Efficient time integration methods for index-1 DAEs
(applied to helicopter simulation)

Elena Kohlwey, Melven Röhrig-Zöllner
Outline

Part 1: Motivational Example & Half-Explicit Methods
• Motivation: multi-disciplinary helicopter simulation
• Different problem formulations and methods
• Half-explicit methods
• Rotor model example
• Numerical results

Part 2: Stability of Exponential Runge-Kutta Methods
• Introduction
• Exponential Butcher-Tableau
• Applications
• Previous work on stability
• “Classical” stability regions
• Even more stability regions

Conclusion
Motivation: multi-disciplinary helicopter simulation

Unsteady aerodynamics

Flight dynamics

Structural dynamics, e.g. flexible rotor blades

- Coupling of many sub-systems, such as the rotor, the rotor wake, the fuselage
- Interaction adds complexity to the behavior of the helicopter model

→ “Coupled” ODE-systems
## Overview of systems and methods

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\begin{align*}
\dot{x}(t) &= f(t, x(t), y(t)) \\
y(t) &= g(t, x(t), y(t))
\end{align*}
\] |
| **Method:** Half-Explicit Runge-Kutta (HERK) |
| **Consistency:** well-known & revisited |
| **Stability:** translates from ERK |
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Half-explicit methods

• **Index-1 DAE** problem:

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), y(t)) \\
y(t) &= g(t, x(t), y(t)) \\
x(t_0) &= x_0
\end{align*}
\]

with \( \bar{g}(t, x, y) := y - g(t, x, y) \) invertible w.r.t. \( y \) and \( \frac{\partial \bar{g}}{\partial y} \) regular near the solution

• **Idea:**

Solve \( \bar{g} = 0 \) (approximately) using (simplified) **Newton**…

→ Apply **explicit ODE** method

• **Assure convergence order of the ODE method:**
  a) Set convergence criterion according to step size and order \( (\epsilon \sim h^k) \)
  b) Use fixed number of simplified Newton steps for each RK-substep

Arnold et. al.: “Half-explicit Runge-Kutta methods […]”, 1993
Rotor model example

Main variables:
• \(\varphi_i(t), i = 1, \ldots, 4\): Lead-lag angle of \(i^{th}\) blade
• \(x_{Fus}(t), y_{Fus}(t)\): Longitudinal and transversal displacement of the fuselage
• \(\varphi_{RH}(t)\): Rotational angle of the rotor head

Attributes:
• Highly oscillatory and stiff behavior
• Stiffness of system given by stiffness of rotational spring (variable \(K_s\))
• Can be written as a “monolithic” ODE system or as an index-1 DAE system:
  • ODE for the rotational spring
  • ODE system for everything else
  • + “coupling equations”

Figure 1: Rotor model sketch (adapted from [SMBA11])

Time evolution of the blades' motion

Initial condition:
• $\varphi_1(t)$: small displacement (0.01 rad)
• $\varphi_2(t), \varphi_3(t), \varphi_4(t)$: at rest

Resulting motion:
• Reciprocal build up of alternate motion of the blades
• Additional high frequency oscillations appear
• The stiffer the mast, the higher the frequency

Figure 2: Motion of the four blades
(blade 1: blue, blade 2: red, blade 3: green, blade 4: pink)
Initial condition:
- $\varphi_1(t)$: small displacement (0.01 rad)
- $\varphi_2(t)$, $\varphi_3(t)$, $\varphi_4(t)$: at rest

Resulting motion:
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Figure 3: Zoom of motion of the four blades
(blade 1: blue, blade 2: red, blade 3: green, blade 4: pink)
Approximation Error

Results:

- More accurate approximation with HEERK4 than HERK4 for $h \in \{10^{-3}, 10^{-4}\}$
- For $h = 10^{-2}$: same approximation quality, but different shapes

<table>
<thead>
<tr>
<th>$h$</th>
<th>HERK4</th>
<th>HEERK4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-2}$</td>
<td>$1.5 \cdot 10^{-1}$</td>
<td>$1.2 \cdot 10^{-1}$</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>$1.4 \cdot 10^{-4}$</td>
<td>$7.3 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>$1.4 \cdot 10^{-8}$</td>
<td>$7.2 \cdot 10^{-10}$</td>
</tr>
</tbody>
</table>

Table 1: RMS Error

Figure 4: Zoom of approximation shapes of $\varphi_1(t)$
(reference solution: blue, approximation HERK4: red, approximation HEERK4: green)
Run time comparison

Findings:
- HEERK4 needs 30% more time than HERK4 for the same problem
- Biggest converging step width is bigger for HEERK4
- HEERK4 needs about 0.3 seconds for the computation of coefficients
- Combinations with red circles are smoothed approximations

Table 2: Run times in seconds for HEERK4 and HERK4 for different $\omega_s$ and $h$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\omega_s = 3.91$</th>
<th>$\omega_s = 12.35$</th>
<th>$\omega_s = 39.06$</th>
</tr>
</thead>
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<tr>
<td></td>
<td>HERK4</td>
<td>HEERK4</td>
<td>HERK4</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>n.a.</td>
<td>n.a.</td>
<td>n.a.</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>0.7</td>
<td>1.0</td>
<td>n.a.</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>7.4</td>
<td>9.8</td>
<td>7.5</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>70.0</td>
<td>93.7</td>
<td>67.1</td>
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Distribution of stiffness

Problem:
• Too much stiffness is still in the ‘nonstiff’ part
• No optimal choice of submodels!
• What is possible?

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<tr>
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<th>$\omega_s = 3.91$</th>
<th>$\omega_s = 12.35$</th>
<th>$\omega_s = 39.06$</th>
<th>$\omega_s = 123.52$</th>
<th>$\omega_s = 390.61$</th>
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<tr>
<td>$10^{-1}$</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>$10^{-2}$</td>
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<td>✔</td>
<td>✔</td>
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<td>✔</td>
<td>✔</td>
<td>✔</td>
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Table 4: Converging step widths of EERK4 with different values of $\omega_s$

<table>
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<tr>
<th>$\text{max. eigenvalues of } S$</th>
<th>$\text{max. eigenvalues of mon. system}$</th>
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<tbody>
<tr>
<td>$\pm 3.91i$</td>
<td>$\pm 114i$</td>
</tr>
<tr>
<td>$\pm 12.35i$</td>
<td>$\pm 326i$</td>
</tr>
<tr>
<td>$\pm 39.06i$</td>
<td>$\pm 1,021i$</td>
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Table 3: Eigenvalues of $S$ and of monolithic system
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Conclusion
Introduction

- **Problem formulation:**
  \[
  \dot{x}(t) = Sx(t) + \tilde{f}(x(t)) \\
  x(0) = x_0
  \]
  - stiff linear part \( S \) (constant matrix)
  - non-stiff nonlinear part \( \tilde{f} \) (with sufficiently small Lipschitz constant)

- **Idea:**
  Approximate the solution of the “variation of constants” formula:
  \[
  x(t) = e^{-tS}x_0 + \int_0^t e^{-(t-\tau)S} \tilde{f}(x(\tau)) \, d\tau
  \]
  → **Linear part handled “exactly”**!

- **Further reading:**
  Hochbruck and Ostermann:
  “Exponential Integrators”, Acta Numerica, 2010
Exponential RK Butcher-Tableau

- **Example:** exponential trapezoidal rule (explicit 2\textsuperscript{nd} order method)

\[
\begin{array}{c|ccc}
0 & & \\
1 & \varphi_{11} & \\
& \varphi_1 - \varphi_2 & \varphi_2 \\
\end{array}
\]

with \( \varphi_{ij} := \varphi_i(c_j h S) \) and

\[
\varphi_i(h S) := \int_0^1 e^{h(1-\theta)S} \frac{\theta^{i-1}}{(i - 1)!} \, d\theta
\]

→ **Coefficients are matrices**
  - depend on \( h \) and \( S \)
  - involve matrix exponentials (linear combinations of \( \varphi_{ij} \))
  - converge to diagonal matrices for \( S \to 0 \)
  → underlying Runge-Kutta scheme
Applications

• Explicit time integration for **parabolic PDE** problems
  • Thoroughly analyzed in the literature

• “Great tool” also for **general stiff ODE** problems
  • Idea: choose a **linear part** $S$ to obtain a less stiff **nonlinear part** $\tilde{f}$ …
  • $S$ can be quite arbitrary (from our experience)

• Drawback: computation of **matrix exponentials** quite **costly**
  • Large sparse matrix $S$
    → Krylov-subspace methods for $x \rightarrow e^{S}x$
  • Fixed step size and “small enough” $S$
    → Precompute $\varphi_{ij}$
  • Exploit special form of $S$:
    block-diagonal, partial Schur decomp. $QSQ^T$, …
Previous work on stability

- **Asymptotic behavior of the stability function**
  Maset and Zennaro:
  “Unconditional stability of explicit exponential RK methods for semilinear ODEs”, 2009
  “Stability properties of explicit exponential RK methods”, 2013

- **Scalar, real negative $S$ (pure damping)**
  Several papers, among others:

→ Open question: How to choose a suitable step size $h$ for a specific setting
“Classical” stability regions (1)

• Scalar test problem:
  \[ \dot{x}(t) = \sigma x + \lambda x \]
  - linear part: \( \sigma \in \mathbb{C} \)
  - “nonlinear” part: \( \lambda \in \mathbb{C} \)

• Notation in the plots with time step size \( h \):
  \[
  w := h\sigma \\
  z := h\lambda
  \]
  (accommodates for the scaling invariance)

• Disclaimer:
  No proof for transferring scalar results to systems for exponential RK methods!

• Hint: inside the marked area, it’s stable…

![Figure 5: Stability regions of 1st (blue), 2nd (red), 3rd (green) and 4th (pink) order underlying RK methods \((w = 0)\)]
“Classical” stability regions (2)

Figure 6: Stability regions of 1st (blue), 2nd (red), 3rd (green) and 4th (pink) order exponential methods with different real part
“Classical” stability regions (3)

Figure 6: Stability regions of 1st (blue), 2nd (red), 3rd (green) and 4th (pink) order exponential methods with different real part

(c) $w = -6$

(d) $w = -100$
“Classical” stability regions (4)

Figure 7: Stability regions of 1st (blue), 2nd (red), 3rd (green) and 4th (pink) order exponential methods with different imaginary part
“Classical” stability regions (5)

Figure 7: Stability regions of 1st (blue), 2nd (red), 3rd (green) and 4th (pink) order exponential methods with different imaginary part.

(c) $w = -6i$

(d) Zoom out for $w = -6i$
“Classical” stability regions (6)

• Observations:
  • For negative real $\sigma$ the stability regions grow.
    $\rightarrow$ Very stable when the linear part contains enough damping
  • For imaginary $\sigma$ the stability regions rotate / distort.

• Difficulties:
  • Previously shown plots change when $h$ changes (fixed $w = h\sigma$)!
  • What happens for complex-conjugate Eigenvalue-pairs?

$\rightarrow$ Let’s try a different perspective…
Even more stability regions (1)

- Real, two-dimensional test problem
  \[ \dot{x}(t) = Sx + Lx \]
  with
  \[
  S = \begin{pmatrix} \sigma_{re} & \sigma_{im} \\ -\sigma_{im} & \sigma_{re} \end{pmatrix}, \quad L = \begin{pmatrix} \lambda_{re} & \lambda_{im} \\ -\lambda_{im} & \lambda_{re} \end{pmatrix}.
  \]

- Plots assume that \( S = c(S + L), c \in \mathbb{R} \)
  (eigenvalues of \( S \) have the “correct angle” in the complex plane)

- Consider different “cuts” of the complex plane:
  - real axis: \( \sigma_{im} = \lambda_{im} = 0 \)
  - 10% damping: \( \sigma_{re} + \lambda_{re} = \frac{1}{9} (\sigma_{im} + \lambda_{im}) \)
  - 1% damping: \( \sigma_{re} + \lambda_{re} = \frac{1}{99} (\sigma_{im} + \lambda_{im}) \)
  - imaginary axis: \( \sigma_{re} = \lambda_{re} = 0 \)
Even more stability regions (2)
Even more stability regions (3)
Even more stability regions (4)
Even more stability regions (5)
Even more stability regions (6)
Even more stability regions (7)
Even more stability regions (8)
Even more stability regions (9)
Even more stability regions (10)

• Observations:
  • Near the **real axis**:
    All methods **stable when linear part dominates** (roughly for \( \frac{\sigma}{\sigma+\lambda} \in (0.5,1.5) \))
    “Overshooting” is fine …
  
  • Near the **imaginary axis**:
    • Exponential Euler and EERK3 variants seem unusable
    • **EERK2** (trapezoidal rule) seems unconditionally stable for \( \frac{\sigma}{\sigma+\lambda} \geq \frac{3}{4} \)
    • EERK4 (HO2005) possibly unstable directly on the imaginary axis
    • EERK4 (variant) promising, but not perfect

• Open points:
  • What if \( \sigma \) and \( \lambda \) point in different directions in the complex plane?
  • “Optimal” choice for EERK4?
Conclusion

- Half-explicit exponential RK methods for stiff index-1 DAEs
  - Work fine in our numerical experiments
    → Interesting & flexible class of explicit methods

- Open question: best way to define the linear stiff part for our application?

- Goal: Paper on the Stability Regions of Explicit Exponential RK methods
  - Not fully analyzed in the literature, yet...
    in particular the behavior near the imaginary axis

- Stability depends on multiple parameters – difficult to visualize…

- Possibly suggest a more stable EERK4 variant?

- Further reading: