COMPUTATIONAL METHODS FOR PERIODIC SYSTEMS - AN OVERVIEW

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Abstract: We present an up-to-date survey of numerical methods for the analysis and design of linear discrete-time periodic systems. The basic tool is the periodic Schur form and its variants, for which a certain form of numerical stability can be ensured.

Keywords: Periodic systems, time-varying systems, discrete-time systems, numerical methods

1. INTRODUCTION

The theory of linear discrete-time periodic systems has received a lot of attention in the last 25 years (Bittanti and Colaneri, 1996). Almost all results for standard discrete-time systems have been extended to periodic systems of the form

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k + D_k u_k \end{aligned} \tag{1}$$

where the matrices $A_k \in \mathrm{R}^{n_{k+1}\times n_k}$, $B_k \in \mathrm{R}^{n_{k+1}\times m}$, $C_k \in \mathrm{R}^{p\times n_k}$, $D_k \in \mathrm{R}^{p\times m}$ are periodic with period $K \geq 1$. Most theoretical results are based on two lifting techniques which reduce the problem for the periodic system (1) to an equivalent problem for a time-invariant system of increased dimensions. The first lifting approach, proposed by Meyer and Burrus (1975), involves forming products of up to K matrices, while the second lifting approach, proposed by Flamm (1991), leads to a large order standard system representation with sparse and highly structured matrices. Although these lifting techniques are useful for their theoretical insight, their sparsity and structure may not be suited for numerical computations. This is why, in parallel to the theoretical developments, numerical methods have been developed that try to exploit this structure. For most analysis and design problems of standard state space systems, there are good numerical algorithms available that meet the standard requirements of speed and accuracy. The purpose of this paper is to present a short overview of recently developed numerical methods for the analysis and design of *periodic systems*. We also mention some open areas where there is still a need for new algorithmic developments.

Notation. To simplify the presentation we introduce first some notation. For a K-periodic matrix X_k we use alternatively the *script* notation

$$\mathcal{X} := \mathrm{diag}\,(X_0, X_1, \ldots, X_{K-1}),$$

which associates the block-diagonal matrix $\mathcal{X}$ to the cyclic matrix sequence X_k, $k = 0, \ldots, K-1$. This notation is consistent with the standard matrix operations as for instance addition, multiplication, inversion as well as with several standard matrix decompositions (Cholesky, SVD). We denote with $\sigma\mathcal{X}$ the K-cyclic shift

$$\sigma\mathcal{X} = \mathrm{diag}\,(X_1, \ldots, X_{K-1}, X_0)$$

of the cyclic sequence X_k, $k = 0, \ldots, K-1$.

By using the script notation, the periodic system (1) will be alternatively denoted by the quadruple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. The transition matrix of the system (1) is defined by the $n_j \times n_i$ matrix $\Phi_A(j,i) = A_{j-1}A_{j-2}\cdots A_i$, where $\Phi_A(i,i) := I_{n_i}$. The state transition matrix over one period $\Phi_A(j+K,j) \in \mathrm{R}^{n_j \times n_j}$ is called the *monodromy matrix* of system (1) at time j and its eigenvalues are called *characteristic multipliers* at time j.

2. DESCRIPTOR PERIODIC SYSTEMS

Descriptor periodic systems of the form

$$\begin{aligned} E_k x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k + D_k u_k \end{aligned} \qquad (2)$$

where the matrices $A_k, E_k \in \mathrm{R}^{n\times n}$, $B_k \in \mathrm{R}^{n\times m}$, $C_k \in \mathrm{R}^{p\times n}$, $D_k \in \mathrm{R}^{p\times m}$ are periodic with period $K \geq 1$, have been considered in (Sreedhar and Van Dooren, 1997; Sreedhar *et al.*, 1998). These systems may also arise in the context of ordinary periodic systems, when for instance forming the inverse or conjugate periodic system. Provided the matrices E_k are invertible we can divide the first equation from the left by E_k which then reduces to a standard periodic model with system quadruple $(\mathcal{E}^{-1}\mathcal{A}, \mathcal{E}^{-1}\mathcal{B}, \mathcal{C}, \mathcal{D})$. The monodromy matrix in this case becomes the $n \times n$ matrix $\Phi_{E^{-1}A}(j,i) := E_{j-1}^{-1}A_{j-1}E_{j-2}^{-1}A_{j-2}\cdots E_i^{-1}A_i$. It should be pointed out that analysis and design algorithms for such systems should nevertheless work even when the matrices E_k are singular, provided these problems are well defined.

3. SATISFACTORY ALGORITHMS

We first briefly discuss three key requirements for a satisfactory numerical algorithm for periodic system: generality, numerical stability, and efficiency. A *general* algorithm is one which has no limitations for its applicability of any technical nature. For the periodic system (1) it should be able to handle the most general class of periodic systems. For example, a pole assignment algorithm for a periodic system able to assign *only* distinct poles should not be considered satisfactory. Since the minimal realization of a periodic system has in general a time-varying state dimension, it is highly desirable to develop algorithms for the analysis and design of periodic systems which are able to handle systems with time-varying dimensions.

Numerical stability (more precisely, *backward stability*) of an algorithm means that the results computed by that algorithm are exact for slightly perturbed original data. As a consequence, a numerically stable algorithm applied to a well conditioned problem will produce guaranteed accurate results. This is why numerical stability is a key feature for a satisfactory algorithm. A basic ingredient to achieve numerical stability is the use of orthogonal transformations wherever possible. The use of these transformations often leads to bounds for perturbations of the initial data which are equivalent to the cumulative effect of round-off errors occurring during the computations. This is a way to prove the numerical stability of such an algorithm. This immediately suggests that one should avoid forming matrix products as those appearing in the lifted formulation of Meyer and Burrus (1975), since these amount to non-orthogonal transformations of the data matrices. The main idea when developing numerically stable algorithms for periodic systems is to exploit the problem structure by applying only orthogonal transformations on the original problem data, and thereby trying to reduce the original problem to an equivalent one which is easier to solve.

Because of the intrinsic complexity of several computational problems in systems theory, it is not always possible to develop numerically stable algorithms for them. Therefore one often imposes this requirement only on the substeps of the algorithm. Although this is not enough to guarantee numerical stability of the global algorithm, one can still expect that it will perform accurately on well-conditioned problems.

The *efficiency* of an algorithm involves two main aspects: avoiding extensive storage use and keeping the computational complexity as low as possible. For periodic systems the first requirement implies that the storage should be proportional to the amount of data defining the system, i.e. it should be $O(K\overline{n}^2) + O(K\overline{n}m) + O(K\overline{n}p)$, where $\overline{n} = \max\{n_i\}$. Explicitly forming the lifted representation of Flamm (1991) should thus be avoided. Concerning the second requirement applied to a periodic system of period K, one would hope for a complexity of at most $O(K\overline{n}^3)$ since the complexity for standard state-space algorithms is typically $O(n^3)$. This implies again that one should not use large dimensional lifted representations.

4. BASIC NUMERICAL INGREDIENTS

The use of condensed forms of the system matrices, obtained under orthogonal transformations, is a basic ingredient for solving many computational problems (Van Dooren and Verhaegen, 1985). The system matrices are transformed to a particular coordinate system in which they are condensed, such that the solution of the computational problem is straightforward. For periodic systems with constant dimensions, the *periodic real Schur form* (PRSF) plays an important role in solving many computational problems. According to Bojanczyk *et al.* (1992), given the matrices $A_k \in \mathrm{R}^{n\times n}$, $k = 0, 1, \ldots, K{-}1$, there exist orthogonal matrices Z_k, $k = 0, 1, \ldots, K{-}1$, $Z_K := Z_0$, such that

$$\widetilde{A}_k := Z_{k+1}^T A_k Z_k \tag{3}$$

where $\widetilde{A}_{K-1}$ is in *real Schur form* (RSF) and the matrices $\widetilde{A}_k$ for $k = 0, \ldots, K{-}2$ are upper triangular. Numerically stable algorithms to compute the PRSF have been proposed in (Bojanczyk *et al.*, 1992; Hench and Laub, 1994). By using these algorithms, we can determine the orthogonal matrices Z_k, $k = 0, \ldots, K{-}1$ to reduce the cyclic product $A_{K-1} \cdots A_1 A_0$ to the RSF without forming explicitly this product. An intermediate condensed form with potential applications in computational algorithms is the *periodic Hessenberg form* (PHF), where $\widetilde{A}_{K-1}$ is in a Hessenberg form, while $\widetilde{A}_k$ for $k = 0, \ldots, K{-}2$ are upper triangular.

For systems with time-varying dimensions, the *extended periodic real Schur form* (EPRSF) represents a generalization of the PRSF which allows to address many problems with varying dimensions. According to Varga (1999), given the matrices $A_k \in \mathrm{R}^{n_{k+1}\times n_k}$, $k = 0, 1, \ldots, K{-}1$, with $n_K = n_0$ there exist orthogonal matrices $Z_k \in \mathrm{R}^{n_k\times n_k}$, $k = 0, 1, \ldots, K-1$, $Z_K := Z_0$, such that the matrices

$$\widetilde{A}_k := Z_{k+1}^T A_k Z_k = \begin{bmatrix} \widetilde{A}_{k,11} & \widetilde{A}_{k,12} \\ 0 & \widetilde{A}_{k,22} \end{bmatrix}, \tag{4}$$

are block upper triangular, where $\widetilde{A}_{k,11} \in \mathrm{R}^{\underline{n}\times\underline{n}}$, $\widetilde{A}_{k,22} \in \mathrm{R}^{(n_{k+1}-\underline{n})\times(n_k-\underline{n})}$ for $k = 0, 1, \ldots, K{-}1$ and $\underline{n} = \min_k\{n_k\}$. Moreover, $\widetilde{A}_{K-1,11}$ is in RSF, $\widetilde{A}_{k,11}$ for $k = 0, \ldots, K{-}2$ are upper triangular and $\widetilde{A}_{k,22}$ for $k = 0, \ldots, K{-}1$ are upper trapezoidal.

For descriptor systems with fixed dimensions, the *generalized periodic real Schur form* (GPRSF) extends the PRSF to so-called regular periodic systems (see e.g. Bojanczyk *et al.* (1992)). Given the matrices $A_k, E_k \in \mathrm{R}^{n\times n}$, $k = 0, 1, \ldots, K{-}1$, there exist orthogonal matrices $Z_k, Q_k \in \mathrm{R}^{n\times n}$, $k = 0, 1, \ldots, K-1$, $Z_K := Z_0$, such that the matrices

$$\widetilde{A}_k := Q_k^T A_k Z_k, \quad \widetilde{E}_k := Q_k^T E_k Z_{k+1}, \tag{5}$$

are all upper triangular, except for $\widetilde{A}_{K-1}$, which is in RSF.

5. PERIODIC MATRIX EQUATIONS

The reduction of a periodic matrix A_k to PRSF and EPRSF is the principal ingredient in solving important linear equations for periodic systems as the periodic Lyapunov and Sylvester equations. Periodic Lyapunov equations appear in solving periodic state-feedback stabilization problems or in computing gradients for optimal periodic output feedback problems (Varga and Pieters, 1998). Consider for example the *discrete-time periodic Lyapunov equations* (DPLE)

$$\mathcal{X} = \mathcal{A}^T \sigma \mathcal{X} \mathcal{A} + \mathcal{V} \tag{6}$$

$$\sigma\mathcal{X} = \mathcal{A} \mathcal{X} \mathcal{A}^T + \sigma\mathcal{W} \tag{7}$$

where V_k, W_k are symmetric K-periodic matrices of appropriate dimensions. To ensure the existence of a unique solution of these equations we assume that the monodromy matrix $\Phi_A(K, 0)$ has no reciprocal eigenvalues.

The orthogonal Lyapunov transformation in (3) or (4) can be expressed as $\widetilde{\mathcal{A}} = \sigma\mathcal{Z}^T\mathcal{A}\mathcal{Z}$. By multiplying equation (6) with $\mathcal{Z}^T$ from left and with $\mathcal{Z}$ from right, and multiplying equation (7) with $\sigma\mathcal{Z}^T$ from left and with $\sigma\mathcal{Z}$ from right, one obtains

$$\widetilde{\mathcal{X}} = \widetilde{\mathcal{A}}^T \sigma \widetilde{\mathcal{X}} \widetilde{\mathcal{A}} + \widetilde{\mathcal{V}}, \tag{8}$$

$$\sigma\widetilde{\mathcal{X}} = \widetilde{\mathcal{A}} \widetilde{\mathcal{X}} \widetilde{\mathcal{A}}^T + \sigma\widetilde{\mathcal{W}}, \tag{9}$$

where $\widetilde{\mathcal{X}} = \mathcal{Z}^T\mathcal{X}\mathcal{Z}$, $\widetilde{\mathcal{V}} = \mathcal{Z}^T\mathcal{V}\mathcal{Z}$ and $\widetilde{\mathcal{W}} = \mathcal{Z}^T\mathcal{W}\mathcal{Z}$. By this transformation the resulted transformed equations (8) and (9) have exactly the same form as the original ones in (6) and (7), but this time the periodic matrix $\widetilde{A}_k$ is in PRSF or EPRSF. After solving these equations for $\widetilde{\mathcal{X}}$, the solution results as $\mathcal{X} = \mathcal{Z}\widetilde{\mathcal{X}}\mathcal{Z}^T$. The reduced DPLEs (8) and (9) can be solved by using special substitution algorithms (Varga, 1997). Important computational subproblems are in this context the efficient and numerically stable solution of low order DPLEs and periodic Sylvester equations. Computational approaches for these subproblems are also described in detail in (Varga, 1997).

The computation of periodic reachability and observability grammians for periodic systems involves the solution of periodic Lyapunov equations having nonnegative solutions. For example, assuming that $\Phi_A(K, 0)$ has only eigenvalues in the interior of the unit circle, the DPLE

$$\mathcal{X} = \mathcal{A}^T \sigma \mathcal{X} \mathcal{A} + \mathcal{R}^T \mathcal{R} \tag{10}$$

can be solved directly for the Cholesky factor $\mathcal{U}$ of the nonnegative definite solution $\mathcal{X} = \mathcal{U}^T\mathcal{U}$. When solving minimal realization or model reduction problems, the periodic matrices A_k and R_k in (10) have often time-varying dimensions. Efficient algorithms to solve nonnegative DPLE are based on transformation techniques involving the PRSF for constant dimensions (Varga, 1997) or the EPRSF for time-varying dimensions (Varga, 1999). Similar comments hold for the dual equation

$$\sigma\mathcal{X} = \mathcal{A}\mathcal{X}\mathcal{A}^T + \sigma\mathcal{R}\sigma\mathcal{R}^T. \tag{11}$$

A class of periodic robust state-feedback pole assignment problems can be reduced to the solution of a *periodic Sylvester equation* (PSE) of the form (Varga, 2000*b*)

$$\mathcal{A}\mathcal{X} + \sigma\mathcal{X}\mathcal{B} = \mathcal{C} \tag{12}$$

where A_k, B_k and C_k are K-periodic matrices with constant dimensions. By reducing A_k and B_k to the PRSF, the periodic solution X_k can be computed using a transformation method which generalizes the well-known Bartels-Stewart method (Byers and Rhee, 1995; Varga, 2000*b*).

Periodic Riccati equations appear when solving periodic LQ-design problems (Sreedhar and Van Dooren, 1994*b*). Using our notation it can be written as follows

$$\mathcal{P} = \mathcal{A}^T[\sigma\mathcal{P} - \sigma\mathcal{P}\mathcal{B}(\mathcal{R} + \mathcal{B}^T\sigma\mathcal{P}\mathcal{B})^{-1}\mathcal{B}^T\mathcal{P}]\mathcal{A} + \mathcal{S}.$$

The solution of this periodic Riccati equation can be obtained from the generalized periodic Schur decomposition of $(\hat{E}_k, \hat{A}_k)$, where

$$\hat{E}_k \doteq \begin{bmatrix} I_n & B_k R_k^{-1} B_k^T \\ 0 & A_k^T \end{bmatrix}, \; \hat{A}_k \doteq \begin{bmatrix} A_k & 0 \\ -S_k & I_n \end{bmatrix} \tag{13}$$

Upon partitioning the $2n \times 2n$ matrices Z_k of the GPRSF of (13) as

$$\hat{Z}_k \doteq \begin{bmatrix} X_k & V_k \\ Y_k & W_k \end{bmatrix}, \tag{14}$$

one obtains the matrices P_k that constitute $\mathcal{P}$ as $P_k = Y_k X_k^{-1}$, provided the GPRSF has ordered eigenvalues, i.e. the stable eigenvalues appear first in the Schur form. Algorithms for this have been proposed in (Bojanczyk *et al.*, 1992; Hench and Laub, 1994; Sreedhar and Van Dooren, 1994*b*). Similar results hold for the dual periodic Riccati equation occurring in the filtering problem :

$$\sigma\mathcal{P} = \mathcal{A}[\mathcal{P} - \mathcal{P}\mathcal{C}^T(\mathcal{R} + \mathcal{C}\mathcal{P}\mathcal{C}^T)^{-1}\mathcal{C}\mathcal{P}]\mathcal{A}^T + \sigma\mathcal{S}.$$

6. ALGORITHMS FOR THE ANALYSIS OF PERIODIC SYSTEMS

Structural properties of stable periodic systems such as reachability, observability, minimality can be analyzed by computing the reachability and observability grammians. With a straightforward scaling, the same techniques can be used to study unstable systems as well. The computation of minimal (i.e., completely reachable and completely observable) realizations can be done using efficient and numerically reliable algorithms proposed by Varga (1999). Order reduction of periodic systems using balancing techniques can be performed by using accuracy enhancing *square-root* and *balancing-free* algorithms developed in (Varga, 2000*a*). The main computation in these algorithms is the solution of two nonnegative definite periodic Lyapunov equations to determine directly the Cholesky factors of the controllability and observability grammians. The *square-root* term signifies that all subsequent computations for determining the projection matrices are based exclusively on square-root information (i.e., Cholesky factors of the grammians). The accuracy can be further enhanced by avoiding the computation of the possibly ill-conditioned balancing transformation. Instead, well-conditioned projection matrices are constructed which leads to so-called *balancing-free* order reduction.

The computation of poles is important in many applications. To compute the poles of a periodic system, the eigenvalues of the *monodromy* matrix product $\Phi_A(K, 0)$ must be determined. The computation can be done without forming this matrix product explicitly, but by reducing the K-periodic matrices A_k to the PRSF in the case of constant dimensions, or to the EPRSF in the case of time-varying dimensions. If only poles are requested one can also use the more economical approach suggested by Van Dooren (1999).

The evaluation of system norms can be done using reliable algorithms. The Hankel-norm of a periodic system can be computed as the maximal singular value of the products of periodic grammians. This computation is part of the algorithm to determine minimal realizations of periodic systems using balancing techniques (Varga, 1999). The computation of the H_2-norm involves the solution of a periodic Lyapunov equation to determine the controllability or observability grammians (Bittanti and Colaneri, 1996). This can be done using algorithms proposed in (Varga, 1997; Varga, 1999). For the computation of the H_∞-norm an algorithm with quadratic convergence is given in (Sreedhar *et al.*, 1997). The main step there is the computation of generalized eigenvalues of a periodic symplectic pencil using the periodic QZ-decomposition (Bojanczyk *et al.*, 1992).

7. ALGORITHMS FOR THE DESIGN OF PERIODIC SYSTEMS

Basic design algorithms for periodic systems with constant state dimension have been proposed by several authors. A Schur method for pole assignment has been proposed by Sreedhar and Van Dooren (1993) and a stabilization algorithm has been proposed by Sreedhar and Van Dooren (1994*a*). A robust pole assignment method relying on Sylvester equations has been proposed in (Varga, 2000*b*).

A computational approach for the periodic LQG methods involves the solution of periodic Riccati equations. Efficient methods have been proposed using the ordered PRSF or *generalized periodic real Schur form* (GPRSF) (Bojanczyk *et al.*, 1992; Hench and Laub, 1994). A computational approach for the solution of the optimal periodic output feedback problem has been developed by Varga and Pieters (1998).

8. ALGORITHMS FOR DESCRIPTOR PERIODIC SYSTEMS

All of the equations for standard periodic systems $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ can be extended to descriptor periodic systems $(\mathcal{E}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ provided $\mathcal{E}$ is invertible. When $\mathcal{E}$ is singular, one has to show that the underlying control problem makes sense and is solved by the corresponding equations. For regular periodic systems this is typically the case (see e.g., Sreedhar and Van Dooren (1999)).

The generalized Lyapunov and Riccati equations are

$$\mathcal{E}\sigma\mathcal{X}\mathcal{E}^T = \mathcal{A}\mathcal{X}\mathcal{A}^T + \sigma\mathcal{W}, \tag{15}$$

and

$$\mathcal{E}\sigma\mathcal{P}\mathcal{E}^T = \mathcal{A}[\mathcal{P} - \mathcal{P}\mathcal{C}^T(\mathcal{R} + \mathcal{C}\mathcal{P}\mathcal{C}^T)^{-1}\mathcal{C}\mathcal{P}]\mathcal{A}^T + \sigma\mathcal{S},$$

respectively. Using the GPRSF (5), which can be written as follows :

$$\widetilde{\mathcal{E}} = \mathcal{Q}^T\mathcal{E}\sigma\mathcal{Z}, \quad \widetilde{\mathcal{A}} = \mathcal{Q}^T\mathcal{A}\mathcal{Z}, \tag{16}$$

these can easily be transformed to the same equations where now $\mathcal{E}$ and $\mathcal{A}$ are in GPRSF form. From this, the required solution then easily follows in much the same way as for the standard periodic equations. It is shown by Sreedhar and Van Dooren (1994*b*) how these equations can been used to solve stabilization and optimal control problems of periodic descriptor systems.

The generalized Sylvester equation has the form

$$\mathcal{E}\sigma\mathcal{X}\mathcal{B} + \mathcal{A}\mathcal{X}\mathcal{F} = \mathcal{C}, \tag{17}$$

where the matrix pairs $\mathcal{E}, \mathcal{A}$ and $\mathcal{B}, \mathcal{F}$ can again be put in GPRSF. Using the latter, a lifting technique for periodic descriptor systems has been introduced by Sreedhar *et al.* (1998) which, in conjunction with the backward/forward decomposition technique of Sreedhar and Van Dooren (1997), allows to determine minimal order representations of descriptor periodic systems.

Further, poles of descriptor periodic systems can be computed using the periodic QZ decomposition proposed by Bojanczyk *et al.* (1992) or the more economical variant described by Van Dooren (1999).

9. SOME EXTENSIONS AND OPEN PROBLEMS

There are several computational problems for periodic systems for which it is in principle straightforward to develop reliable computational methods by extending algorithms for standard systems. Frequency-domain methods for the analysis of periodic systems rely on the $Kp \times Km$ *transfer-function matrix* (TFM) of the associated lifted systems (Meyer and Burrus, 1975; Flamm, 1991). The computation of frequency responses can be done by computing first the corresponding TFM and then evaluating the frequency response using the resulting rational matrix. A method to compute the TFM can be devised along the lines of the poles/zeros approach proposed by Varga and Sima (1981) for standard systems and in by Varga (1989) for descriptor systems. Alternatively, the frequency response can be computed by exploiting the sparse structure of the lifted representation of the periodic system. Here the periodic Hessenberg form can play potentially the same role as the Hessenberg form in the case of standard systems (Laub, 1981).

Recursive Schur techniques to compute coprime factorizations (Varga, 1998) can be extended to the periodic case along the line of the periodic Schur form method for pole assignment proposed by Sreedhar and Van Dooren (1993). This has been done for the coprime factorization with inner denominator by Varga (2001) in the context of balancing and model reduction of unstable periodic systems. A computational approach to determine normalized coprime factorizations can probably be developed based on results for standard systems of Bongers and Heuberger (1990).

There are several open problems for which still efficient algorithms are to be developed, as for example, the computation of controllability and observability canonical forms, zeros of the associated TFM, Kronecker-structure of associated system pencil, inner-outer and spectral factorization, and so on. Recent developments also look at the solution of periodic Linear Matrix Inequalities for solving various design problems (Bittanti and Cuzzola, 2000). The use of periodic matrix decompositions could be useful here as well.

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