$$K = (A + BF + LC + LDF, L, F, 0).$$

The following result extends Lemma 1 of Liu et al. (1990) to the case of possibly unstable controllers.

Corollary 2. For a given n th order system $\mathbf{G} = (A, B, C, D)$ suppose F is a state feedback gain and L is a state estimator gain, such that $A + BF$ and $A + LC$ are stable. Then the frequency-weighted controllability and observability grammians for the method of Enns (1984) applied to the frequency-weighted controller reduction problems with weights defined in (3), (4), or (5) can be computed by solving Lyapunov equations of order at most $2n$.

In the case of state feedback and observer based controllers important computational effort saving results if we further exploit the problem structure. In this case

$$A_w = \begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix}$$

and this matrix can be put in an upper block diagonal form using the transformation matrix

$$T = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}.$$

We obtain the transformed matrices $\tilde{A}_w := T^{-1}A_w T$, $\tilde{B}_i := T^{-1}B_i$, and $\tilde{C}_o := C_o T$, where

$$\tilde{A}_w = \begin{bmatrix} A + BF & BF \\ 0 & A + LC \end{bmatrix}.$$

If \tilde{P}_i and \tilde{Q}_o satisfy

$$\begin{aligned} \tilde{A}_w \tilde{P}_i + \tilde{P}_i \tilde{A}_w^T + \tilde{B}_i \tilde{B}_i^T &= 0 \\ \tilde{A}_w^T \tilde{Q}_o + \tilde{Q}_o \tilde{A}_w + \tilde{C}_o^T \tilde{C}_o &= 0 \end{aligned} \quad (27)$$

then P_i in (25) and Q_o in (24) are given by $P_i = T\tilde{P}_i T^T$ and $Q_o = T^{-T}\tilde{Q}_o T^{-1}$, respectively. The computational saving arises from the need to reduce A_w to a real Schur form (RSF) when solving the Lyapunov equations (24) and (25). Instead of reducing the $2n \times 2n$ matrix A_w , we can reduce two $n \times n$ matrices $A + BF$ and $A + LC$ to obtain \tilde{A}_w in a RSF. This means a 4 times speedup of computations for this step.

4. Accuracy-enhancing techniques

4.1. Square-root and balancing-free techniques

The emphasis on improving the accuracy of computations in model reduction has led to the so-called model reduction algorithms with enhanced accuracy. Several techniques can be employed to enhance the accuracy of computed reduced models. The square-root (SR) method, introduced by Tombs and Postlethwaite (1987), determines the truncation matrices L and T according to (11). These matrices are

then used to compute the reduced order model via (8). Thus, the computation of the matrices of the reduced model can be done entirely on the basis of the Cholesky factors S and R of the positive semidefinite grammians, satisfying $P = SS^T$ and $Q = R^T R$. This approach is numerically appealing because the upper triangular Cholesky factors R and S can be computed directly by solving the Lyapunov equations satisfied by the grammians (Hammarling, 1982). For well-equilibrated systems this approach is usually numerically very accurate. The alternative to compute first the grammians and then their Cholesky factors is numerically unsatisfactory, because roundoff errors can lead to the loss of positive definiteness of the computed grammians and thus make the computation of factors numerically unreliable.

Potential accuracy losses can be induced in the reduced model if either of the truncation matrices L or T is numerically nearly rank deficient. Recall that T and L in the **SR** method, are the first r columns and first r rows of the balancing transformation matrix Z in (7) and its inverse, respectively. If the original system is highly unbalanced, Z can become very ill-conditioned and thus the accuracy of computations can be adversely affected. To avoid ill-conditioned truncation matrices, a *balancing-free* (**BF**) approach has been proposed by Safonov and Chiang (1989) in which L and T are always well-conditioned if a sufficient gap exists between the singular values σ_r and σ_{r+1} (i.e., $\sigma_r \gg \sigma_{r+1}$). The truncation matrices are computed from two matrices whose orthogonal columns span bases for the right and left eigenspaces of the product PQ corresponding to the first r largest eigenvalues $\sigma_1^2, \dots, \sigma_r^2$. However, because of the need to compute explicitly P and Q as well as their product, this approach is usually less accurate for moderately ill-balanced systems than the **SR** approach.

A *balancing-free square-root* (**BFSR**) algorithm which combines the advantages of the **BF** and **SR** approaches has been proposed by Varga (1991b). Among several ways to choose L and T , a convenient choice is

$$L = (Y^T X)^{-1} Y^T, \quad T = X,$$

where X and Y are $n \times r$ matrices with orthonormal columns computed from two QR decompositions

$$SV_1 = XW, \quad R^T U_1 = YZ$$

with W and Z non-singular and upper-triangular. Note that by this choice, the reduced model computed by the BT method is related by a system similarity transformation to the reduced model obtained by the **SR** method. The above choice of truncation matrices involves two QR-decompositions and the inversion of an $r \times r$ matrix and is cheaper than the choice suggested by Safonov and Chiang (1989) involving, instead inversion, the SVD of the product $Y^T X$. The accuracy of the **BFSR** algorithm is usually better than either of **SR** or **BF** approaches.

The SPA method can be used directly on a balanced minimal order realization of the original system computed with

the **SR** method. A **BFSR** method to compute SPAs has been proposed by Varga (1991a). The essence of this method is to construct the truncation matrices L and T such that the system (LAT, LB, CT, D) is a minimal realization of the original system and the product of corresponding grammians is block-diagonal, allowing thus the application of the SPA formulas.

The effectiveness of the **SR** or **BFSR** techniques depends mostly on the accuracy of the computed Cholesky factors of the grammians. Having the Cholesky factors for the frequency-weighted grammians, we can compute in a numerically reliable way the truncation matrices L and T using either the **SR** or **BFSR** technique. In what follows, we develop explicit formulas to compute the Cholesky factors of the frequency-weighted grammians for the balancing-related frequency-weighted model and controller reduction. With these formulas, the new **SR** or **BFSR** formulations of both the FWBT and FWSPA methods are entirely satisfactory from a numerical point of view and can serve as a basis for robust software implementations of the frequency-weighted balancing-related approach.

4.2. Square-root techniques for FWMR

In this section we show how square-root formulas can be employed to compute the frequency-weighted grammians for the specific choices described in the Section 2. Assume \bar{S} and \bar{R} are the Cholesky factors of \bar{P} and \bar{Q} in (14), respectively, satisfying $\bar{P} = \bar{S} \bar{S}^T$ and $\bar{Q} = \bar{R}^T \bar{R}$. These factors can be computed with the method of Hammarling (1982) by solving directly for the Cholesky factors the Lyapunov equations

$$\begin{aligned} \bar{A}_1 \bar{P} + \bar{P} \bar{A}_1^T + \bar{B}_1 \bar{B}_1^T &= 0, \\ \bar{A}_0^T \bar{Q} + \bar{Q} \bar{A}_0 + \bar{C}_0^T \bar{C}_0 &= 0. \end{aligned} \quad (28)$$

If we partition \bar{S} and \bar{R} as

$$\bar{S} = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$

then the Cholesky factors of $P_E = S_E S_E^T$ and $Q_E = R_E^T R_E$ corresponding to Enns' method are given by

$$\begin{aligned} S_E S_E^T &= S_{11} S_{11}^T + S_{12} S_{12}^T = [S_{11} \quad S_{12}] \begin{bmatrix} S_{11}^T \\ S_{12}^T \end{bmatrix}, \\ R_E^T R_E &= R_{22}^T R_{22} + R_{12}^T R_{12} = [R_{22}^T \quad R_{12}^T] \begin{bmatrix} R_{22} \\ R_{12} \end{bmatrix}. \end{aligned}$$

Thus, to compute S_E the RQ-factorization of the matrix $[S_{11} \quad S_{12}]$ must be performed, while for computing R_E the QR-factorization of $[R_{22}^T \quad R_{12}^T]^T$ must be performed. Both these factorizations can be computed using factorization updating techniques (Gill, Golub, Murray, & Saunders, 1974)

which fully exploit the upper triangular shapes of S_{11} and R_{22} .

The Cholesky factors of $P_L = S_L S_L^T$ and $Q_L = R_L^T R_L$ for the method of Lin and Chiu (1992) are

$$S_L = S_{11}, \quad R_L = R_{22}.$$

Thus, by working directly with the Cholesky factors we completely avoid the matrix inversions necessary when forming P_L and Q_L directly from (16). Therefore, no difficulties are expected to be encountered when $P_{22} = S_{22} S_{22}^T$ or $Q_{11} = R_{11}^T R_{11}$ are ill-conditioned or even exactly singular (i.e., the state space representations of W_i and/or W_o are not minimal).

For the *combination method*, the Cholesky factors of $P_{EL} = S_{EL} S_{EL}^T$ and $Q_{EL} = R_{EL}^T R_{EL}$ are given by

$$S_{EL} S_{EL}^T = S_{11} S_{11}^T + (1 - \alpha_c^2) S_{12} S_{12}^T,$$

$$R_{EL}^T R_{EL} = R_{22}^T R_{22} + (1 - \alpha_c^2) R_{12}^T R_{12}$$

and can be computed from the RQ-decomposition of $[S_{11} \ \sqrt{1 - \alpha_c^2} S_{12}]$ and QR-decomposition of $[R_{22}^T \ \sqrt{1 - \alpha_c^2} R_{12}^T]^T$. For the *modified combination method*, the Cholesky factors of $P_V = S_V S_V^T$ and $Q_V = R_V^T R_V$ result by solving (21) directly for these factors using the algorithm of Hammarling (1982).

The solution of the Lyapunov equations (28) involves the orthogonal reduction of both \bar{A}_i and \bar{A}_o to RSF. This reduction can automatically be achieved by performing appropriate orthogonal similarity transformations on the state-space realizations \mathbf{G} , \mathbf{W}_i and \mathbf{W}_o to reduce independently the state matrices A , A_i and A_o to RSF.

4.3. Square-root techniques for controller reduction

We can employ the method of Hammarling (1982) to solve (25) and (24) directly for the Cholesky factors S_i of $P_i = S_i S_i^T$ and R_o of $Q_o = R_o^T R_o$, respectively. In the case of an unstable controller, we assume a state space realization of \mathbf{K} as in (22) with the $n^- \times n^-$ matrix A_{c2} containing the stable eigenvalues of A_c . If we partition S_i and R_o in the form

$$S_i = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad R_o = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix},$$

where both S_{22} and R_{22} are $n^- \times n^-$, then the Cholesky factor of the trailing block of P_i in (25) corresponding to the stable part of \mathbf{K} is simply $S_E = S_{22}$, while the Cholesky factor R_E of the trailing block of Q_o in (24) satisfies $R_E^T R_E = R_{22}^T R_{22} + R_{12}^T R_{12}$. Thus the computation of R_E involves an additional QR-decomposition of $[R_{22}^T \ R_{12}^T]^T$ and can be computed using standard updating techniques (Gill et al., 1974). Updating can be avoided in the case of one-sided weight $W_o = (I + GK)^{-1} G$, by using alternative state-space realizations of

W_o and \mathbf{K} . For details, see (Varga & Anderson, 2002). Still in the case of two-sided weighting with $W_o = (I + GK)^{-1} G$ and $W_i = (I + GK)^{-1}$ we prefer the approach of the Theorem 1 with W_i and W_o sharing the same state matrix A_w , because the computation of both grammians can be done with a single reduction of this $(n + n_c) \times (n + n_c)$ matrix to the RSF. In this case the cost to compute the two grammians is practically the same as for one grammian.

For a state-feedback and full-observer based controller, let \tilde{S}_i be the Cholesky factor of \tilde{P}_i in (27) partitioned as

$$\tilde{S}_i = \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ 0 & \tilde{S}_{22} \end{bmatrix}.$$

The $n^- \times n^-$ Cholesky factor S_E corresponding to the trailing $n^- \times n^-$ part of P_i is the trailing $n^- \times n^-$ block of an upper triangular matrix \hat{S}_{22} which satisfies

$$\hat{S}_{22} \hat{S}_{22}^T = \tilde{S}_{11} \tilde{S}_{11}^T + (\tilde{S}_{12} + \tilde{S}_{22})(\tilde{S}_{12} + \tilde{S}_{22})^T$$

\hat{S}_{22} can be computed easily from the RQ-decomposition of $[\tilde{S}_{11} \ \tilde{S}_{12} + \tilde{S}_{22}]$ using standard factorization updating formulas (Gill et al., 1974). No difference appears in the computation of the Cholesky factor R_E .

5. Numerical example

We present an example which is only intended to illustrate the proposed enhancements of Enns' method. This is the fourth-order model used in Example 1 of Wang et al. (1999) with the frequency weights

$$\mathbf{W}_i = \mathbf{W}_o = (-4.5I_2, 3I_2, 1.5I_2, I_2).$$

We computed approximations \mathbf{G}_r of orders $r = 1, 2$ and 3 using the combination method for values of $\alpha = \alpha_c = \alpha_o, 0, 0.5$ and 1 . Recall that $\alpha = 0$ corresponds to Enns' method while $\alpha = 1$ corresponds to the method of Lin and Chiu (1992). In Table 1 we present the resulting weighted approximation error $\|W_o(G - G_r)W_i\|_\infty$ for the FWBT and FWSPA methods. For comparison purposes, we have also added in the last column the results reported for the method of Wang et al. (1999) (WSL).

The modified combination method, with $\alpha = 0$ and the pair (P_V, Q_V) computed as in (21), leads for both FWBT and FWSPA to almost the same errors as those in Table 1. In all cases the errors are smaller than those for the method of Wang et al. (1999). The FWSPA method for $\alpha = 0$ (see values in bold face) behaves uniformly better on this example than all other approaches. We postulate that the good behaviour is probably due to the fact that, by forcing a zero error in the steady-state gain, lower error results also in the low frequency domain which partly overlaps with the frequency region where the weights are most active.

Table 1
Weighted errors for the FWBT, FWSPA and WSL methods

| r | FWBT | | | FWSPA | | | WSL |
|-----|--------------|----------------|--------------|--------------|----------------|--------------|-------|
| | $\alpha = 0$ | $\alpha = 0.5$ | $\alpha = 1$ | $\alpha = 0$ | $\alpha = 0.5$ | $\alpha = 1$ | |
| 1 | 2.112 | 2.116 | 2.566 | 1.405 | 1.495 | 2.035 | 2.121 |
| 2 | 0.265 | 0.261 | 0.560 | 0.250 | 0.256 | 0.687 | 0.272 |
| 3 | 0.112 | 0.110 | 0.164 | 0.065 | 0.069 | 0.121 | 0.115 |

6. Conclusions

We proposed three enhancements of Enns' method which substantially enlarge the applicability of balancing related FWMR approaches: (1) the FWSPA approach; (2) new choices of the frequency-weighted controllability and observability grammians ensuring the stability of approximations for two-sided frequency weighting; and (3) increased numerical accuracy of overall computations by extending the balancing-free square-root techniques to the frequency-weighted balancing-related model reduction. The proposed enhancements can be immediately applied to solve frequency-weighted controller reduction problems as well. For a special class of stability/performance preserving controller reduction problems, we have shown that by applying the FWMR approach, a significant reduction of the computational burden can be achieved by exploiting the particular structure of the problem. All these results apply in the discrete-time case as well (Varga & Anderson, 2001, 2002). For the newly developed model and controller reduction methods, robust numerical software has been implemented by Varga (2002) as part of the freely available Fortran 77 library SLICOT. To facilitate the usage of these tools, easy-to-use, flexible and user friendly interfaces have been developed to integrate them in MATLAB.

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