SINGULAR VALUE AND STRUCTURED SINGULAR VALUE BOUNDS IN PARAMETER SPACE

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Abstract

This contribution discusses the mapping of singular value ($\sigma-$) bounds and structured singular value ($\mu-$) bounds into parameter space. The $\sigma-$ and $\mu-$ mapping problem is defined to enclose robust design and analysis in parameter space. Real, complex and mixed uncertainty structures are considered. While the singular values enjoy nice continuity properties and bound mapping is rather principally straightforward, the structured singular value bound mapping is more involved. Discontinuity properties of $\mu-$measures w.r.t. parametric uncertainties are further discussed in terms of bifurcation mapping conditions.

1 Introduction

Singular value bound specifications reside at the heart of many robust design and analysis techniques [7]. Consider the interaction of a given stable and proper system P and some uncertainty Δ, which may be a complex, real (parametric) or mixed uncertainty. The basic paradigm of all techniques is phrased as follows: given that the norm (H_∞) of the uncertainty is smaller than some $1/\gamma$, ($\gamma > 0$), the stability/performance robustness specifications are fulfilled if

$$\sup_\omega \overline{\sigma}(P) \leq \gamma,$$

holds. While this is known to be a fairly conservative requirement, the structured singular value measure [3] is introduced to include the uncertainty structure information on the interaction between the system and the uncertainty, thus leading to less conservative constraint. The robustness requirement reads now

$$\sup_\Delta \mu_\Delta(P) \leq \gamma.$$

However less conservativeness has its price, since μ may be discontinuous for systems with real parametric uncertainties [2],[6].

This paper is about mapping of singular value (H_∞) and structured singular value bounds in parameter space. To this end, the $\sigma-$ and $\mu-$mapping problems are defined, which may be used for analysis and synthesis of robust controllers for MIMO systems with real, complex and/or mixed uncertainties. Based on two necessary conditions, the σ-mapping problem into a plane of two parameters is solved for input-output and state-space system representation. Also the set of necessary and sufficient mapping conditions is derived for the input-output system representations. Continuity properties of μ are discussed in the parameter space framework. It is shown that mapping singularities such as bifurcation singular and branching frequencies may cause the discontinuity of μ w.r.t. to the 'problem data', [2]. The $\mu-$discontinuity may also appear in the frequency response. A continuity condition of μ for real parameter uncertainties is derived. It is shown that a linear fractional transformation exerts a shifting and scaling effect on some uncertainty ball. Based on this we propose a solution for the $\mu-$mapping problem as an analysis and design tool for the systems with mixed uncertainties.

The notation used in the paper is standard.

2 $\sigma-$Mapping problem

Let P be a matrix in $C^{m\times n}$. By definition, a singular value σ of P fulfills the conditions

$$PP^*u = \sigma^2 u, \ P^*Pv = \sigma^2 v, \tag{1}$$

i.e. the squares of the singular values of P represent the eigenvalues of the matrix PP^* and P^*P. Obviously, P^*P is a hermitian matrix with a real nonnegative eigenvalue spectrum.

Let $P = P(j\omega, \boldsymbol{q}) \in \mathcal{RH}_\infty$ with $\boldsymbol{q} \in \boldsymbol{R}^p$, be any proper and stable transfer matrix of a MIMO system with $m-$inputs and $n-$outputs: $\boldsymbol{q}$ is a vector of uncertain system parameters and/or controller parameters. Define the $\sigma-$mapping problem as follows,

Definition 2.1 *Let $\gamma > 0$. Find the set $\boldsymbol{Q}_\sigma$ of all parameters $\boldsymbol{q}$, such that the maximal singular value of the system is smaller than γ, i.e.*

$$\boldsymbol{Q}_\sigma = \{\boldsymbol{q} \in \boldsymbol{R}^p : \sup_\omega \overline{\sigma}\left(P(j\omega, \boldsymbol{q})\right) < \gamma\}. \tag{2}$$

3 Continuity of singular values

This section is about continuity properties of the singular values. The following is a reformulation of the well-known polynomial root-continuity theorem.

Definition 3.1 ($\sigma-$**Continuity**) *Let $P(j\omega, \boldsymbol{q}) \in \mathcal{RH}_\infty$ and $\sigma^* = \sigma(P(j\omega, \boldsymbol{q}^*))$ its singular value. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\|\boldsymbol{q} - \boldsymbol{q}^*\| < \delta \Rightarrow |\sigma(P(j\omega, \boldsymbol{q})) - \sigma^*| < \varepsilon$.*

On the basis of this definition, the Boundary Crossing Theorem of Frazer and Duncan [5] for the singular values is restated in the following theorem.

Theorem 3.2 (Boundary Crossing) *Let $\gamma > 0$ and $P(j\omega, \boldsymbol{q}) \in \mathcal{RH}_\infty$. Then (2) is equivalent to the conjunction of the conditions*

$$\begin{array}{ll} (\exists \boldsymbol{q} \in \boldsymbol{Q}_\sigma \ \ s.t. & \sup_\omega \overline{\sigma}\left(P(j\omega, \boldsymbol{q})\right) < \gamma) \\ (\forall \boldsymbol{q} \in \boldsymbol{Q}_\sigma, & \sup_\omega \overline{\sigma}\left(P(j\omega, \boldsymbol{q})\right) \neq \gamma) . \end{array}$$

Hence, $\boldsymbol{q}-$boundaries should be searched for parameters $\boldsymbol{q}$, s.t. $\sigma(P(j\omega, \boldsymbol{q})) = \gamma$. Regions in the parameter space (i.e. plane) result which are then checked if the condition $\sigma(P(j\omega, \boldsymbol{q}) < \gamma$ is fulfilled. The solution of the $\sigma-$mapping problem, $\boldsymbol{Q}_\sigma$, is the region where this inequality is valid for each singular value of the system.

4 Transfer matrix based mapping

Obviously, (2) is equivalent to

$$\boldsymbol{Q}_\sigma = \{\boldsymbol{q} \in \boldsymbol{R}^p : \sigma\left(P(j\omega, \boldsymbol{q})\right) < \gamma, \ \forall \sigma \in \mathcal{S}\} \tag{3}$$

where $\mathcal{S} = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ is the set of all singular values, i.e.

$$\sigma_i^2 \leq \gamma^2, \tag{4}$$

with $i = 1, 2, \ldots, m$, since σ_i is nonnegative. Considering any singular value σ_i, we search for the boundaries in $\boldsymbol{q}-$parameter space where the following two conditions apply

$$\sigma_i(\omega, \boldsymbol{q}) = \gamma, \ \frac{\partial \sigma_i}{\partial \omega}(\omega, \boldsymbol{q}) = 0, \tag{5}$$

with $i = 1, 2, \ldots, m$. These conditions are usually called the point and tangent condition and represent the necessary conditions for a single singular value to be maximal at frequency ω.

Consider the matrix,

$$\gamma^2 I - P(j\omega, \boldsymbol{q}) P^T(-j\omega, \boldsymbol{q}) \tag{6}$$

and define,

$$e = \det\left(\gamma^2 I - P(j\omega, \boldsymbol{q}) P^T(-j\omega, \boldsymbol{q})\right). \tag{7}$$

The singular value decomposition (SVD) of (6) yields

$$\gamma^2 I - PP^* = U \ \left(\gamma^2 I - \Sigma\right) \ U^{-1}. \tag{8}$$

Therefore,

$$e = \prod_{j=1}^{m} (\gamma^2 - \sigma_j^2). \quad (9)$$

and the following two equations

$$e(\omega, \boldsymbol{q}) = 0, \ \ \frac{\partial e}{\partial \omega}(\omega, \boldsymbol{q}) = 0, \quad (10)$$

represent necessary conditions for the set of m pair conditions (5), i.e. (2).

5 State space representation based mapping

Let (A, B, C, D) be a state-space representation of an uncertain system $P(j\omega, \boldsymbol{q})$ (the argument dependency is dropped). Define the Hamiltonian matrix

$$H_\gamma = \begin{bmatrix} A - BR^{-1}D^TC & -\gamma BR^{-1}B^T \\ \gamma C^TS^{-1}C & -A^T + C^TDR^{-1}B^T \end{bmatrix}, \quad (11)$$

where $R = D^TD - \gamma^2 I$ and $S = DD^T - \gamma^2 I$ and $\gamma > 0$. The following theorem is standard, [4].

Theorem 5.1 *Assume A has no imaginary eigenvalues, $\omega_0 \in \mathbf{R}$ and $\gamma > 0$ is not a singular value of D. Then, γ is a singular value of $P(j\omega)$ iff $j\omega_0$ is an eigenvalue of H_γ.*

This theorem relates the singular values of the system with the eigenvalues of its Hamiltonian matrix. A straightforward consequence is the following well-known theorem, [4].

Theorem 5.2 (Boyd et. al.) *Let A be stable and $\bar{\sigma}(D) < \gamma$. Then the two statements are equivalent,*

1. $||H||_\infty \geq \gamma$
2. *H_γ has pure imaginary eigenvalues.*

This theorem has been used to obtain a numerical bisection method to compute the H_∞ norm of a given transfer matrix. We use the above theorem in its symbolic form to derive mapping equations for the H_∞ norm. The boundary of a parameter region with $||H||_\infty < \gamma$ is given by parameters for which H_γ defined in (11) has at least a pair of pure imaginary eigenvalues. The eigenvalue equation $e(s, \boldsymbol{q}) = \det(sI - H_\gamma) = 0$ can be used to check if s is an eigenvalue of H_γ. From this follows that

$$e(\omega, \boldsymbol{q}) = \det(j\omega I - H_\gamma) = 0, \quad (12)$$

is a necessary condition for parameters with $||H||_\infty = \gamma$. Since (11) is a Hamiltonian, pure imaginary eigenvalues will exist as pairs. Therefore the following double root condition for polynomial equations applies,

$$\frac{\partial e}{\partial \omega}(\omega, \boldsymbol{q}) = 0. \quad (13)$$

Equations (12) and (13) define two polynomial equations which can be used to map a given H_∞ norm condition into parameter space. Note that Theorem 5.1 guarantees that these equations are identical to (10).

6 The necessary and sufficient mapping conditions

The derived mapping conditions in the past two sections represent the necessary conditions for the solution of σ−mapping problem and they are suitable for the mapping of the singular value bounds in the parameter plane of two parameters. In this section we generalize the mapping equations by deriving the set of necessary and sufficient conditions for the σ−mapping problem.

Equation (8) says that σ−mapping conditions are equivalent to positivity of the matrix (6), i.e.

$$\gamma^2 I - P(j\omega, \boldsymbol{q})P^*(j\omega, \boldsymbol{q}) > 0. \quad (14)$$

The following theorem is standard in linear algebra.

Theorem 6.1 *A hermitian matrix $H(j\omega) = (h_{ij})$ is positive definite iff,*

$$\Delta_i^{(H)}(j\omega) > 0, \ \forall \omega > 0, \ i = 1, 2, \ldots, m$$

with

$$\Delta_i^{(H)}(j\omega) = \begin{vmatrix} h_{11}(j\omega) & \cdots & h_{1i}(j\omega) \\ \vdots & & \vdots \\ h_{i1}(j\omega) & \cdots & h_{ii}(j\omega) \end{vmatrix},$$

Now apply this theorem to (14) to get the set of necessary and sufficient conditions which guarantee the singular value bound condition

$$e_i(j\omega, \boldsymbol{q}) > 0, \ i = 1, \ldots, m. \tag{15}$$

with

$$e_i(j\omega, \boldsymbol{q}) = \Delta_i^{(\gamma^2 I - PP^*)}(j\omega), \ i = 1, 2, \ldots, m. \tag{16}$$

This is a system of parametric nonlinear algebraic inequalities. Its solution is the set $\boldsymbol{Q}_\sigma$.

7 μ–Mapping problem

Nonconservative robust control analysis and design criteria can be formulated using the *structured singular value* measure [3], [6], [7]. The structured singular value (μ) of a system reflects the interaction of the nonconservative uncertainty set with the nominal (unperturbed) system.

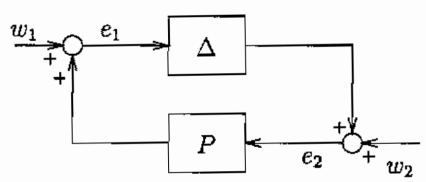

Figure 1: To illustrate the definition of μ.

To recall the definition of μ, consider the system in the above figure with $P(s) \in \mathcal{RH}_\infty$ representing the nominal system and Δ the uncertainty. Let $P \in \boldsymbol{C}^{n\times n}$ and $\Delta \in \boldsymbol{C}^{n\times n}$. The uncertainty Δ is a diagonal matrix with structured uncertainty blocks of the forms $q_j I_j$, $\delta_j(s) I_j$ and unstructured uncertainty $\Delta_j(s) \in \boldsymbol{C}^{m_j\times m_j}$. Here q_j is a scalar uncertain parameter. All of the entries in Δ are assumed to be in $\mathcal{RH}_\infty$. Such entries are said to be allowed uncertainties. Let $\boldsymbol{\Delta}$ denote the set of all such uncertainties.

The structured singular value of P w.r.t. the structured uncertainty Δ is defined as

$$\mu_{\boldsymbol{\Delta}}(P) = \frac{1}{\min\{\overline{\sigma}(\Delta) : \det(I - P\Delta) = 0\}},$$

unless no $\Delta \in \boldsymbol{\Delta}$ makes $I - P\Delta$ singular, in which case, $\mu_{\boldsymbol{\Delta}}(P) = 0$, [6], [7]. That is, μ represents the reciprocal of the smallest (in the singular value sense) structured uncertainty $\Delta \in \boldsymbol{\Delta}$, which destabilizes the loop in Fig. 1.

The μ–mapping problem is defined as follows,

Definition 7.1 *Find the set $\boldsymbol{Q}_\mu$ of all parameters $\boldsymbol{q}$ defined by,*

$$\boldsymbol{Q}_\mu = \{\boldsymbol{q} \in \boldsymbol{R}^p : \sup_\omega \overline{\sigma}(\Delta(j\omega, \boldsymbol{q})) \sup_\Delta \mu_\Delta(P(j\omega)) \leq 1\}. \tag{17}$$

Some conventions are needed for the generalization of the μ–mapping problem to framework the structured robust stability and performance analysis and synthesis. Consider the general setup in

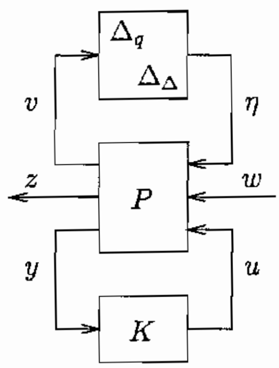

Figure 2: General μ framework.

the figure above. The uncertainty block Δ is split into two blocks Δ_q and Δ_Δ. The block Δ_q includes real parametric uncertainties $q_j I_j$, and/or structured complex uncertainties $\delta_j(s) I_j$, while Δ_Δ includes unstructured complex uncertainties Δ_j. All entries are assumed to be allowable (i.e. stable and proper). By $\mathcal{D}_\Delta$ we will denote the set of all Δ_Δ uncertainties and by $\mathcal{D}_q$ that of Δ_q's. Clearly $\mathbf{\Delta} = \mathcal{D}_q \cup \mathcal{D}_\Delta$.

We will further discriminate between three sets of plants: $\mathcal{P}_N$ includes the set of all nominal plants, $\mathcal{P}_\Delta$ the set of all plant LFTs w.r.t. some unstructured uncertainty Δ_Δ, i.e. $\mathcal{F}_u(P, \Delta_\Delta)$ and $\mathcal{P}_q$ the set of all plant LFTs w.r.t. the controller K, $\mathcal{F}_l(P, K)$, or some structured uncertainty Δ_q, $\mathcal{F}_l(P, \Delta_q)$. Define also $\mathcal{P} = \mathcal{P}_N \cup \mathcal{P}_\Delta \cup \mathcal{P}_q$.

8 Continuity of μ

The continuity properties of μ, on its domain of definition are essential to the μ–mapping problem. By definition, $\mu : \mathcal{P} \times \mathbf{\Delta} \mapsto \boldsymbol{R}$. Formally we say that μ is defined on an interacting pair $(P, \Delta) \in (\mathcal{P} \times \mathbf{\Delta})$, where P is a generalized plant and Δ some structured or unstructured uncertainty. The continuity of μ is defined naturally as follows,

Definition 8.1 *Let $\mu_\Delta(P) = \mu$. μ is said to be continuous on (P, Δ) if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\|(P_1, \Delta_1) - (P, \Delta)\| < \delta \Rightarrow |\mu_{\Delta_1}(P_1) - \mu| < \varepsilon$.*

Continuity properties of μ depend primarily on the interaction pair type. We will differ two interaction types. An $q-q$ interaction includes the $(\mathcal{P}_q \times \mathcal{D}_q)$ pairs, while $q - \Delta$ the remainder cases $(\mathcal{P}_q \times \mathcal{D}_\Delta)$ and $(\mathcal{P}_\Delta \times \mathcal{D}_q)$. A short discussion of the continuity properties of these interaction types follows.

8.1 $q - \Delta$ Interaction

A pair $(\mathcal{P}_\Delta, \mathcal{D}_q)$ appears typically during a robustness analysis step, e.g. when applying the main-loop-theorem [6], [7]. On the other hand, $(\mathcal{P}_q, \mathcal{D}_\Delta)$ is met when performing μ-design w.r.t. to some Δ_Δ.

Lemma 8.2 *Suppose $\mu_{\Delta_q}(P_\Delta) = \beta > 0$. Then μ is continuous on the pair (P_Δ, Δ_q).*

Proof. In [6] it has been shown that μ is always upper-semicontinuous, i.e. $\forall \varepsilon_1 > 0$, $\exists \delta_1 > 0 : \|P^{(1)}_{\Delta_q} - P_{\Delta_q}\| < \delta_1 \Rightarrow \mu_{\Delta_q}(P^{(1)}_\Delta) < \beta - \varepsilon_1$. On the other hand, since $\overline{\sigma}(\cdot)$ is continuous on Δ_q, $\forall \varepsilon_2 > 0$, $\exists \delta_2 > 0 : \|\Delta^{(2)}_q - \Delta_q\| < \delta_2 \Rightarrow \overline{\sigma}(\Delta^{(2)}_q) \leq \frac{1}{\beta - \varepsilon_2}$. Define $F(X, \Delta^{(2)}_q) = \det(I - X\Delta^{(2)}_q) = 0$, where $\forall \omega > 0$, $\min \overline{\sigma}(P_\Delta) \leq \overline{\sigma}(X) \leq \max \overline{\sigma}(P_\Delta)$. F is continuous in X, so μ is well-defined in $\|\Delta^{(2)}_q - \Delta_q\| < \delta_2$. But therein $\mu_{\Delta^{(2)}_q}(X) \geq \beta - \varepsilon_2$. So μ is also lower-semicontinuous. This completes the proof.

Analogously it can be shown that μ is continuous on $(\mathcal{P}_q, \mathcal{D}_\Delta)$. Therefore it can be concluded that μ is continuous on $q - \Delta$ interactions.

8.2 $q - q$ Interaction

This type of interaction describes the systems with real parameter uncertainties. In [2] it was shown that μ may be discontinuous in this case. Note that an ODE

$$\boldsymbol{J}(\boldsymbol{q}, \omega)\boldsymbol{q}' + \boldsymbol{e}_0(\boldsymbol{q}, \omega) = 0, \tag{18}$$

can be associated to the mapping equation $\boldsymbol{e}(\omega, \boldsymbol{q}) = 0$, whereby $\boldsymbol{q}' = (\mathrm{d}q_1/\mathrm{d}\omega, \mathrm{d}q_2/\mathrm{d}\omega)^T$ and

$$\boldsymbol{J}(\boldsymbol{q}, \omega) = \partial \boldsymbol{e}/\partial \boldsymbol{q}, \quad \boldsymbol{e}_0(\boldsymbol{q}, \omega) = \partial \boldsymbol{J}/\partial \omega. \tag{19}$$

The peculiar mapping condition appear at $(\boldsymbol{q}_b, \omega_b)$ for which

$$rank\,(\boldsymbol{J}) = rank\,(\boldsymbol{J}, \boldsymbol{e}_0) = 1. \tag{20}$$

At such a $(\boldsymbol{q}_b, \omega_b)$ (18) is consistent but the solution uniqueness is lost. At frequency ω_b a *bifurcation* occurs, i.e. new curve branches emerge or disappear. The bifurcation frequency ω_b may be a *singular frequency* or a *branching frequency*. Geometrically the difference is that in an environment of a singular frequency ω_b real solutions for $\boldsymbol{q}_b$ exist, whereas at a branching frequency the complex solutions fall down to the real $\boldsymbol{q}$ plane, or 'disappear' into complex solutions at $\boldsymbol{q}_b$. Another difference is that branching frequencies appear only for a nonlinear parameter dependency, whereas singular frequencies are present also for linear parameter dependency.

Clearly such bifurcation events affect substantially the $\mu-$ mapping regions. Actually new boundaries (regions) emerge in the parameter space which may cause discontinuity of the robustness margin, see Example 9.7.

9 μ–Mapping equations

9.1 Basics

Consider the primitive definition of the μ–mapping problem in (17). Note that this definition includes a (P_N, Δ_q) interaction pair and it does not fit into any of the robust control framework pair type just defined, since P_N is fixed. However its solution will illustrate the basic approach.

The μ–mapping problem is split into three subproblems.

1. The boundary condition

$$e(j\omega, q) = \det (I - P(j\omega)\Delta(j\omega, q)) = 0, \quad (21)$$

is *mapped* in the q–parameter plane. A curve ∂_μ results in the parameter plane.

2. Jointly to mapping, an *optimization* algorithm for the singular value of $\Delta_q = \Delta(j\omega, q)$ on the boundary (21) is executed. The minimal singular value of Δ_q, say

$$\min_{\partial_\mu} \overline{\sigma}(\Delta(j\omega, q)) = \frac{1}{\beta}. \quad (22)$$

is returned ($\beta > 0$).

3. The third subproblem is a σ–mapping problem,

$$\text{search for } q: \ \sup_{\omega} \overline{\sigma}(\Delta(j\omega, q)) < \beta \quad (23)$$

in (the same) q-plane.

Remark 9.1 *If the entries of the uncertainty Δ_q are scalars, the σ–mapping problem is rather straightforward, since $\overline{\sigma}(\Delta_q) = max(|q_1|, |q_2|, \ldots)$.*

Example 9.2 *The nominal system P_n is defined by $P_n(s) = C(sI - A)^{-1}B$, with*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5\,q_a{}^2 & -20 - 8\,q_a & -44 - 2\,q_a & -20 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \ C = \begin{bmatrix} -10\,q_a{}^2 & -20\,q_a & -10 & 0 \\ 0 & 20 & 40 & 20 \end{bmatrix}$$

while the uncertainty is described with two uncertain parameters $\Delta_q = diag[q_1, q_2]$. The boundary mapping equation (21) for this pair is $e(\omega, q_a, q) = J(\omega, q_a)q + e_0$, with

$$J(\omega, q_a) = \begin{bmatrix} -10\,\omega^2 + 10\,q_a{}^2 & 40\,\omega^2 \\ 20\,q_a\omega & 20\,\omega^3 - 20\,\omega \end{bmatrix},$$

and

$$e_0(\omega, q_a) = \begin{bmatrix} \omega^4 + (-44 - 2\,q_a)\,\omega^2 + 5\,q_a{}^2 \\ -20\,\omega^3 + (20 + 8\,q_a)\,\omega \end{bmatrix}.$$

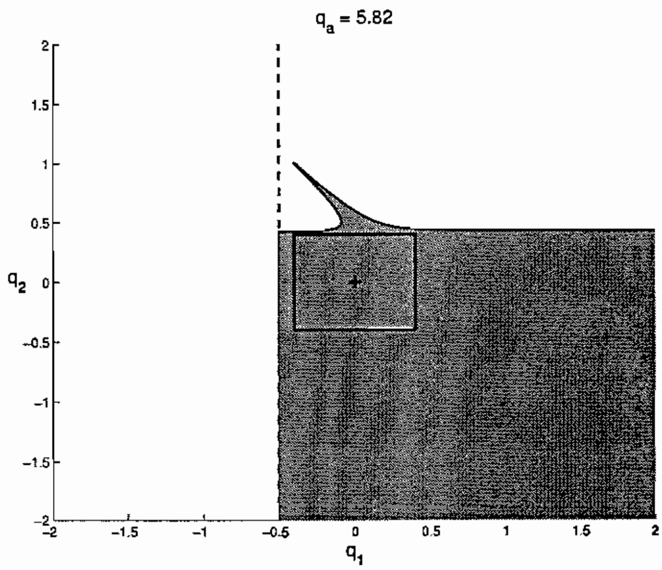

Figure 3: Solution of the μ–mapping problem.

The solution of the μ–mapping problem is shown in Fig. 3. Note that the line $q_1 = -0.5$ is the singular line, which corresponds to the singular frequency $\omega = 0$, [1]. Q_μ is represented by the rectangle.

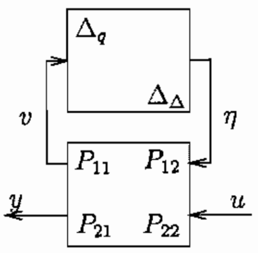

Figure 4: $q-\Delta$ Interaction.

9.2 $q-\Delta$ Interaction

9.2.1 Main Loop Theorem

Consider Fig. 4 and assume $\overline{\sigma}(\Delta_\Delta) \leq 1$. Recall the main loop theorem: given that $\mu_{\Delta_\Delta}(P_{22}) < 1$, then $\mu_\Delta(P) < 1$ iff $\mu_{\Delta_q}(\mathcal{F}_l(P, \Delta_\Delta)) < 1$, with $\Delta = \text{diag}\,[\Delta_q, \Delta_\Delta]$, and $P_{i,j}$ being submatrices of a conveniently partitioned P. Note that in this case the $\mu-$mapping problem can be defined as,

$$\text{check whether} \quad \mu_{\Delta_q}(\mathcal{F}_l(P, \Delta_\Delta)) < 1. \tag{24}$$

This is a more involved problem compared to the example above, since the boundaries of a family of infinite plants have to be mapped to the parameter plane. However, as we showed, the $q-\Delta$ interactions enjoy the powerful continuity conditions, which might be of use to overcome this difficulty.

An intuitive idea is to define boundaries which enclose the set of all plants $\mathcal{F}_l(P, \Delta_\Delta)$ and then to map these boundaries into a parameter plane. As the enclosure criterion should be used the size of the uncertainty Δ_Δ, since it is the only available information. In addition, the enclosure should be *nonconservative.*

Definition 9.3 *Define the* lower-boundary-plant *(LBP)* $\mathcal{L}(j\omega)$ *and the* upper-boundary-plant *(UBP)* $\mathcal{U}(j\omega)$ *as follows*

$$\begin{aligned}\underline{\sigma}(\mathcal{L}(j\omega)) &= \min_{\Delta_\Delta \in \boldsymbol{B\Delta}} \underline{\sigma}(\mathcal{F}_l(P, \Delta_\Delta)),\ \forall \omega > 0\\ \overline{\sigma}(\mathcal{U}(j\omega)) &= \max_{\Delta_\Delta \in \boldsymbol{B\Delta}} \overline{\sigma}(\mathcal{F}_l(P, \Delta_\Delta)),\ \forall \omega > 0.\end{aligned}$$

9.2.2 Obtaining the LBP and the UBP boundaries

By definition $\mathcal{F}_l(P, \Delta_\Delta)$ is a homographic transformation of the uncertainty Δ_Δ into some $\widehat{\Delta}_\Delta$. In the scalar case such a transformation may map a circle onto a circle. It is therefore apparent to investigate whether the same holds also in the matrix spaces (i.e. $\mathcal{RH}_\infty$).

Indeed, consider the plant uncertainty

$$\widetilde{\Delta}_\Delta = \Delta_\Delta(I - P_{22}\Delta_\Delta)^{-1}, \tag{25}$$

which is, by definition, well defined in the ball Δ_Δ. Solving this equation for Δ_Δ, a homographic transformation results, as well

$$\Delta_\Delta = \widetilde{\Delta}_\Delta(I + \widetilde{\Delta}_\Delta P_{22})^{-1}. \tag{26}$$

The uncertainties 'lying' on the ball boundary, $\partial\boldsymbol{B\Delta}$ are unitary, i.e. they satisfy the equation $\Delta_\Delta\Delta_\Delta^* = I$. Therefore the ball boundary $\partial\boldsymbol{B\Delta}$ is mapped to the $\widetilde{\Delta}_\Delta$ uncertainties which satisfy

$$\widetilde{\Delta}_\Delta(I - P_{22}P_{22}^*)\widetilde{\Delta}_\Delta^* - \widetilde{\Delta}_\Delta P_{22}^* - P_{22}\widetilde{\Delta}_\Delta^* - I = 0. \tag{27}$$

Remark 9.4 *Since* $\rho(P_{22}) < \mu_{\Delta_\Delta}(P_{22}) < 1$, *the eigenvalues of the hermitian* $P_{22}P_{22}^*$ *are smaller than unity. Therefore* $I - P_{22}P_{22}^*$ *is always* non-singular *and* positive-definite. *These properties of* $I - P_{22}P_{22}^*$ *are essential for the existence (and uniqueness) of the LBP and UBP.*

Consider a sphere in $\widetilde{\Delta}_\Delta$ space, with the center at β and the radius ρ. Move to its center using a linear transformation (α nonsingular)

$$\widehat{\Delta}_\Delta = \alpha\widetilde{\Delta}_\Delta + \beta, \tag{28}$$

and substitute (28) in $\widehat{\Delta}_\Delta\widehat{\Delta}_\Delta^* = \rho\rho^*$ to get the sphere equation in $\widetilde{\Delta}_\Delta$ space

$$\widetilde{\Delta}_\Delta(\alpha\alpha^*)\widetilde{\Delta}_\Delta^* + \widetilde{\Delta}_\Delta\alpha\beta^* + \beta\alpha^*\widetilde{\Delta}_\Delta^* + \beta\beta^* - \rho\rho^* = 0. \tag{29}$$

Compare (29) with (27) to conclude that $\Delta_\Delta(I - P_{22}\Delta_\Delta)$ is indeed mapped to a ball in $\widetilde{\Delta}_\Delta$ space.

That is, *the LFT $\mathcal{F}_l(P, \Delta_\Delta)$ shifts and scales the uncertainty ball Δ_Δ*. The following are the equations to determine its center and radius,

$$\begin{aligned} \alpha\alpha^* &= I - P_{22}P_{22}^* && (30) \\ \alpha\beta^* &= -P_{22} && (31) \\ \beta\beta^* - \rho\rho^* &= -I && (32) \end{aligned}$$

Since, $I - P_{22}P_{22}^*$ is a positive definite and non-singular hermitian, a *unique* solution for α will always exist, which is

$$\alpha = (I - P_{22}P_{22}^*)^{1/2}. \quad (33)$$

The same is valid also for ρ and β,

$$\begin{aligned} \rho &= (I - P_{22}P_{22}^*)^{-1/2}, && (34) \\ \beta &= (\rho\rho^* - I)^{1/2}. && (35) \end{aligned}$$

Therefore, the uncertainty ball, which Δ_Δ is mapped to by $\mathcal{F}_l(P, \Delta_\Delta)$, is found to be

$$\mathcal{F}_l(P, \Delta_\Delta) = Q_1 + Q_2\widehat{\Delta}_\Delta Q_3 \quad (36)$$

where,

$$\begin{aligned} Q_1 &= P_{11} - P_{12}\beta P_{21} && (37) \\ Q_2 &= P_{12}\alpha^{-1} && (38) \\ Q_3 &= P_{21} && (39) \end{aligned}$$

with,

$$\overline{\sigma}(\widehat{\Delta}_\Delta) = \overline{\sigma}(\rho). \quad (40)$$

Finally define,

$$\begin{aligned} l(\omega) &= 1 + \frac{\overline{\sigma}(Q_2)\ \overline{\sigma}(\rho)\ \overline{\sigma}(Q_3)}{\underline{\sigma}(Q_1)} && (41) \\ u(\omega) &= 1 + \frac{\overline{\sigma}(Q_2)\ \overline{\sigma}(\rho)\ \overline{\sigma}(Q_3)}{\overline{\sigma}(Q_1)} && (42) \end{aligned}$$

to complete the following theorem.

Theorem 9.5 *If $\mu_{\Delta_\Delta}(P_{22}) < 1,\ \Delta_\Delta \in \mathrm{B}\Delta$, than the lower-boundary-plant and upper-boundary-plant of the LFT $\mathcal{F}_l(P, \Delta_\Delta)$ are uniquely defined to be,*

$$\begin{aligned} \mathcal{L}(j\omega) &= l(\omega)Q_1(j\omega) && (43) \\ \mathcal{U}(j\omega) &= u(\omega)Q_1(j\omega). && (44) \end{aligned}$$

Proof Indeed $\forall\omega > 0$,

$$\begin{aligned} \max_{\Delta_\Delta \in B\Delta} & \overline{\sigma}(\mathcal{F}_l(P, \Delta_\Delta)) \\ &= \max_{\Delta_\Delta \in B\Delta} \overline{\sigma}(Q_1 + Q_2\widehat{\Delta}_\Delta Q_3) \\ &= \overline{\sigma}(Q_1) + \overline{\sigma}(Q_2)\overline{\sigma}(\rho)\overline{\sigma}(Q_3) \\ &= u(\omega)\overline{\sigma}(Q_1) \\ &= \overline{\sigma}(\mathcal{U}(j\omega)). \end{aligned}$$

Analogously,

$$\begin{aligned} \min_{\Delta_\Delta \in B\Delta} & \underline{\sigma}(\mathcal{F}_l(P, \Delta_\Delta)) \\ &= \min_{\Delta_\Delta \in B\Delta} \underline{\sigma}(Q_1 + Q_2\widehat{\Delta}_\Delta Q_3) \\ &= \underline{\sigma}(Q_1) + \overline{\sigma}(Q_2)\overline{\sigma}(\rho)\overline{\sigma}(Q_3) \\ &= l(\omega)\overline{\sigma}(Q_1) \\ &= \overline{\sigma}(\mathcal{L}(j\omega)). \end{aligned}$$

Note that in both cases it is assumed that $\widehat{\Delta}_\Delta$ is in the 'direction' of Q_1. Hence $\mathcal{U}(j\omega)$ and $\mathcal{L}(j\omega)$ represent non-conservative upper and lower boundaries for the LFT.

9.2.3 The μ-region in parameter space

The continuity property of $q-\Delta$ interaction guarantee that the set of equations

$$\det(I - \mathcal{F}_l(P, \Delta_\Delta)\Delta_q(j\omega, \boldsymbol{q})) = 0, \quad (45)$$

is mapped onto a closed set. In this section, we derive the mapping equations which define the boundaries of this set. Form (45) it can be read, that the set of all parameters $\boldsymbol{q}$ is to be found, such that $\lambda = 1$ is an eigenvalue of $\mathcal{F}_l(P, \Delta_\Delta)\Delta_q(j\omega, \boldsymbol{q})$. Note that this is equivalent to the requirement for $\sigma = 1$ to be a singular value of the hermitian,

$$\begin{aligned} \mathcal{H}_\Delta = {} & \mathcal{F}_l(P, \Delta_\Delta)\Delta_q + \Delta_q^*\mathcal{F}_l^*(P, \Delta_\Delta) - \\ & \mathcal{F}_l(P, \Delta_\Delta)\Delta_q\Delta_q^*\mathcal{F}_l^*(P, \Delta_\Delta). \end{aligned}$$

Formally, we are looking for $Q_{\sigma=1}$ defined by,

$$Q_{\sigma=1} = \{\boldsymbol{q} \in \boldsymbol{R}^p : \exists\omega > 0, \sigma(\mathcal{H}_\Delta) = 1\}. \quad (46)$$

Next we try to convert this problem into a $\sigma-$mapping problem. To this end, define further the UBP and LBP hermitians,

$$\begin{aligned} \mathcal{H}_u = {} & \mathcal{F}_l(P, \mathcal{U})\Delta_q + \Delta_q^*\mathcal{F}_l^*(P, \mathcal{U}) - && (47) \\ & \mathcal{F}_l(P, \mathcal{U})\Delta_q\Delta_q^*\mathcal{F}_l^*(P, \mathcal{U}), \\ \mathcal{H}_l = {} & \mathcal{F}_l(P, \mathcal{L})\Delta_q + \Delta_q^*\mathcal{F}_l^*(P, \mathcal{L}) && (48) \\ & -\mathcal{F}_l(P, \mathcal{L})\Delta_q\Delta_q^*\mathcal{F}_l^*(P, \mathcal{L}). \end{aligned}$$

Several straightforward linear algebra steps lead to the following crucial boundary conditions for these hermitians, $\forall \omega > 0$,

$$\underline{\sigma}(\mathcal{H}_l) = \min_{\Delta_q} \underline{\sigma}(\mathcal{H}_\Delta),\ \max_{\Delta_q} \overline{\sigma}(\mathcal{H}_\Delta) = \overline{\sigma}(\mathcal{H}_u), \quad (49)$$

Define further the two σ-mapping problems,

$$\begin{aligned} \boldsymbol{Q}_{\sigma<1} &= \{\boldsymbol{q} \in \boldsymbol{R}^p : \sup_\omega \overline{\sigma}(\mathcal{H}_u) < 1\}, && (50) \\ \boldsymbol{Q}_{\sigma>1} &= \{\boldsymbol{q} \in \boldsymbol{R}^p : \inf_\omega \underline{\sigma}(\mathcal{H}_l) > 1\}. && (51) \end{aligned}$$

The solution of the problem (46) is given by,

Lemma 9.6

$$\boldsymbol{Q}_{\sigma=1} = \boldsymbol{R}^p \setminus \boldsymbol{Q}_{\sigma>1} \setminus \boldsymbol{Q}_{\sigma<1} \quad (52)$$

The proof of this lemma is rather straightforward, and is omitted.

After mapping the conditions, the two steps described in (22) and (23) are to be done, to complete the μ–mapping problem for $q - \Delta$ interaction.

9.2.4 μ–Synthesis

Note that principally the $q - \Delta$ paradigm may be used also for the synthesis of the controller K w.r.t. some unstructured uncertainties Δ. The structure of the controller is assumed to be predefined and the plant P and the uncertainty Δ are mixed in an LFT, $\mathcal{F}_l(P, \Delta)$. The set K_μ is searched,

$$\begin{aligned} \boldsymbol{K}_\mu = \{&\boldsymbol{k} \in \boldsymbol{R}^p : && (53) \\ &\sup_\omega \overline{\sigma}(K(j\omega, \boldsymbol{k})) \sup_K \mu_K \mathcal{F}_l(P, \Delta) \le 1\}. \end{aligned}$$

9.3 The $q - q$ Interaction

In this section the μ–mapping problem of a $\boldsymbol{q} - \boldsymbol{q}$ interaction is defined. Just its usage for the analysis purposes is discussed here.

A $q - q$ interaction is defined on $(\mathcal{P}_q, \Delta_q)$ pairs, that is, a controller K is to be found, such that robust stability and performance criteria are satisfied w.r.t. parameter uncertainties. Principally any of the blocks Δ_q or K (see Fig. 2) may be mixed with the plant P in an LFT, to define $\mathcal{P}_q$, e.g. $P_q = \mathcal{F}_u(P, K)$. The μ–mapping problem in this case reads: find the set of all controllers K, i.e. controller parameters $\boldsymbol{k}$, such that $\forall \boldsymbol{q} \in \boldsymbol{Q} - \text{Box}$

$$\sup_\omega \overline{\sigma}(\Delta_q(j\omega, \boldsymbol{q})) \sup_{\Delta_q} \mu_{\Delta_q}(\mathcal{F}_u(P, K)) \le 1. \quad (54)$$

In [2] it was shown that μ– (i.e. robustness margin) of these systems may be discontinuous on its domain of definition. The μ– mapping approach provides a very nice framework to illustrate this discontinuity.

Consider the mapping equations

$$\boldsymbol{e}(j\omega, \boldsymbol{q}, \boldsymbol{k}) = 0. \quad (55)$$

Now observe the relation between the infinitesimal increments of $\mathrm{d}\boldsymbol{q}$ and $\mathrm{d}\boldsymbol{k}$, by applying the total differential on (55)

$$\boldsymbol{J}_q\, \mathrm{d}\boldsymbol{q} + \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{k}}\, \mathrm{d}\boldsymbol{k} + \frac{\partial \boldsymbol{e}}{\partial \omega}\, \mathrm{d}\omega = 0, \quad (56)$$

where $\boldsymbol{J}_q = \partial \boldsymbol{e}/\partial \boldsymbol{q}$ stands for the jacobian matrix w.r.t. $\boldsymbol{q}$. As long as this jacobian is non-singular, the vicinity of $(\boldsymbol{q}, \boldsymbol{k})$ is governed by continuity properties and

$$\mathrm{d}\boldsymbol{q} = -\boldsymbol{J}_q^{-1} \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{k}}\, \mathrm{d}\boldsymbol{k} - J_q^{-1} \frac{\partial \boldsymbol{e}}{\partial \omega}\, \mathrm{d}\omega. \quad (57)$$

Hence, if μ is defined on a $(\boldsymbol{q}, \boldsymbol{k})$, i.e. it is a solution of the equation

$$\boldsymbol{e} := \det(I - P_q(\boldsymbol{k})\Delta_q(j\omega, \boldsymbol{q})) = 0,$$

then it is continuous in a vicinity of $(\boldsymbol{q}, \boldsymbol{k})$ if the jacobians

$$\boldsymbol{J}_q = \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{q}} \text{ and } \boldsymbol{J}_k = \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{k}}$$

are nonsingular at $(\boldsymbol{q}, \boldsymbol{k})$.

Example 9.7 *The determinant of the jacobian $\boldsymbol{J}_q$ in example 9.2 is*

$$\begin{aligned} det(\boldsymbol{J}_q) &= -200\,\omega\, \left(\omega^2 - (1 - q_a)\omega + q_a\right) \\ &\quad \times \left(\omega^2 + (1 - q_a)\omega + q_a\right) \end{aligned}$$

Note that singular frequencies *may appear for a certain* q_a. *Indeed, observe that if for* $q_a = 3 + 2\sqrt{2}$ *a double singular frequency exists at* $\omega = 1 + \sqrt{2}$. *This bifurcation event generates two singular lines that cause the discontinuity in parameter plane. This is exactly what was reported in [2] to represent the discontinuity of robustness margin on 'problem data'. In Fig. 5*

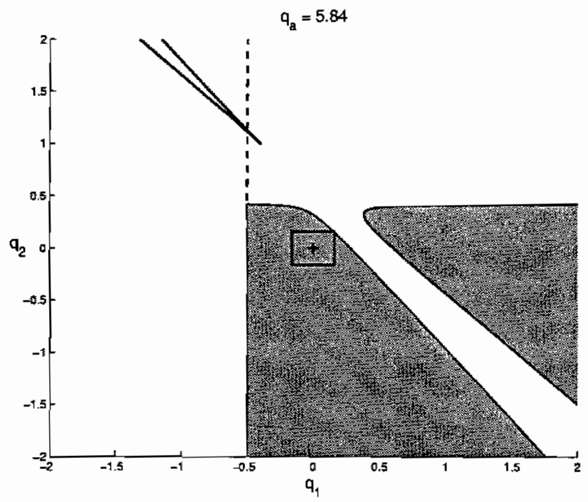

Figure 5: $\mu-$ Mapping solution for the 'problem data', $q_a = 5.84$.

the $\mu-$*mapping problem is solved for the 'problem data'* $q_a = 5.84$, *and in Fig. 6 both* $\mu-$*curves are shown to visualize the effect of the discontinuity.*

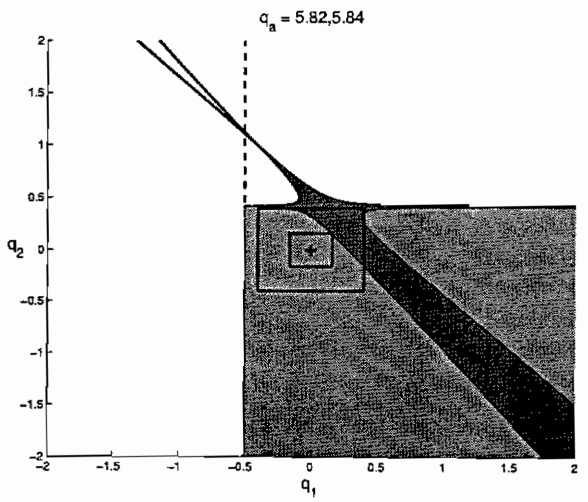

Figure 6: Illustrating the discontinuity of μ, due to a double singular frequency.

Conclusions

Singular value and structured singular value bounds mapping in parameter space is discussed. A paradigm is developed, which may be used for the H_∞ and μ analysis and synthesis in parameter space. The discontinuity of μ is shown to be related to the singularities in the solution of the mapping equations. The approach presented in this paper calls for further research efforts.

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