Computing generalized inverse systems using matrix pencil methods

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Abstract

We address the numerically reliable computation of generalized inverses of rational matrices in descriptor state space representation. We put a particular emphasis on two classes of inverses: the weak generalized inverse and the Moore-Penrose pseudoinverse. By combining the underlying computational techniques, other types of inverses of rational matrices can be computed as well. The main computational ingredient to determine generalized inverses is the orthogonal reduction of the system matrix pencil to appropriate Kronecker-like forms.

Keywords: System inversion, rational matrices, descriptor systems, numerical methods.

1 Introduction

Inverse systems have many important applications in areas such as control theory, filtering and coding theory. The computation of so-called *zero initial state* system inverses for linear time-invariant state-space systems is essentially equivalent to compute generalized inverses of the associated transfer-function matrices. For square and invertible systems, the computation of inverses can be done by explicit formulas either in the standard state-space or in a descriptor system formulation. For non-square systems, explicit formulas can be employed only in the full-rank case to determine left or right inverses, provided the system feedthrough matrix has also full rank. However, these direct formulas do not allow to arbitrarily choose the spurious poles which appear in the computed left or right inverses. In the more general case of systems with transfer-function matrices of arbitrary rank, no explicit formulas can be used.

In this paper we address the numerically reliable computation of generalized inverses of rational matrices by using orthogonal matrix pencil reduction techniques. We put a particular emphasis on the computation of two classes of inverses: the weak generalized inverse, known also as the (1,2)-inverse, and the Moore-Penrose pseudoinverse. The (1,2)-inverses can be computed using a numerically reliable approach based on the reduction of the system matrix pencil to a particular Kronecker-like form. The computation of the Moore-Penrose pseudoinverse is done by employing full rank factorizations resulted from appropriate column/row compressions with all-pass factors. By combining the underlying computational techniques, other types of inverses of rational matrices can be computed as well. We present some numerical examples to illustrate our methods and report on recently developed MATLAB software developed by the author to compute some generalized inverses.

2 Generalized inverses

Consider a $p \times m$ rational matrix $G(\lambda)$ with real coefficients. Throughout the paper, we assume that $\operatorname{rank} G(\lambda) = r$ over rationals (i.e., $G(\lambda)$ has rank r for almost all values of λ). The *zeros* of $G(\lambda)$ are those values of λ (finite or infinite) where $G(\lambda)$ looses its maximal rank r. The *poles* of $G(\lambda)$ are those values of λ (finite or infinite) where the elements of $G(\lambda)$ become infinite. We call $G(\lambda)$ *proper* if it has only finite zeros (i.e., $G(\infty)$ is finite). In a system theoretical setting, $G(\lambda)$ can be interpreted as the transfer-function matrix either of a continuous-time system, if $\lambda = s$ is the complex variable appearing in the Laplace transform, or of a discrete-time system, if $\lambda = z$ is the complex variable appearing in the Z-transform. Accordingly, we call $G(\lambda)$ *stable* if all its poles lie in the appropriate stability domain (i.e., the left open half complex plane for a continuous-time system or the interior of the unit circle for a discrete-time system). Depending on the type of the system, the *conjugate* of $G(\lambda)$ is the matrix $G^{\sim}(\lambda)$ defined as $G^{\sim}(s) = G^T(-s)$ for a continuous-time system or $G^{\sim}(z) = G^T(1/z)$ for a discrete-time system. We say that $G(\lambda)$ is *all-pass* if $G^{\sim}(\lambda)G(\lambda) = I_m$. A stable all-pass matrix is called an *inner* matrix. Throughout the paper we will work exclusively with *square* all-pass/inner matrices.

Let $X(\lambda)$ be an $m \times p$ real rational matrix. Consider the following Moore-Penrose relations (see e.g., [2]):

$$
\begin{array}{llcl}
(1) & G(\lambda)X(\lambda)G(\lambda) & = & G(\lambda) \\
(2) & X(\lambda)G(\lambda)X(\lambda) & = & X(\lambda) \\
(3) & G(\lambda)X(\lambda) & = & (G(\lambda)X(\lambda))^{\sim} \\
(4) & X(\lambda)G(\lambda) & = & (X(\lambda)G(\lambda))^{\sim}
\end{array}
$$

The well-known *Moore-Penrose pseudoinverse* $X(\lambda) = G^{\#}(\lambda)$ is unique and satisfies all four Moore-Penrose relations. Depending on the interpretation of λ, we have for the same rational matrix $G(\lambda)$ different pseudoinverses $G^{\#}(s)$ and $G^{\#}(z)$ for a continuous-time and a discrete-time system, respectively.

In solving practical problems, the uniqueness of the Moore-Penrose pseudoinverse is rather a disadvantage, since no flexibility is provided, for example, in assigning the poles of the corresponding inverse system to desired locations. Therefore, often other types of generalized inverses are preferred. The *weak generalized inverse* $X(\lambda) = G^{+}(\lambda)$ satisfies only the first two Moore-Penrose conditions and is therefore called a (1,2)-inverse. Weak generalized inverses are useful in solving rational matrix equations or in computing inner-outer factorizations [15].

In what follows we will use the notation $G^{+}(\lambda)$ for all types of inverses satisfying one or more Moore-Penrose conditions. Of particular importance for applications are [2, 5]: the (1)-inverse to solve systems of equations with rational matrices, the (1,3)-inverse to compute least square solutions of rational matrix equations, the (1,4)-inverse, known also as the *minimum norm inverse*, to compute minimum norm solutions of matrix equations. With this nomenclature, the Moore-Penrose pseudoinverse is an (1,2,3,4)-inverse. The *left* and *right inverses*, frequently used in the control literature, are particular (1,2)-inverses of full column rank or full row rank matrices, respectively.

Note: Our approach to determine generalized inverses relies on the standard system inversion concepts used in the control literature. Therefore, to define the generalized inverses using the Moore-Penrose conditions we will not assume that $G(\lambda)$ has constant rank for all λ (as is done elsewhere [4]). It follows, that the generalized inverses computed by our methods satisfy conditions (1)-(4) (or part of them) for almost all values of λ, excepting a finite set of points, which includes certainly the poles and zeros of $G(\lambda)$.

3 Computation of weak inverses

Weak or (1,2)-inverses of proper rational matrices appear frequently in control, filtering and coding applications. An important class of (1,2)-inverses form the left and right inverses. Their computations and properties received considerable attention in the control literature (see e.g. [18] and the references cited therein). The properties of more general weak inverses have been studied in [12] (for an application of these concepts in solving differential-algebraic equations see also [6]).

In this section we consider the most general case of computing the (1,2)-inverse of an arbitrary rank rational matrix $G(\lambda)$, without the restriction that $G(\lambda)$ is proper. Thus, the proposed approach is applicable to general improper $G(\lambda)$, and in particular, to polynomial matrices. Important associated problems are the computation of stable (1,2)-inverses (whenever exist) or of least McMillan degree (1,2)-inverses. A still unsolved problem is the computation of stable (1,2)-inverses with least McMillan degree.

Let $G(\lambda)$ be a real rational matrix of rank r, having a n-th order regular descriptor representation $\mathbf{G}$, denoted as $\mathbf{G} := (A - \lambda E, B, C, D)$, which satisfies

$$G(\lambda) = C(\lambda E - A)^{-1}B + D.$$

We assume that the descriptor representation of $G(\lambda)$ is minimal, thus n is the least integer for which the above relation holds. Let $S(\lambda)$ be the system pencil matrix associated to the descriptor system $\mathbf{G} = (A - \lambda E, B, C, D)$

$$S(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right].$$

From the following straightforward formula

$$\left[\begin{array}{cc} A - \lambda E & 0 \\ 0 & G(\lambda) \end{array}\right] = \left[\begin{array}{cc} I_n & 0 \\ -C(A - \lambda E)^{-1} & I_p \end{array}\right] S(\lambda) \left[\begin{array}{cc} I_n & -(A - \lambda E)^{-1}B \\ 0 & I_m \end{array}\right]$$

it follows that $S(\lambda)$ has rank $n+r$ over rationals. Further, the computation of (1,2)-inverses of $G(\lambda)$ can be done as

$$G^{+}(\lambda) = \left[\begin{array}{cc} 0 & I_m \end{array}\right] S^{+}(\lambda) \left[\begin{array}{c} 0 \\ I_p \end{array}\right], \tag{1}$$

where $S^+(\lambda)$ is an (1,2)-inverse of $S(\lambda)$. By using this formula the computation of the (1,2)-inverse of any rational matrix can be accomplished by computing the (1,2)-inverse of the associated system pencil. Note that if $m = p = r$, then the (1,2)-inverse of $S(\lambda)$ is the ordinary inverse of this matrix, and thus (1) represents the descriptor inverse of an invertible rational matrix. Further, if D is invertible, then the inverse system can be alternatively expressed as

$$\mathbf{G^{-1}} := (A - BD^{-1}C - \lambda E,\ -BD^{-1},\ D^{-1}C,\ D^{-1})$$

In the general case, we will use the following result to compute (1,2)-inverses [2]:

Lemma 1 *Let $S(\lambda)$ be a rational matrix of rank $n+r$ and let P_l and P_r permutation matrices such that*

$$P_l S(\lambda) P_r = \begin{bmatrix} S_{11}(\lambda) & S_{21}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{bmatrix}$$

where $\operatorname{rank} S_{11}(\lambda) = n + r$. Then, an (1,2)-inverse of $S(\lambda)$ is

$$S^+(\lambda) = P_r \begin{bmatrix} S_{11}^{-1}(\lambda) & 0 \\ 0 & 0 \end{bmatrix} P_l.$$

The computation of $S^+(\lambda)$ can be done by reducing $S(\lambda)$ to an appropriate Kronecker-like form from which a maximal rank regular sub-pencil can be easily separated. Let Q and Z be orthogonal matrices to reduce $S(\lambda)$ to the Kronecker-like form

$$\overline{S}(\lambda) := QS(\lambda)Z = \left[\begin{array}{c|c} \overline{S}_{11} & \overline{S}_{12}(\lambda) \\ \hline 0 & \overline{S}_{22} \end{array}\right] = \left[\begin{array}{c|cccc} B_r & A_r - \lambda E_r & A_{r,reg} - \lambda E_{r,reg} & A_{r,l} - \lambda E_{r,l} \\ 0 & 0 & A_{reg} - \lambda E_{reg} & A_{reg,l} - \lambda E_{reg,l} \\ 0 & 0 & 0 & A_l - \lambda E_l \\ \hline 0 & 0 & 0 & C_l \end{array}\right]$$

where the regular part $A_{reg} - \lambda E_{reg}$ contains the finite and infinite system zeros, the pair $(A_r - \lambda E_r, B_r)$ is controllable with E_r nonsingular and the pair $(C_l, A_l - \lambda E_l)$ is observable with E_l nonsingular (see [14] for how to obtain such a Kronecker-like form). The controllability of the pair $(A_r - \lambda E_r, B_r)$ and the observability of the pair $(C_l, A_l - \lambda E_l)$ are the consequences of full row rank and full column rank conditions involving these pairs.

Since

$$\operatorname{rank} \overline{S}(\lambda) = \operatorname{rank} \overline{S}_{12}(\lambda),$$

by applying Lemma 1 with an obvious choice of the permutation matrices, we obtain an (1,2)-inverse of $G(\lambda)$ as

$$\mathbf{G}^+ := (\overline{A}_{12} - \lambda \overline{E}_{12}, \overline{B}_1, \overline{C}_2, 0),$$

where

$$\overline{A}_{12} - \lambda \overline{E}_{12} := \overline{S}_{12}(\lambda), \quad \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} := Q \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad [\, \overline{C}_1\ \overline{C}_2 \,] := [\, 0\ \ -I_m \,] Z$$

The eigenvalues of the inverse are

$$\Lambda(\overline{A}_{12}, \overline{E}_{12}) = \Lambda_{fixed} \cup \Lambda_{spurious},$$

where $\Lambda_{fixed} = \Lambda(A_{reg}, E_{reg})$ are the eigenvalues of the regular part an thus contains the system zeros, and $\Lambda_{spurious} = \Lambda(A_r, E_r) \cup \Lambda(A_l, E_l)$ are the finite "spurious" zeros originating from the column/row singularities of $G(\lambda)$.

While the system zeros are always among the poles of the inverse, the spurious poles can be arbitrarily chosen. To show this, consider two transformation matrices U and V of the form

$$U = \left[\begin{array}{ccc|c} I & 0 & 0 & K_1 \\ 0 & I & 0 & K_2 \\ 0 & 0 & I & K_3 \\ \hline 0 & 0 & 0 & I \end{array}\right] := \left[\begin{array}{c|c} I & K \\ \hline 0 & I \end{array}\right]$$

$$V = \left[\begin{array}{c|ccc} I & F_1 & F_2 & F_3 \\ \hline 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right] := \left[\begin{array}{c|c} I & F \\ \hline 0 & I \end{array}\right]$$

Then the transformed system pencil is given by

$$\begin{aligned} \widehat{S}(\lambda) &:= U\overline{S}(\lambda)V = \left[\begin{array}{c|c} \overline{S}_{11} & \widehat{S}_{12}(\lambda) \\ \hline 0 & \overline{S}_{22} \end{array}\right] \\ &= \left[\begin{array}{c|ccc} B_r & A_r + B_rF_1 - \lambda E_r & A_{r,reg} + B_rF_2 - \lambda E_{r,reg} & A_{r,l} + B_rF_3 + K_1C_l - \lambda E_{r,l} \\ 0 & 0 & A_{reg} - \lambda E_{reg} & A_{reg,l} + K_2C_l - \lambda E_{reg,l} \\ 0 & 0 & 0 & A_l + K_3C_l - \lambda E_l \\ \hline 0 & 0 & 0 & C_l \end{array}\right] \end{aligned} \tag{2}$$

where $\widehat{S}_{12}(\lambda) = \overline{S}_{12}(\lambda) + \overline{S}_{11}F + K\overline{S}_{22}$. The inverse is defined by

$$\mathbf{G}^+ := (\widehat{A}_{12} - \lambda\widehat{E}_{12}, \widehat{B}_1, \widehat{C}_2, 0),$$

where

$$\begin{aligned} \widehat{A}_{12} - \lambda\widehat{E}_{12} &:= \overline{A}_{12} + \overline{S}_{11}F + K\overline{S}_{22} - \lambda\overline{E}_{12}, \\ \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \end{bmatrix} &:= UQ\begin{bmatrix} 0 \\ I_p \end{bmatrix} = \begin{bmatrix} \overline{B}_1 + K\overline{B}_2 \\ \overline{B}_2 \end{bmatrix}, \\ [\, \widehat{C}_1 \ \widehat{C}_2 \,] &:= [\, 0 \ -I_m \,]ZV = [\, \overline{C}_1 \ \overline{C}_2 + \overline{C}_1F \,]. \end{aligned} \tag{3}$$

An important aspect in some applications is to obtain inverses with the spurious poles lying in a "good" domain $\mathbb{C}_g$ of the complex plane (e.g., stable or antistable domain). For the transformed pencil $\widehat{S}(\lambda)$, the spurious poles of the inverse are

$$\Lambda_{spurious} = \Lambda(A_r + B_rF_1, E_r) \cup \Lambda(A_l + K_3C_l, E_l)$$

Thus, with $F_2 = 0$, $F_3 = 0$, $K_1 = 0$, $K_2 = 0$ and by choosing F_1 such that $\Lambda(A_r + B_rF_1, E_r) \subset \mathbb{C}_g$ and K_3 such that $\Lambda(A_l + K_3C_l, E_l) \subset \mathbb{C}_g$ all spurious poles can be moved to $\mathbb{C}_g$. This is always possible because the pair $(A_r - \lambda E_r, B_r)$ is controllable and the pair $(C_l, A_l - \lambda E_l)$ is observable. Thus, our approach provides, in the most general setting, a complete solution to the stable left/right inverse problems solved in [1] .

Another interesting aspect is the computation of inverses with least McMillan degree. This aspect can be addressed by our approach in the following way. We can choose

F and/or K to make some of spurious poles of $G^+(\lambda)$ unobservable or uncontrollable. For example, for a fixed K, choosing F such that all spurious poles corresponding to the pair $(A_r + B_r F_1, E_r)$ becomes unobservable is the disturbance rejection (or output-nulling) problem (DRP) [20]. Similarly, cancelling the spurious poles of the pair $(A_l + K_3 C_l, E_l)$ is the dual of a DRP (unknown-input problem). The cancellation of all spurious poles is usually not possible, but to find the (1,2)-inverse with least McMillan degree, we can always cancel a maximum number of poles by appropriate choice of F and K. The computational problem is essentially of finding an (A,B)-invariant subspace of least dimension which contains a given subspace [8] and reliable algorithms for its solution can be devised. A much more difficult problem is to compute least McMillan degree inverses with the additional constraint that the spurious poles lie in a given domain $\mathbb{C}_g$. As far as we know, there is no complete solution of this problem, this being equivalent to the difficult question of stabilizing a linear system via constant gain output feedback (see also [19]).

An important advantage of our computational approach to compute the (1,2)-inverse is that a descriptor representation of inverse is determined without explicitly inverting the system matrix $S(\lambda)$. The reduction to the Kronecker-like form can be done by using a structure preserving numerically stable reduction algorithm similar to that proposed in [7] (see also [14]). The eigenvalue placement problems can be solved by using stabilization or pole assignment techniques for descriptor systems as those proposed in [13]. Note that this computation must be performed only if the spurious eigenvalues must lie in a specific region $\mathbb{C}_g$ of the complex plane, as for instance, in the stability domain as necessary in computing certain inner-outer factorizations [15, 17].

4 Computation of pseudoinverse

The numerical computation of Moore-Penrose pseudoinverses has been only recently addressed in the control literature [10]. A numerically reliable approach to compute pseudoinverses of rational matrices can be devised along the lines of the recently developed general methods to compress rational matrices to full row/column-rank matrices by using left/right multiplication with inner factors. This computation is the main part of the recently developed algorithms to compute inner-outer factorizations of rational matrices [10, 9]. By using "orthogonal" compression techniques with inner factors, the computation of the Moore-Penrose pseudoinverse of a given $G(\lambda)$ can be done as follows:

1. Compute the full row-rank factorization

$$G(\lambda) = U(\lambda) \begin{bmatrix} G_1(\lambda) \\ O \end{bmatrix}$$

   where $G_1(\lambda)$ has full row-rank and $U(\lambda)$ is square inner ($U^{\sim}(\lambda)U(\lambda) = I_p$).

2. Compute the full row-rank factorization

$$G_1^T(\lambda) = V^T(\lambda) \begin{bmatrix} G_2^T(\lambda) \\ O \end{bmatrix}$$

where $G_2(\lambda)$ is invertible and $V(\lambda)$ is square inner ($V^{\sim}(\lambda)V(\lambda) = I_m$).

Note. We have now the overall "orthogonal" decomposition of $G(\lambda)$ as

$$G(\lambda) = U(\lambda) \begin{bmatrix} G_2(\lambda) & O \\ O & O \end{bmatrix} V(\lambda). \tag{4}$$

3. Compute
$$G^{\#}(\lambda) := V^{\sim}(\lambda) \begin{bmatrix} G_2^{-1}(\lambda) & O \\ O & O \end{bmatrix} U^{\sim}(\lambda). \tag{5}$$

This computational approach employing the recent compression techniques developed in [10, 9] is applicable to an arbitrary rank rational matrix $G(\lambda)$ regardless it is polynomial, proper or improper. Furthermore, $G(\lambda)$ can have stable or unstable poles and zeros, and even poles and zeros on the imaginary axis or on the unit circle.

The main computations in this approach are the row compressions performed at steps 1 and 2. The methods proposed in [10, 9] to perform this compression are based on reducing the system pencil $S(\lambda)$ to a particular Kronecker-like form which isolates the left singular structure of $G(\lambda)$. Then the compression is achieved by solving for the stabilizing solution a standard algebraic Riccati equation of order n_ℓ, where n_ℓ is the sum of the left minimal indices of $G(\lambda)$. Note that n_ℓ is usually much smaller than the order n of a minimal descriptor realization of $G(\lambda)$.

5 Computation of other type of inverses

In this section we show how we can compute (1,2,3)- and (1,2,4)-inverses by combining the underlying techniques to determine (1,2)- and (1,2,3,4)-inverses. We also discuss shortly a general approach to compute inverses using row/column compression techniques. For notational convenience, we will denote in this section a rational matrix $G(\lambda)$ simply by G.

In the previous section we discussed compression techniques of rational matrices by premultiplying them with all-pass factors. The following procedure can be used to compute an (1,2,3)-inverse of a given G by combining row compression with inner factors and (1,2)-inverse computation.

1. Compute the full row-rank factorization
$$G = U \begin{bmatrix} R \\ O \end{bmatrix}$$
where R has full row-rank and U is square inner.

2. Compute R^+, an (1,2)-inverse (right inverse) of R using the approach of section 3.

3. Compute
$$G^+ := \begin{bmatrix} R^+ & O \end{bmatrix} U^{\sim}.$$

It is easy to check that G^+ is an (1,2,3)-inverse. Indeed,

$$GG^+ = U \begin{bmatrix} R \\ O \end{bmatrix} \begin{bmatrix} R^+ & O \end{bmatrix} U^\sim = U \begin{bmatrix} I & O \\ O & O \end{bmatrix} U^\sim$$

and the first three Moore-Penrose relations are satisfied. To compute an (1,2,4)-inverse, the above procedure can be applied to the transposed matrix G^T.

An alternative approach to compute generalized inverses can be devised using exclusively row/column compression techniques. With a row compression followed by a column compression we can determine a full rank factorization of G in the form

$$G = L \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} T \tag{6}$$

with L, T and R invertible rational matrices. This decomposition of G allows to determine several inverses with a specific choice of the transformation matrices L and T. In general, an (1,2)-inverse of G can be simply computed as

$$G^+ = T^{-1} \begin{bmatrix} R^{-1} & 0 \\ 0 & 0 \end{bmatrix} L^{-1}$$

If we use for the row compression an inner matrix L and for column compression a general matrix T, then the corresponding inverse G^+ is an (1,2,3)-inverse, while for L general and T inner the corresponding inverse is an (1,2,4)-inverse. If L and T are inner then the inverse G^+ is the Moore-Penrose pseudoinverse.

Compression techniques with inner matrices have been recently developed in [10, 9]. For compression with general matrices, methods proposed in [11] can be employed. These methods also rely on pencil algorithms using the equivalent descriptor representation of the rational matrix G. A particular advantage of this approach is the flexibility in choosing the poles and zeros of the compressing factor.

6 Numerical issues

All presented computational methods to compute a generalized inverse of a rational matrix $G(\lambda)$ have an overall computational complexity $0(n^3)$, where n is the order of the underlying descriptor representation $\mathbf{G}$. The main advantage of using pencil techniques to solve computational problems involving rational matrices is that the whole arsenal of well developed linear algebra techniques to manipulate matrix pencils can be employed to devise numerically reliable algorithms for these rather complex problems. In contrast, rational matrix manipulations using polynomial techniques are generally considered to be numerically less robust than methods based on state-space representations. It is to be expected that all presented techniques can be extended to more general systems, as for instance, periodic time-varying, and even general time-varying systems.

For the computation of weak generalized inverses with arbitrarily assigned spurious poles and for the computation of Moore-Penrose pseudoinverses MATLAB functions are available in a recently developed DESCRIPTOR SYSTEMS toolbox [16]. The main functions rely on a collection of *mex*-functions providing easy to use gateways to highly optimized Fortran codes from the SLICOT library [3] developed within the NICONET project[1]. The

[1]see `http://www.win.tue.nl/niconet/niconet.html`

computational layer of basic *mex*-functions provides efficient and numerically robust computational tools to perform reductions to Kronecker-like forms, reordering of generalized real Schur form, balancing of descriptor systems, minimal realization of rational matrices, and other computations.

7 Example

Consider the rational matrix from [18]

$$G(s) = \begin{bmatrix} \dfrac{1}{s+2} & \dfrac{1}{s+1} \\ \dfrac{s+3}{s^2+3s+2} & \dfrac{s}{s+1} \\ \dfrac{s^2+3s}{s^2+3s+2} & 0 \end{bmatrix}$$

This matrix has full column rank and has no zeros. A state space realization for $G(s)$ is given by

$$A = \begin{bmatrix} -3 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 0 & -2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For the given $G(s)$ we compute three left-inverses with particular properties and the unique Moore-Penrose pseudoinverse.

First we compute the left-inverses using the approach to compute (1,2)-inverses in section 3. For notation we also refer to this section. To allow an easy reproducibility of the results, we illustrate our method by using non-orthogonal transformation matrices Q and Z to obtain the Kronecker-like form of the system pencil $S(\lambda)$ with "nice" numbers. With the following transformation matrices

$$U = \begin{bmatrix} I_5 & K \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad V = I,$$

where

$$K = \left[\begin{array}{c} K_2 \\ \hline K_3 \end{array}\right] = \left[\begin{array}{c} k_1 \\ k_2 \\ \hline k_3 \\ k_4 \\ k_5 \end{array}\right],$$

we obtain the reduced pencil $\widehat{S}(\lambda) = UQS(\lambda)ZV$ in (2) such that

$$\widehat{S}(\lambda) = \left[\begin{array}{c|c} A_{reg} & A_{reg,l} + K_2C_l \\ \hline 0 & A_l + K_3C_l - \lambda I \\ \hline 0 & C_l \end{array}\right] = \left[\begin{array}{cc|ccc} 1 & 0 & 2 & -2 & k_1 \\ 0 & 1 & -2 & 4 & k_2 - 1 \\ \hline 0 & 0 & -3-\lambda & 0 & k_3 \\ 0 & 0 & -2 & -\lambda & k_4 \\ 0 & 0 & 0 & -3 & -\lambda + k_5 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

From (3) we obtain

$$\left[\begin{array}{c} \widehat{B}_1 \\ \hline \widehat{B}_2 \end{array}\right] = \left[\begin{array}{ccc} k_1 & 0 & 1 \\ k_2 & 1 & 0 \\ k_3 & 0 & -1 \\ k_4 & 0 & -1 \\ k_5 & -1 & -1 \\ \hline 1 & 0 & 0 \end{array}\right], \quad \widehat{C}_2 = \left[\begin{array}{ccccc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{array}\right]$$

The (1,2)-inverse corresponding to $\widehat{S}(\lambda)$ has a standard state space realization $(A_{inv}, B_{inv}, C_{inv}, D_{inv})$ of order 3, with

$$A_{inv} = \left[\begin{array}{ccc} -3 & 0 & k_3 \\ -2 & 0 & k_4 \\ 0 & -3 & k_5 \end{array}\right], \quad B_{inv} = \left[\begin{array}{ccc} k_3 & 0 & -1 \\ k_4 & 0 & -1 \\ k_5 & -1 & -1 \end{array}\right]$$

$$C_{inv} = \left[\begin{array}{ccc} 2 & -2 & k_1 \\ -2 & 4 & k_2 - 1 \end{array}\right], \quad D_{inv} = \left[\begin{array}{ccc} k_1 & 0 & 1 \\ k_2 & 1 & 0 \end{array}\right]$$

This realization can be easily retrieved from the particular forms of $\widehat{S}(\lambda)$, $\widehat{B}_1$ and $\widehat{C}_2$.

With $K = 0$, the corresponding (1,2)-inverse is

$$G^{+}(s) = \left[\begin{array}{ccc} 0 & 0 & \dfrac{s^2+3s+2}{s(s+3)} \\ 0 & \dfrac{s+1}{s} & -\dfrac{s+1}{s^2} \end{array}\right]$$

and has McMillan degree 3. This inverse is not stable, having poles in the origin.

With

$$K = \left[\begin{array}{ccccc} 0 & 0 & 0 & 2 & -5 \end{array}\right]^T$$

we can assign the spurious poles of the inverse to $\{-2,-3,-3\}$. The corresponding stable (1,2)-inverse

$$G^{+}(s)=\left[\begin{array}{ccc}\dfrac{-4s}{(s+2)(s+3)} & \dfrac{4}{(s+2)(s+3)} & \dfrac{s^3+8s^2+27s+28}{(s+2)(s+3)^2}\\[2ex] \dfrac{13s+6}{(s+2)(s+3)} & \dfrac{s^2+6s-2}{(s+2)(s+3)} & -\dfrac{s^2+22s+35}{(s+2)(s+3)^2}\end{array}\right]$$

has McMillan degree equal to 3.

A least McMillan order stable left inverse can be obtained by choosing K

$$K=\left[\begin{array}{ccccc}-\frac{2}{3} & 1 & \frac{4}{3} & 1 & 0\end{array}\right]^T.$$

The corresponding inverse is

$$G^{+}(s)=\left[\begin{array}{ccc}-\frac{2}{3}\,\dfrac{s}{s+1} & \frac{2}{3}\,\dfrac{1}{s+1} & \frac{1}{3}\,\dfrac{3\,s+5}{s+1}\\[2ex] \frac{1}{3}\,\dfrac{10\,s+3\,s^2+3}{s^2+2\,s+1} & \frac{1}{3}\,\dfrac{3\,s^2+6\,s-1}{s^2+2\,s+1} & -\frac{2}{3}\,\dfrac{3\,s+5}{s^2+2\,s+1}\end{array}\right]$$

and has McMillan degree 2. This is the same inverse as that of [18, p. 224], where $b_{01}=0$ and $b_{02}=1$ have been chosen.

We also computed the unique Moore-Penrose pseudoinverse of $G(s)$

$$G^{\#}(s)=\left[\begin{array}{ccc}\frac{-s^5-3s^4+s^3+9s^2+6s}{p(s)} & \frac{s^4+3s^3-s^2-9s-6}{p(s)} & \frac{s^3+8s^2+27s+28}{p(s)}\\[1ex] \frac{s^6+s^5-10s^4-11s^3+2s^2+6s+3}{p(s)} & \frac{-s^5+13s^3+9s^2-12s-9}{p(s)} & -\frac{-s^5+8s^3+4s^2-3s}{p(s)}\end{array}\right].$$

where $p(s)=s^6-11s^4+15s^2-9$. $G^{\#}(s)$ has McMillan degree equal to 6 and is unstable.

8 Conclusions

We proposed numerically reliable methods to compute two classes of generalized inverses of rational matrices: the (1,2)- or weak inverse and the Moore-Penrose pseudoinverse. The proposed methods are completely general, being applicable to arbitrary rank rational matrices regardless they are polynomial, proper or improper. A particular emphasis has been put on reliably computing (1,2)-inverses, because of their relevance to many practical applications. The proposed approach provides flexibility to cope with various conditions on the spurious poles of the computed (1,2)-inverses. For instance, a stable (1,2)-inverse can be easily computed whenever exists. The computation of other type of inverses has been also addressed, by combining "orthogonal" compression techniques using inner factors with left/right inverse computation. For the proposed methods, robust numerical software has been implemented.

Interesting open computational problems in the context of determining various types of inverses is the exploitation of intrinsic parametric freedom. Two aspects could be relevant for control applications: (1) determining particular inverses with special properties (e.g., minimal Mc-Millan degree or stable with minimal Mc-Millan degree), and (2) generating the whole class of inverses by exploiting the parametrizations of inverses (see e.g., [5]). In this paper we partially addressed only the first aspect. However, a detailed elaboration of algorithms to compute least order inverses is still necessary.

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