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Mapping MIMO control system specifications into parameter space

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Abstract

This paper considers the mapping of design objectives for parametric multi-input multi-output systems into parameter space. To this end mapping equations for standard norm specifications are derived. Design objectives under consideration include H_∞ performance, H_2 performance and dissipativity. The mapping equations are derived from a novel uniform framework. These presented mapping equations are similar to the well-known mapping equations for pole location specifications. The theory which extends the parameter space approach to MIMO systems is illustrated by an example.

1 Introduction

The parameter space approach has been one of the earliest concepts to analyze and design control systems subject to structured (parametric) perturbations. The classical purpose of parameter space methods is to establish a direct correlation between roots of the characteristic equation and adjustable or uncertain parameters appearing in coefficients of the equation. Robust controller design based on this method is thoroughly presented in [1] and supported by publicly available software [2].

Recently there has been an increased interest in mapping frequency-domain specifications into a parameter plane. Incorporating these specifications into the controller synthesis enables the designer to take specifications into account which are difficult to express by time-domain specifications. Furthermore a robust design with respect to unstructured uncertainties such as high-frequency parasitic modes is possible. H_∞ norm specifications have been mapped into parameter space in [3] for SISO systems using the magnitude of a single I/O channel, while [4] considered specifications arising from the Nyquist and Popov planes. A Routh-Hurwitz type criterion for the MIMO H_∞ norm was presented in [5] and frequency restricted magnitude bounds for interval rational functions were investigated.

The purpose of this paper is to derive the mapping equations of control design objectives for parametric multi-input multi-output systems into parameter space. We consider general well accepted and widely used norm specifications, namely the H_∞ and H_2 norm. In addition the derivation of the mapping equations for dissipativity (passivity), generalized H_2 norm, Hankel norm and the complex structured stability radius fits into the same framework.

This paper is organized as follows. MIMO specifications are reviewed in Section 2. Section 3 is a brief presentation of algebraic Riccati equations (ARE) and their basic properties. This section also introduces a theorem for parameter dependent algebraic Riccati equations due to [6], which will be used to establish the mapping equations. The main result of this paper, namely the mapping equations for MIMO specifications are presented in Section 4, followed by a robust control example in Section 5. Finally in Section 6 we summarize the most important conclusions.

2 MIMO specifications in control theory

This section reviews the various specifications and objectives relevant for design and analysis of multivariable control systems. All specifications will be formulated using algebraic Riccati equations or Lyapunov equations. While there will be no special notation for parametric dependencies, the considered systems might depend on several real parameters $q \in \mathbb{R}^d$. Since the parameter space approach does not favor controller over plant uncertainties we will not discriminate these.

Consider uncertain, time-invariant systems with state-space realization

$$\dot{x} = A(q)x + B(q)u, \qquad y = C(q)x + D(q)u \tag{1}$$

or transfer matrix $H(s,q)$, i.e.

$$H(s,q) = C(q)(sI - A(q))^{-1}B(q) + D(q). \tag{2}$$

Our main objective is to map specifications relevant for dynamic systems (1) and (2) into the parameter space or a parameter plane.

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2.1 H_∞ performance

The H_∞ norm of (2) with A being stable and $\sigma_{\max}(D) < \gamma$, satisfies

$$||H(j\omega)|| < \gamma, \quad \forall \omega \in \mathbb{R}$$

if and only if the algebraic Riccati equation

$$\begin{aligned} \gamma X B S_r^{-1} B^* X \quad &+ \quad \gamma C^* S_l^{-1} C - X(A - BS_r^{-1} D^* C) \\ &- \quad (A - BS_r^{-1} D^* C)^* X = 0, \end{aligned} \tag{3}$$

has a hermitian solution X_0 such that all eigenvalues of $A - BB^* X_0$ lie in the open left half-plane, where $S_r = (D^*D - \gamma^2 I)$ and $S_l = (DD^* - \gamma^2 I)$.

2.2 Passivity and Dissipativity

Passivity is equivalent to the transfer matrix H being positive-real, i.e. $H(s) + H(s)^* \geq 0$, $\forall$ Re $s > 0$, which can be expressed in the time-domain as

$$\int_0^T u^T y \, dt \geq 0, \quad \forall x(0) = 0, T \geq 0. \tag{4}$$

Condition (4) is equivalent to the following statement [7]: There exists $X = X^*$ satisfying the ARE

$$A^*X + XA + (XB - C^*)(D + D^*)^{-1}(XB - C^*)^* = 0. \tag{5}$$

A system is said to have dissipation η if

$$\int_0^T (u^T y - \eta u^T u) \, dt \geq 0, \quad \forall x(0) = 0, T \geq 0$$

Thus passivity corresponds to nonnegative dissipation. A system has dissipativity η if the following ARE has a hermitian solution

$$\begin{aligned} (XB - C^*)(2\eta I - (D + D^*))^{-1}(XB - C^*)^* \\ + A^*X + XA = 0. \end{aligned} \tag{6}$$

Remark 1 *Both H_∞ and dissipativity specifications fit into the more general framework of quadratic constraints of the form*

$$\int_0^\infty \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} P & R \\ R^T & Q \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} dt \leq 0, \quad \begin{matrix} \forall x(0) = 0, \\ T \geq 0. \end{matrix} \tag{7}$$

2.3 Structured complex stability radius

The *complex structured stability radius* of the system

$$\dot{x}(t) = (A + U\Delta V)x(t) \tag{8}$$

is defined by

$$r_\mathbb{C} = \inf \{||\Delta|| : \sigma(A + U\Delta V) \cap \mathbb{C}_+ \neq \varnothing\}, \tag{9}$$

where Δ is a complex matrix of appropriate dimension, $\mathbb{C}_+$ denotes the closed right half plane and $||\Delta||$ the spectral norm of Δ. Following [8], $r_\mathbb{C}$ can be computed using the equality

$$r_\mathbb{C}(A, U, V) = ||V(sI - A)^{-1}U||_\infty^{-1}.$$

Thus the determination of the complex structured stability radius is equivalent to the computation of the H_∞ norm of a related transfer function.

2.4 H_2 performance

The H_2 norm is defined as

$$||H||_2 \doteq \left(\mathrm{Tr} \, \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) H(j\omega)^* d\omega \right)^{1/2}.$$

This norm is only finite if $D = 0$ (resp, $D(q) = 0$). Hence we assume $D = 0$ in this subsection.

The H_2 norm can be expressed as

$$||H||_2^2 = \mathrm{Tr}\,(B^T W_{obs} B), \tag{10}$$

where W_{obs} is the observability Gramian of the realization, which can be computed by solving the Lyapunov equation

$$A^T W_{obs} + W_{obs} A + C^T C = 0. \tag{11}$$

The H_2 norm is different from the specifications presented so far in that a specification $||H||_2 \leq \gamma$ cannot be expressed by an ARE. In that sense the H_2 norm does not really fit into the ARE framework. But this specification can be formulated by means of the more special Lyapunov equation, which is affine in the unknown W_{obs}.

2.5 Generalized H_2 norm

In the scalar case, the H_2 norm can be interpreted as the system gain, when the input are L_2 functions and the output bounded L_∞ time functions. Thus the scalar H_2 norm is a measure of the peak output amplitude for energy bounded input signals. Low values for this quantity are especially desirable if we want to avoid saturation in the system. Unfortunately this interpretation does not hold for the H_2 norm in the vector case. The so-called *generalized H_2 norm*, defined in [9], can be expressed as

$$||H||_{2,g} = \lambda_{\max}^{1/2}(B^T W_{obs} B), \text{ if } ||y||_\infty = \sup_{0 \leq t \leq \infty} ||y(t)||_2$$

or

$$||H||_{2,g} = d_{\max}^{1/2}\,(B^T W_{obs} B) \text{ if } ||y||_\infty = \sup_{0 \leq t \leq \infty} ||y(t)||_\infty$$

depending on the type of L_∞ norm chosen for the vector valued output y. Here, $\lambda_{\max}(\cdot)$ and $d_{\max}(\cdot)$ denote the maximum eigenvalue and maximum diagonal entry of a nonnegative matrix respectively.

2.6 Hankel norm

The Hankel norm of a system is a measure of the effect of the past system input on the future output. It is known that the Hankel norm is given by

$$||H||_{hankel} = \lambda_{\max}^{1/2}(W_{obs} W_{contr}),$$

where the controllability gramian W_{contr} is the solution of $AW_{contr} + W_{contr}A^T + BB^T = 0$.

The Gramian W_{obs} measures the energy that can appear in the output and W_{contr} measures the amount of energy that can be stored in the system state using an excitation with a given energy.

3 Algebraic Riccati equations

This section gives an overview of algebraic Riccati equations. The general algebraic matrix Riccati equation is given by

$$XRX - XP - P^*X - Q = 0, \tag{12}$$

where P, R and Q are given $n \times n$ complex matrices with Q and R hermitian. Although in most applications in control theory P, R and Q will be real, the results will be given for complex matrices were possible.

An ARE has in general many solutions. Real symmetric solutions, and especially the maximal solution, play a crucial role in the classical continuous time quadratic control problems.

For $R = 0$ the ARE reduces to an affine matrix equation in X. These so called Lyapunov equations have proven to be very useful in analyzing stability and controllability.

Associated with (12) is a $2n \times 2n$ Hamiltonian matrix:

$$M = \begin{bmatrix} -P & R \\ Q & P^* \end{bmatrix}. \tag{13}$$

The matrix M in (13) can be used to obtain the solutions of (12). For the parameter space approach these particular solutions are not relevant, but we will use some properties of (13). Namely the set of all eigenvalues of M is symmetric about the imaginary axis. To see that, introduce

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

It follows that $J^{-1}MJ = -JMJ = -M^*$. Thus M and $-M^*$ are similar and λ is an eigenvalue of M if and only if $-\bar{\lambda}$ is. The following theorem provides an important link between solutions of AREs and the Hamiltonian M.

Theorem 1 *Suppose that $R \geq 0$, $Q = Q^*$, (P, R) is stabilizable, and there is a hermitian solution of (12). Then for the maximal hermitian solution X_+ of (12), $P - RX_+$ is stable if and only if the matrix M from (13) has no eigenvalues on the imaginary axis.*

Thus the non-existence of pure imaginary eigenvalues is a necessary and sufficient condition that the ARE (12) has a maximal, stabilizing, hermitian solution.

Since the parameter space approach deals with uncertain parameters we will extend the previous results for invariant matrices to matrices with uncertain parameters in the next section.

3.1 Continuous and analytic dependence

Suppose now that the matrices P, Q and R are real analytic functions of a real parameter $q \in \mathbb{R}$, i.e. $P = P(q), Q = Q(q), R = R(q)$.

Theorem 2 *[Lancaster,Rodman,[6]] Let $P = P(q)$, $Q = Q(q)$, and $R = R(q)$ be analytic $n \times n$ matrix functions of q on a real interval $[q^-; q^+]$, with $R(q)$ positive semidefinite hermitian, $Q(q)$ hermitian, and $(P(q), R(q))$ stabilizable for every $q \in [q^-; q^+]$. Assume that for all $q \in [q^-; q^+]$, the Riccati equation*

$$X(q)R(q)X(q) - X(q)P(q) - P(q)^*X(q) - Q(q) = 0 \tag{14}$$

has a hermitian solution. Further assume that the number of pure imaginary or zero eigenvalues of

$$M(q) := \begin{bmatrix} -P(q) & R(q) \\ Q(q) & P(q)^* \end{bmatrix} \tag{15}$$

is constant. Then the maximal solution $X_+(q)$ of (14) is an analytic function of $q \in [q^-; q^+]$. Conversely, if $X_+(q)$ is an analytic function of $q \in [q^-; q^+]$, then the number of pure imaginary or zero eigenvalues of $M(q)$ is constant.

The previous result can be generalized to the case when $P(q), Q(q)$ and $R(q)$ are analytic functions of several real variables $q = (q_1, \ldots, q_d) \in \Omega$, where Ω is an open connected set in $\mathbb{R}^d$.

4 Mapping specifications into parameter space

In this section we present the mapping equations for the specifications given in Section 2 for systems with uncertain parameters.

In general, the parameter space approach maps specifications into the space of parameters. We are thus seeking the subspace for which the specifications are fulfilled. Especially important are parameter planes, since the resulting subspaces are simple regions which are easily visualized by their boundaries. This allows intuitive, interactive design of robust control systems.

For eigenvalue specifications the boundary of the desired region in the eigenvalue-plane is mapped into a parameter plane by the characteristic polynomial. Using the real and imaginary part of the characteristic polynomial we get two mapping equations which depend on a generalized frequency and the uncertain parameters. The mapping equations presented in this section will have a similar structure.

4.1 ARE based mapping

While we provided the definition of the H_∞, dissipativity and complex stability radius specifications, we showed that all of these specifications are equivalent to the existence of a maximal, hermitian solution of an ARE. Using Theorem 1 we can in turn formulate the adherence of the given specifications as the non-existence of pure imaginary eigenvalues of an associated Hamiltonian matrix.

Consider now the uncertain parameter case. Using Theorem 2 we can extend this equivalence to systems with analytic dependence on uncertain parameters. Given a specific parameter $q^* \in \mathbb{R}^n$ for which a maximal, hermitian solution $X_+(q^*)$ exists, we know from Theorem 1 that the Hamiltonian matrix (15) has no pure imaginary eigenvalues. Using Theorem 2 we can extend this property as long as the number of eigenvalues on the imaginary axis is constant. In other words, having found a parameter for which a specification described by an ARE holds, the same specification holds as long as the number of imaginary eigenvalues of the associated Hamiltonian (15) is zero and does not change. Hence the boundary of the subspace for which the desired specification holds is given by all parameters for which the number of pure imaginary eigenvalues of (15) changes. A new pair of imaginary eigenvalues of (15) only arises if either two complex eigenvalue pairs become a double eigenvalue pair on the imaginary axis or if a double real pair becomes a pure imaginary pair. Note: Another possibility is a drop in the rank of M, which corresponds to eigenvalues which go through infinity.

Let us first discuss the appearance of pure imaginary eigenvalues through a double pair on the imaginary axis. The matrix $M(q)$ has a double eigenvalue at $\lambda = j\omega$ if and only if

$$\begin{aligned} |j\omega I - M(q)| &= 0, \\ \frac{\partial}{\partial \omega}|j\omega I - M(q)| &= 0. \end{aligned} \tag{16}$$

A necessary condition for a real eigenvalue pair which becomes a pure imaginary pair through parameter changes is

$$|j\omega I - M(q)|_{\omega=0} = |M(q)| = 0. \tag{17}$$

Additionally the opposite end of the imaginary axis has to be considered

$$|j\omega I - M(q)|_{\omega=\infty}. \tag{18}$$

Equation (18) is just the coefficient of the term with the highest degree in s of $|sI - M|$.

Equation (17) is not sufficient, since it determines all parameters for which (15) has a pair of eigenvalues at the origin. This includes real pairs which are just interchanging on the real axis. To get sufficiency we have to check all parameters satisfying (17), if there are only real eigenvalues.

The mapping equations (16), (17), and (18) have a similar structure like the familiar equations for pole location specifications. Actually (16), (17), and (18) can be interpreted as the complex, real, and infinite root boundary, respectively. Using the above approach and Lyapunov's famous ARE for Hurwitz stability

$$A^T P + PA = -Q,\; Q = Q^T > 0,$$

we get mapping equations for the CRB and RRB which have the same solution set as the equations derived from the characteristic polynomial.

Example: H_∞ Norm

Using (3) and Theorem 2 the following result holds for the prominent H_∞ norm: Let A be stable and $\gamma > \sigma_{\max}(D)$. Then $\|H\|_\infty < \gamma$ if and only if

$$M_\gamma = \begin{bmatrix} A - BS_r^{-1}D^T C & -\gamma B S_r^{-1} B^T \\ \gamma C^T S_l^{-1} C & -A^T + C^T D S_r^{-1} B^T \end{bmatrix}$$

has no pure imaginary eigenvalues. Thus M_γ is the Hamiltonian matrix needed in the mapping equations (16), (17) and (18). Two remarks have to be made about the H_∞ norm mapping. The H_∞ norm requires the transfer function $H(s)$ being stable, thus we have to additionally map the Hurwitz stability condition. Without this additional condition we are actually mapping a L_∞ norm condition. Second, using a frequency domain formulation it can be shown that the condition $\gamma > \sigma_{\max}(D)$ is implicitly mapped by (17) and (18).

4.2 H_2 Norm

We will now present the mapping equation for the H_2 norm. Using (10) a specification on the H_2 norm like $\|H(q)\|_2 \le \gamma$ can be mapped into the parameter space. In order to use (10) as a mapping equation we have to compute the parameter dependent observability Gramian W_{obs}. This can be done using the Lyapunov equation (11) which is a linear equation in the unknown matrix elements w_{ij} of W_{obs}. Thus even for a system with uncertainties $W_{obs}(q)$ can be readily computed. The resulting mapping equation

$$\|H\|_2^2 = \mathrm{Tr}(B(q)^T W_{obs}(q) B(q)) = \gamma^2, \tag{19}$$

is a single equation which depends only on the system parameters q. This is in line with the fact that the general definition of the H_2 norm includes an integral over all frequencies.

If the parameters q enter in a polynomial fashion into $A(q), B(q), C(q)$, the mapping equation (19) is a polynomial equation.

Note: There are some special cases, when (19) is affine in one or more parameters, but in general this equation is polynomial in q, even if $A(q), B(q), C(q)$ are affine in q.

4.3 Maximal eigenvalue based mapping

Both the Hankel and the generalized H_2 norm can be expressed as a function of a parametric matrix. These associated matrices can be computed using the solution of parametric Lyapunov equations.

To get mapping equations for the Hankel and generalized H_2 norms, we apply standard results for mapping eigenvalue specifications. Namely a condition $\lambda_{\max}(M) = \gamma$, where M is a nonnegative matrix leads to the mapping equation

$$|\gamma I - M| = 0.$$

Accordingly the condition $d_{\max}(M) = \gamma$, $M \geq 0$ leads to the mapping equations

$$m_{ii} = \gamma, \qquad i = 1..n.$$

5 Example

We consider a well-known satellite control problem to demonstrate the incorporation of H_∞ and H_2 norm specifications into a robust control design. The satellite model consists of two rigid bodies (main body and sensor module) connected by a flexible link which is modeled as a spring with torque constant k and viscous damping f. These real parameter are uncertain in the ranges $k \in [.09; .4]$, and $f \in [.0038; .04]$.

The equations of motion are

$$\begin{aligned} J_1\ddot{\theta}_1 + f(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) &= T_c + w \\ J_2\ddot{\theta}_2 + f(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) &= 0 \end{aligned}$$

where T_c is the control torque and w is a torque disturbance on the main body. The inertias are given as $J_1 = 1$ and $J_2 = .1$.

The purpose of the control system is to minimize the influence of the disturbance w on the angular position θ_2 of the sensor module. The following specifications are used to achieve this goal:

- minimize the RMS gain from w to θ_2 while keeping the H_2 norm of the transfer function from w to $(\theta_1, \theta_2, T_c)$ small (LQG cost of control).
- guarantee a minimum decay rate of 10 seconds and closed-loop damping $D = 0.5$.

In this example we design a robust state-feedback controller $T_c = -c^T\theta$ which satisfies these requirements for the entire operating range, where $c^T = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$ and $\theta^T = \begin{bmatrix} \theta_1 & \theta_2 & \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}$.

To solve this problem we use the invariance plane concept developed in [10] to sequentially shift the poles of the system. In the first step we move the rigid body poles from the origin to $\lambda = -.2 \pm .35j$. This assigns the desired dynamics to the rigid body. Actually this can be done in a robust manner without affecting the flexible modes of the system.

In the final step we have to modify the flexible modes such that all design goals are satisfied. We proceed as follows: Choose a representative point for which an invariance plane is computed which shifts only the flexible modes without affecting the rigid body modes. We choose a vertex of the operating range, namely $k = .09, f = .0038$. The invariance plane for this operating point is given as

$$\begin{aligned} c^T &= \begin{bmatrix} .167 & .0167 & .412 & .0412 \end{bmatrix} \\ &\quad + \begin{bmatrix} \kappa_a & \kappa_b \end{bmatrix} \begin{bmatrix} .98 & -.80 & 0 & .45 \\ .41 & -.41 & 1 & -.82 \end{bmatrix} \end{aligned} \tag{20}$$

with κ_a, κ_b as free parameters. While (20) guarantees that the rigid body modes remain unaffected for the chosen operating point, this is not true in general. Thus we will map the eigenvalue specifications for the four vertices of the operating range in order to guarantee robust satisfaction of the eigenvalue requirements. Figure 1 shows the resulting boundaries, where the set of good parameters is shaded grey.

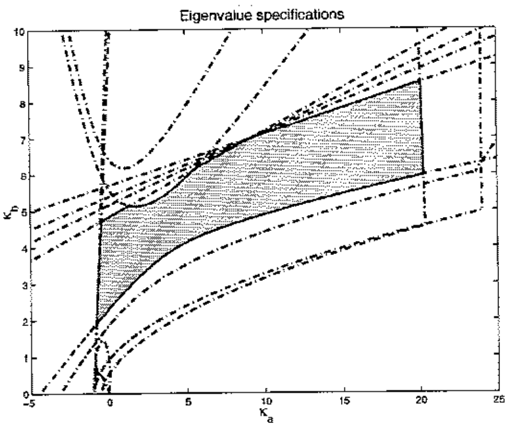

Figure 1: Boundaries with good set for eigenvalue specifications.

Using (19) and the mapping equation for the H_∞ we map the following specifications

$$||G_{w \to (\theta_1, \theta_2, T_c)}||_2 = \{2, 2.3\}, \quad ||G_{w \to \theta_2}||_\infty = \{.25, .5, 1\} \tag{21}$$

into the κ_a, κ_b invariance plane for the four vertices of the uncertain parameter set (f, k). Figure 2 depicts the set of parameters which satisfies the LQG cost specification. Darker regions are used for specifications which are more difficult to achieve. From this figure and the RMS gain mapping not shown here, we can conclude that the concept of first moving the rigid body mode to desired position without affecting the elastic mode leads to quite similar regions in the subsequent invariance plane.

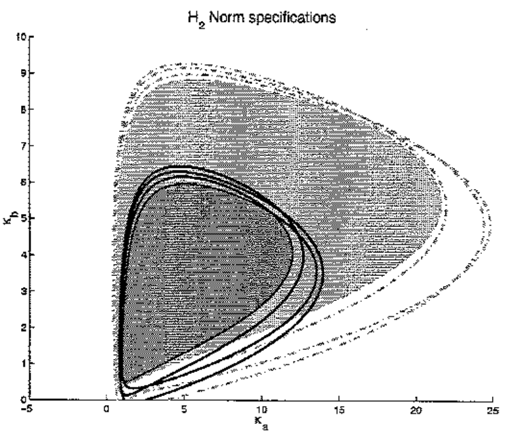

Figure 2: LQG cost of control in invariance plane.

Figure 3 depicts the set of good parameters for the eigenvalue, H_2 and H_∞ norm specifications in a single plot which allows selection of controllers which robustly satisfy all specifications. Actually it turns out that for the given eigenvalue specifications pareto-optimal controllers for the H_2 and H_∞ Norm are given by parameters on the lower edge of the eigenvalue polygon in the invariance plane. Thus by using controllers from this lower edge we can tradeoff the H_2 and H_∞ norm while robustly satisfying the eigenvalue requirements. We have therefore found a robust controller which results in similar closed-loop dynamics.

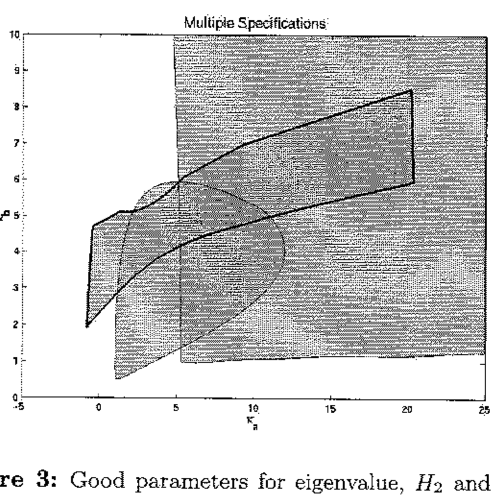

Figure 3: Good parameters for eigenvalue, H_2 and H_∞ norm specifications.

6 Conclusions

In this paper, the mapping equations of design objectives for parametric multi-input multi-output systems into parameter space were presented. Algebraic Riccati and the more special Lyapunov equations provide a uniform framework in which all presented specifications fit.

A robust control example showed that incorporating standard MIMO specifications into the parameter space approach is both possible and rewarding.

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