Efficient time integration methods for index-1 DAEs
(applied to helicopter simulation)

Elena Kohlwey, Melven Röhrig-Zöllner

DLR

Outline

Part 1: Motivational Example & Half-Explicit Methods

- Motivation: multi-disciplinary helicopter simulation
- Different problem formulations and methods
- Half-explicit methods
- Rotor model example
- Numerical results

Part 2: Stability of Exponential Runge-Kutta Methods

- Introduction
- Exponential Butcher-Tableau
- Applications
- Previous work on stability
- “Classical” stability regions
- Even more stability regions

Conclusion

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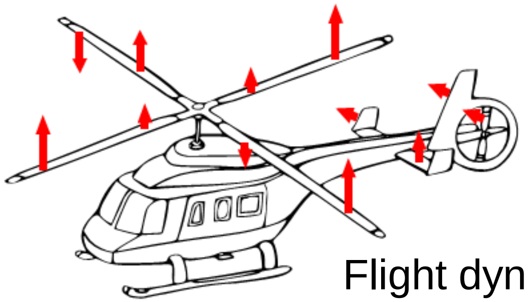
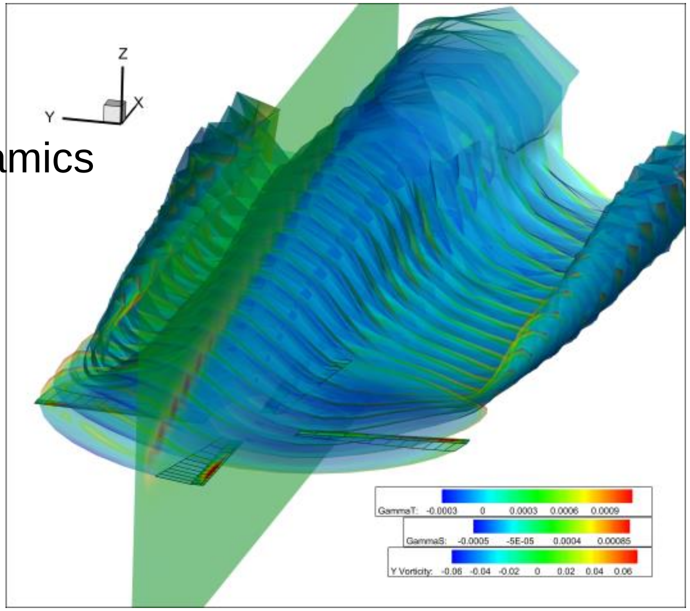
Motivation: multi-disciplinary helicopter simulation

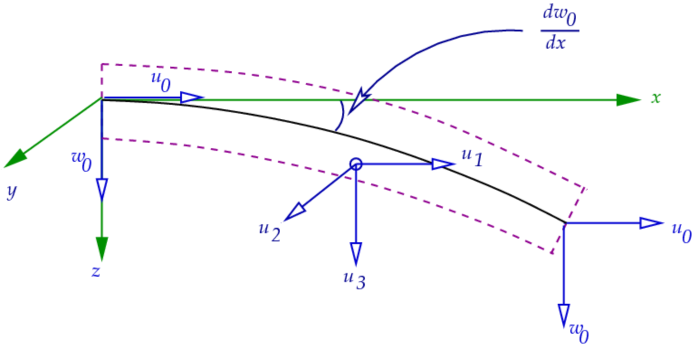


Flight dynamics

Unsteady aerodynamics



Structural dynamics, e.g. flexible rotor blades



- Coupling of many sub-systems, such as the rotor, the rotor wake, the fuselage
- Interaction adds complexity to the behavior of the helicopter model

→ “Coupled” ODE-systems

Overview of systems and methods

	General nonstiff systems / Ordinary Runge-Kutta methods	
ODE systems / Explicit methods	System: $\dot{x}(t) = f(t, x(t), y(t))$ Method: Explicit Runge-Kutta (ERK) Consistency: well-known Stability: well-known	

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Overview of systems and methods

	General nonstiff systems / Ordinary Runge-Kutta methods	Stiff systems with linear stiff part / Exponential Runge-Kutta methods
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Half-explicit methods

- **Index-1 DAE** problem:

$$\begin{cases} \dot{x}(t) = f(t, x(t), y(t)) \\ y(t) = g\big(t, x(t), y(t)\big) \end{cases}$$
$$x(t_0) = x_0$$

  with $\bar{g}(t, x, y) := y - g(t, x, y)$ invertible w.r.t. y and $\frac{\partial \bar{g}}{\partial y}$ regular near the solution

- Idea:
  Solve $\bar{g} = 0$ (approximately) using (simplified) **Newton**…
  → Apply **explicit ODE** method

- Assure convergence order of the ODE method:
  a) Set convergence criterion according to step size and order ($\epsilon \sim h^k$)
  b) Use fixed number of simplified Newton steps for each RK-substep
  Arnold et. al.: “Half-explicit Runge-Kutta methods […]”, 1993

Rotor model example

Main variables:

- $\varphi_i(t), i = 1, \dots, 4$: Lead-lag angle of i^{th} blade
- $x_{Fus}(t)$, $y_{Fus}(t)$: Longitudinal and transversal displacement of the fuselage
- $\varphi_{RH}(t)$: Rotational angle of the rotor head

Attributes:

- Highly oscillatory and stiff behavior
- Stiffness of system given by stiffness of rotational spring (variable K_s)
- Can be written as a "monolithic" ODE system or as an index-1 DAE system:
  - ODE for the rotational spring
  - ODE system for everything else
  - + "coupling equations"

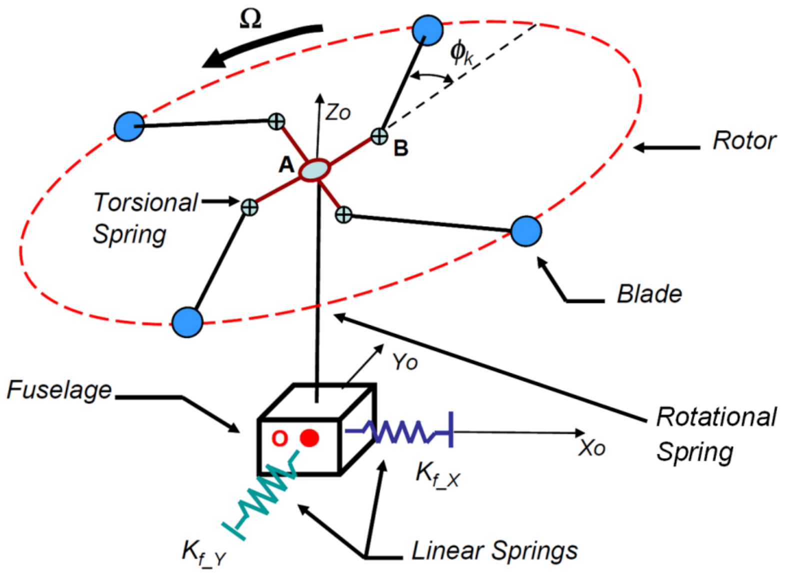

Figure 1: Rotor model sketch (adapted from [SMBA11])

Time evolution of the blades‘ motion

Initial condition:

- $\varphi_1(t)$: small displacement $(0.01\, rad)$
- $\varphi_2(t),\ \varphi_3(t),\ \varphi_4(t)$: at rest

Resulting motion:

- Reciprocal build up of alternate motion of the blades
- Additional high frequency oscillations appear
- The stiffer the mast, the higher the frequency

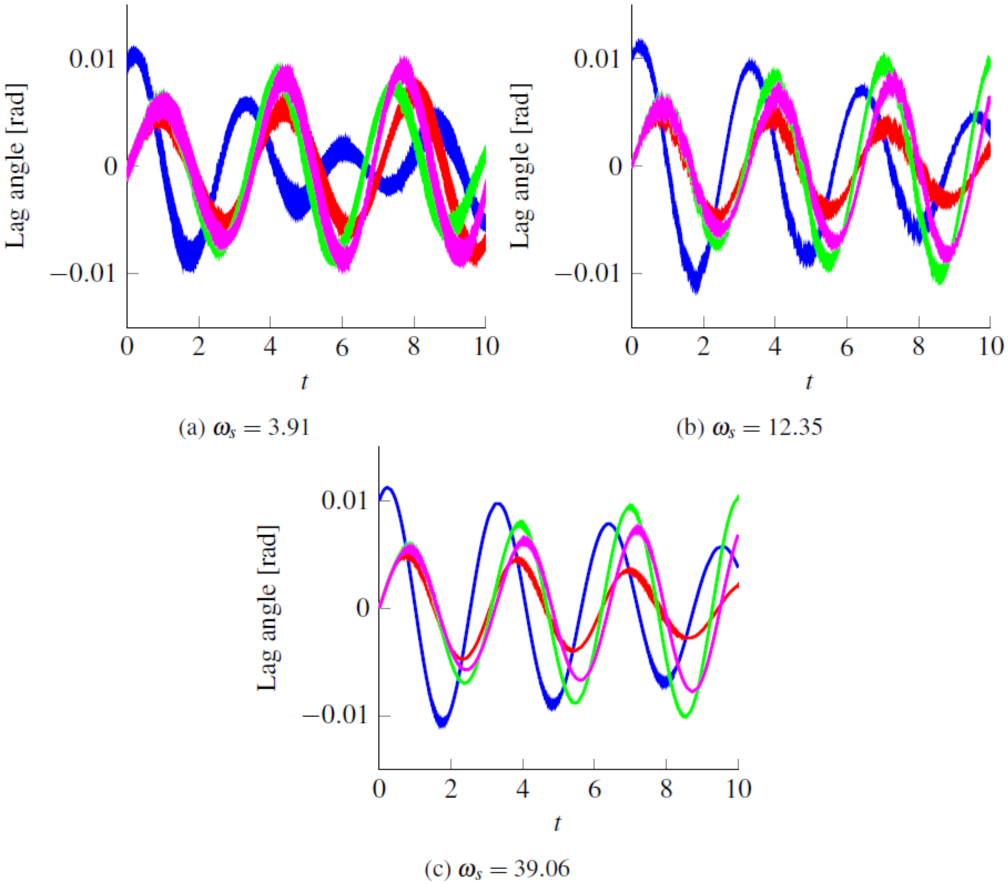

Figure 2: Motion of the four blades
(blade 1: blue, blade 2: red, blade 3: green, blade 4: pink)

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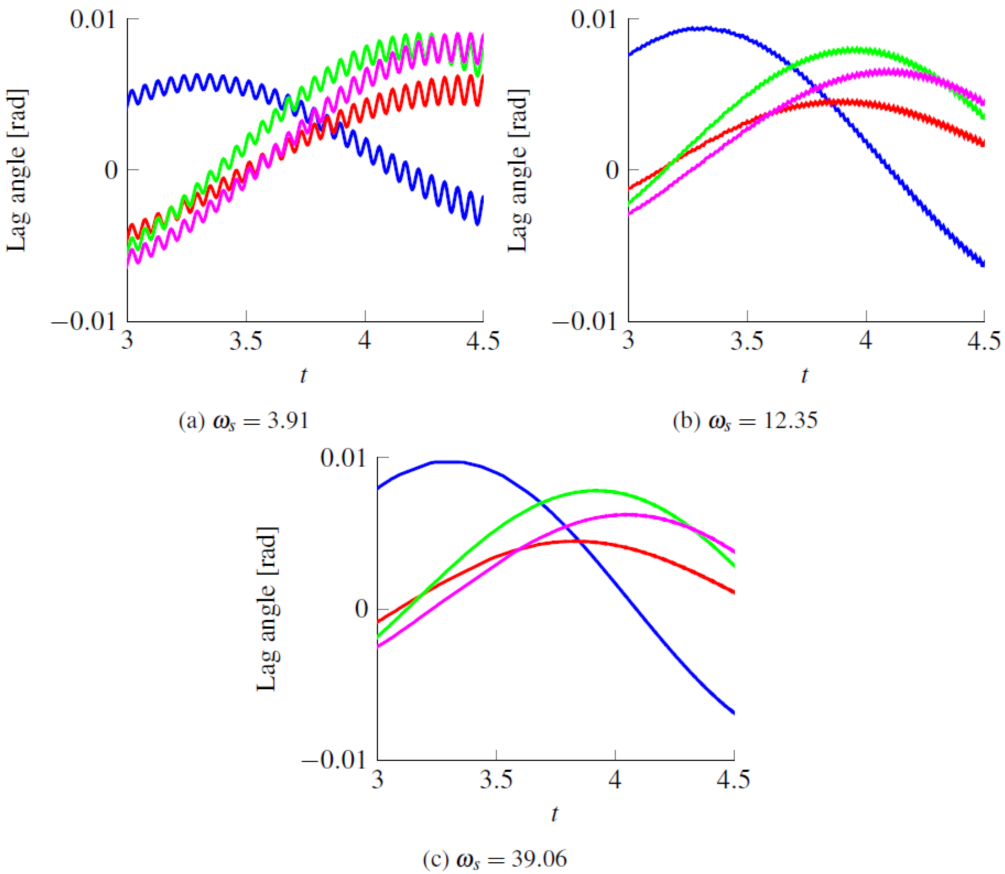


Figure 3: Zoom of motion of the four blades
(blade 1: blue, blade 2: red, blade 3: green, blade 4: pink)

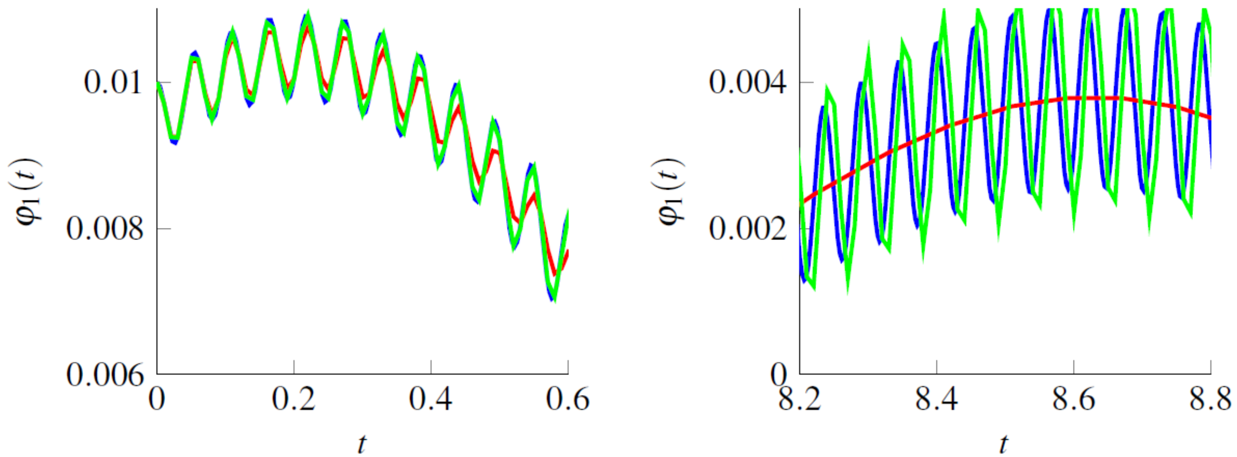
Approximation Error

Results:

- More accurate approximation with HEERK4 than HERK4 for $h \in \{10^{-3}, 10^{-4}\}$
- For $h = 10^{-2}$: same approximation quality, but different shapes

h	HERK4	HEERK4
10^{-2}	$1.5 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$
10^{-3}	$1.4 \cdot 10^{-4}$	$7.3 \cdot 10^{-6}$
10^{-4}	$1.4 \cdot 10^{-8}$	$7.2 \cdot 10^{-10}$

Table 1: RMS Error



(a) $t \in [0, 0.6]$ (b) $t \in [8.2, 8.8]$

Figure 4: Zoom of approximation shapes of $\varphi_1(t)$
(reference solution: blue, approximation HERK4: red, approximation HEERK4: green)

Run time comparison

Findings:

- HEERK4 needs 30% more time than HERK4 for the same problem
- Biggest converging step width is bigger for HEERK4
- HEERK4 needs about 0.3 seconds for the computation of coefficients
- Combinations with red circles are smoothed approximations

h	$\omega_s = 3.91$		$\omega_s = 12.35$		$\omega_s = 39.06$	
	HERK4	HEERK4	HERK4	HEERK4	HERK4	HEERK4
10^{-1}	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
10^{-2}	0.7	1.0	n.a.	0.9	n.a.	n.a.
10^{-3}	7.4	9.8	7.5	9.7	7.4	10.8
10^{-4}	70.0	93.7	67.1	93.0	74.0	104.0

Table 2: Run times in seconds for HEERK4 and HERK4 for different ω_s and h

Distribution of stiffness

Problem:

- Too much stiffness is still in the 'nonstiff' part
- No optimal choice of submodels!
- What is possible?

max. eigenvalues of S	max. eigenvalues of mon. system
$\pm 3.91i$	$\pm 114i$
$\pm 12.35i$	$\pm 326i$
$\pm 39.06i$	$\pm 1,021i$

Table 3: Eigenvalues of S and of monolithic system

h	$\omega_s = 3.91$	$\omega_s = 12.35$	$\omega_s = 39.06$	$\omega_s = 123.52$	$\omega_s = 390.61$
10^{-1}	✓	✓	✓	✓	✗
10^{-2}	✓	✓	✓	✓	✓
10^{-3}	✓	✓	✓	✓	✓
10^{-4}	✓	✓	✓	✓	✓

Table 4: Converging step widths of EERK4 with different values of ω_s

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Introduction

- **Problem formulation:**

$$\dot{x}(t) = Sx(t) + \tilde{f}\big(x(t)\big)$$
$$x(0) = x_0$$

  - stiff linear part S (constant matrix)
  - non-stiff nonlinear part $\tilde{f}$ (with sufficiently small Lipschitz constant)

- Idea:
  Approximate the solution of the “variation of constants” formula:

$$x(t) = e^{-tS}x_0 + \int_0^t e^{-(t-\tau)S}\ \tilde{f}\big(x(\tau)\big)\, d\tau$$

  → **Linear part handled “exactly”!**

- **Further reading:**
  Hochbruck and Ostermann:
  “Exponential Integrators”, Acta Numerica, 2010
  “Exponential Runge-Kutta methods for parabolic problems”, SIAM J. Numer. Anal., 2005

Exponential RK Butcher-Tableau

- **Example:** exponential trapezoidal rule (explicit 2nd order method)

$$
\begin{array}{c|cc}
0 & & \\
1 & \varphi_{11} & \\
\hline
 & \varphi_1 - \varphi_2 & \varphi_2
\end{array}
$$

with $\varphi_{ij} \coloneqq \varphi_i(c_j hS)$ and

$$\varphi_i(hS) \coloneqq \int_0^1 e^{h(1-\theta)S} \frac{\theta^{i-1}}{(i-1)!} \, d\theta$$

→ **Coefficients are matrices**

- depend on h and S
- involve matrix exponentials (linear combinations of φ_{ij})
- converge to diagonal matrices for $S \to 0$
  → **underlying Runge-Kutta scheme**

Applications

- Explicit time integration for **parabolic PDE** problems
  - Thoroughly analyzed in the literature

- “Great tool” also for **general stiff ODE** problems
  - Idea: choose a linear part S to obtain a less stiff nonlinear part $\tilde{f}$…
  - S can be quite arbitrary (from our experience)

- Drawback: computation of **matrix exponentials** quite **costly**
  - Large sparse matrix S
    → Krylov-subspace methods for $x \to e^S x$
  - Fixed step size and “small enough” S
    → Precompute φ_ij
  - Exploit special form of S:
    block-diagonal, partial Schur decomp. $Q\bar{S}Q^{\mathrm{T}}$, …

Previous work on stability

- **Asymptotic behavior of the stability function**
  Maset and Zennaro:
  “Unconditional stability of explicit exponential RK methods for semilinear ODEs”, 2009
  “Stability properties of explicit exponential RK methods”, 2013

- **Scalar, real negative** S (pure damping)
  Several papers, among others:
  Zhu: “Efficient and stable exponential RK methods for parabolic equations”, 2017

→ Open question: How to choose a suitable step size h for a specific setting

“Classical” stability regions (1)

- Scalar test problem:
$$\dot{x}(t) = \sigma x + \lambda x$$
  - linear part: $\sigma \in \boldsymbol{C}$
  - “nonlinear” part: $\lambda \in \boldsymbol{C}$

- Notation in the plots with time step size h:
$$w \coloneqq h\sigma$$
$$z \coloneqq h\lambda$$
(accommodates for the scaling invariance)

- Disclaimer:
No proof for transferring scalar results to systems for exponential RK methods!

- Hint: inside the marked area, it‘s stable…

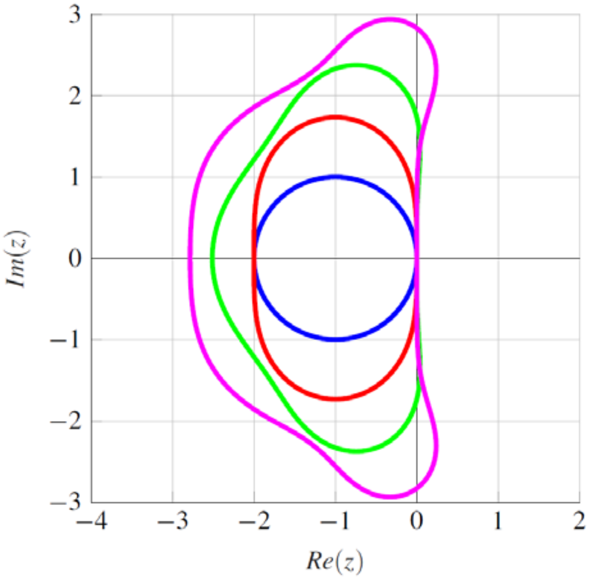


Figure 5: Stability regions of 1st (blue), 2nd (red), 3rd (green) and 4th (pink) order underlying RK methods ($w = 0$)

“Classical” stability regions (2)

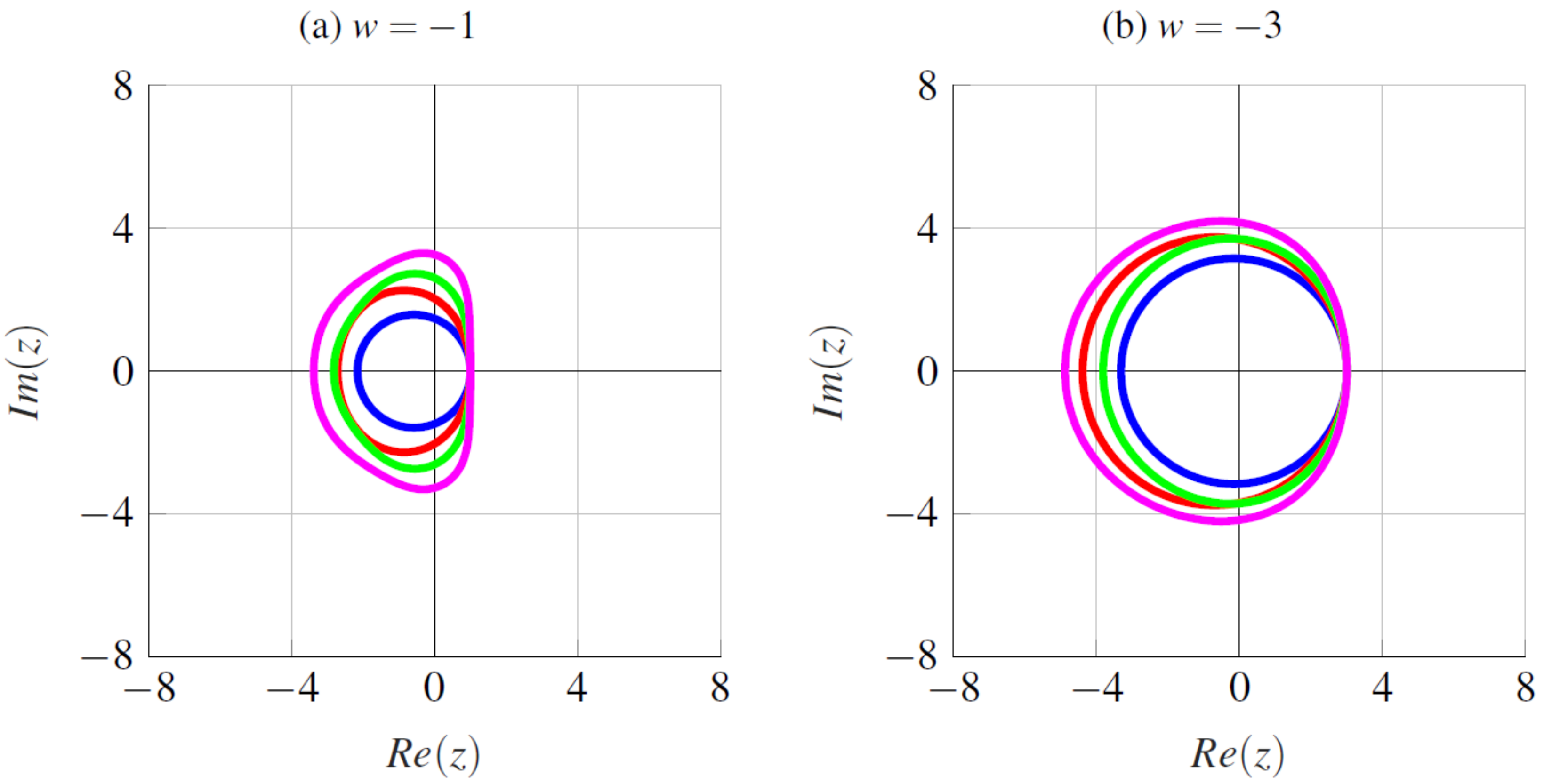
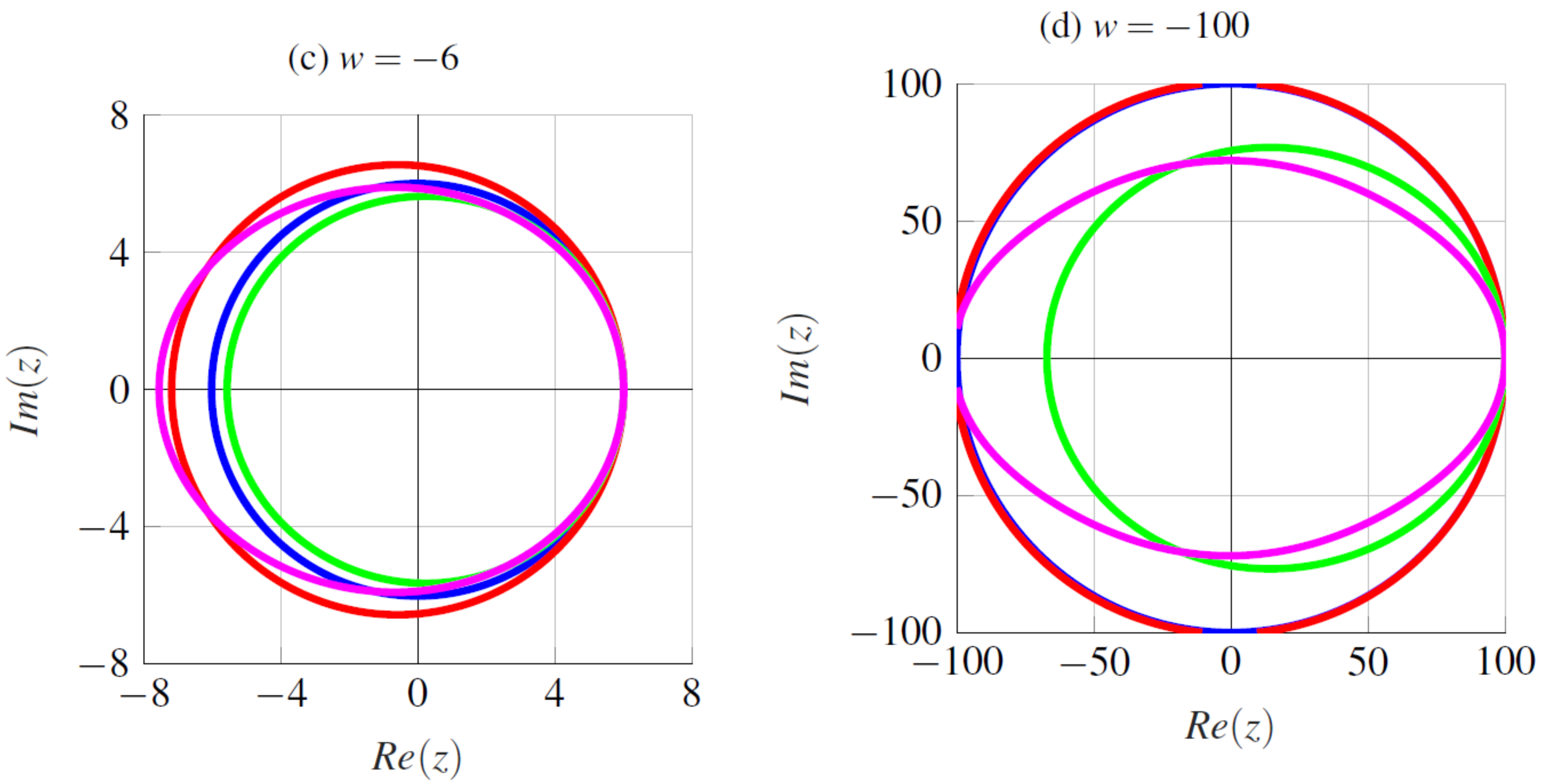


Figure 6: Stability regions of 1st (blue), 2nd (red), 3rd (green) and 4th (pink) order exponential methods with different real part

“Classical” stability regions (3)



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“Classical” stability regions (4)

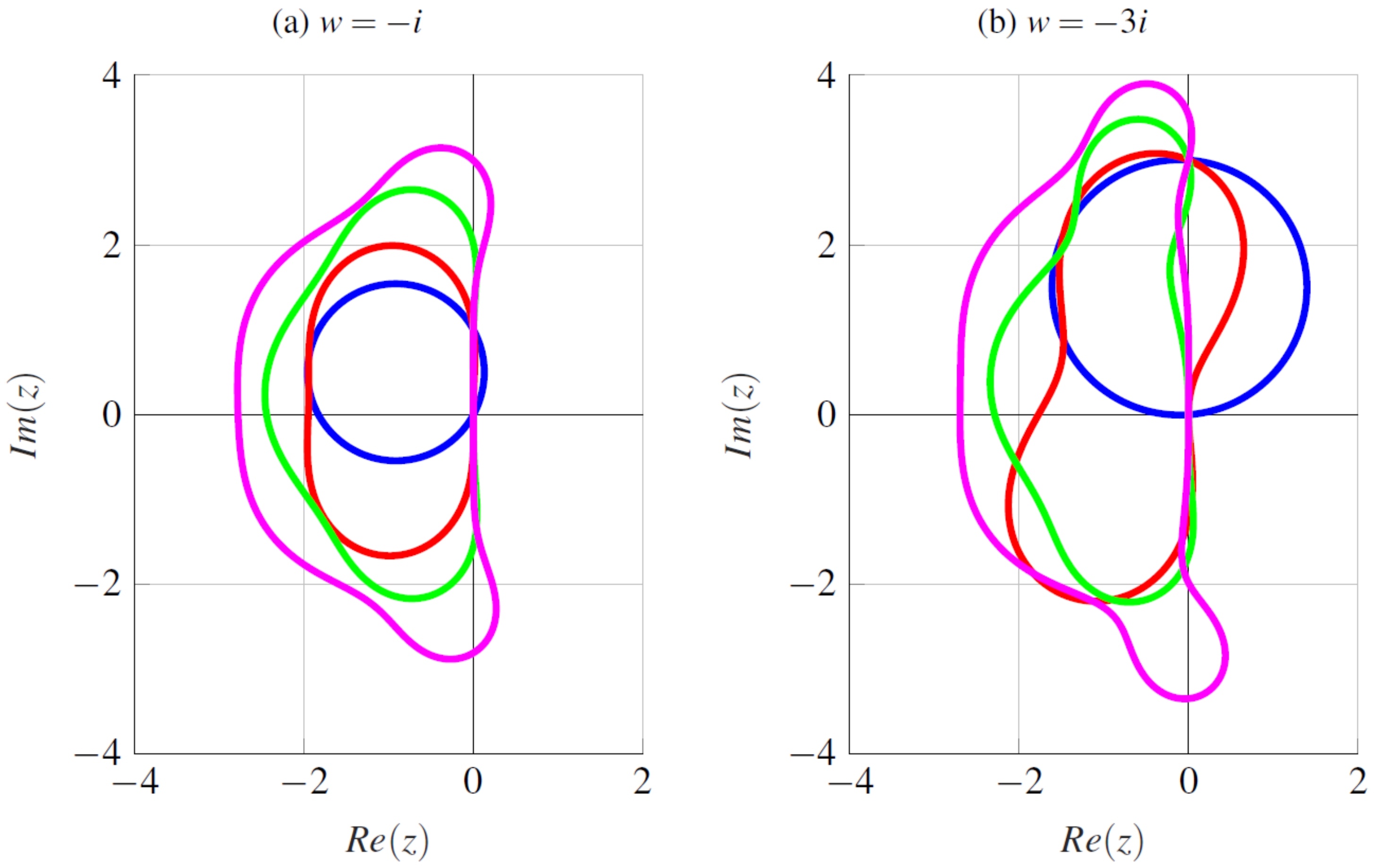

Figure 7: Stability regions of 1st (blue), 2nd (red), 3rd (green) and 4th (pink) order exponential methods with different imaginary part

“Classical” stability regions (5)

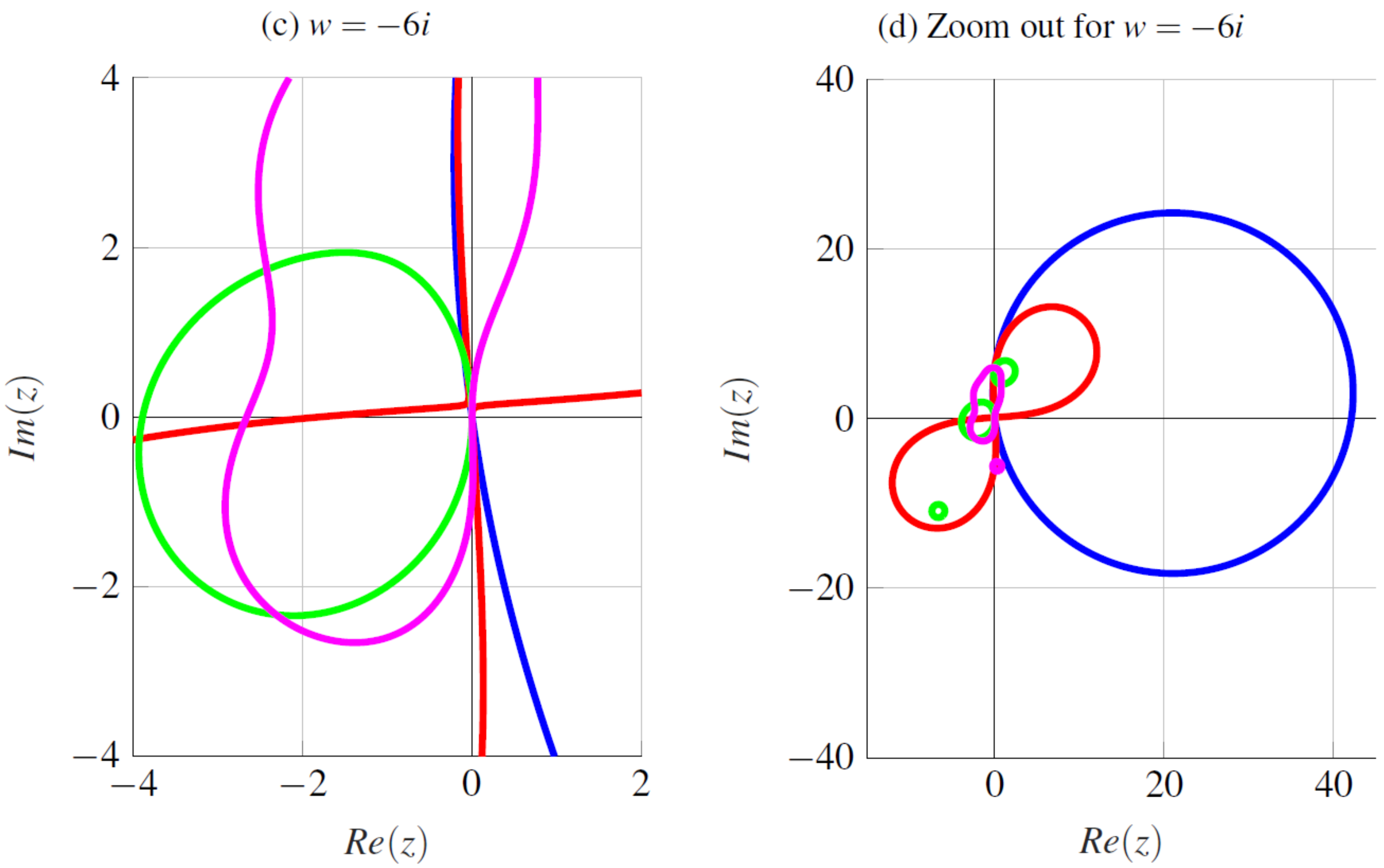

Figure 7: Stability regions of 1st (blue), 2nd (red), 3rd (green) and 4th (pink) order exponential methods with different imaginary part

"Classical" stability regions (6)

- Observations:
  - For negative real σ the stability regions grow.
    $\rightarrow$ Very stable when the linear part contains enough damping
  - For imaginary σ the stability regions rotate / distort.

- Difficulties:
  - Previously shown plots change when h changes (fixed $w = h\sigma$)!
  - What happens for complex-conjugate Eigenvalue-pairs?

$\rightarrow$ Let's try a different perspective…

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Even more stability regions (1)

- Real, two-dimensional test problem

$$\dot{x}(t) = Sx + Lx$$

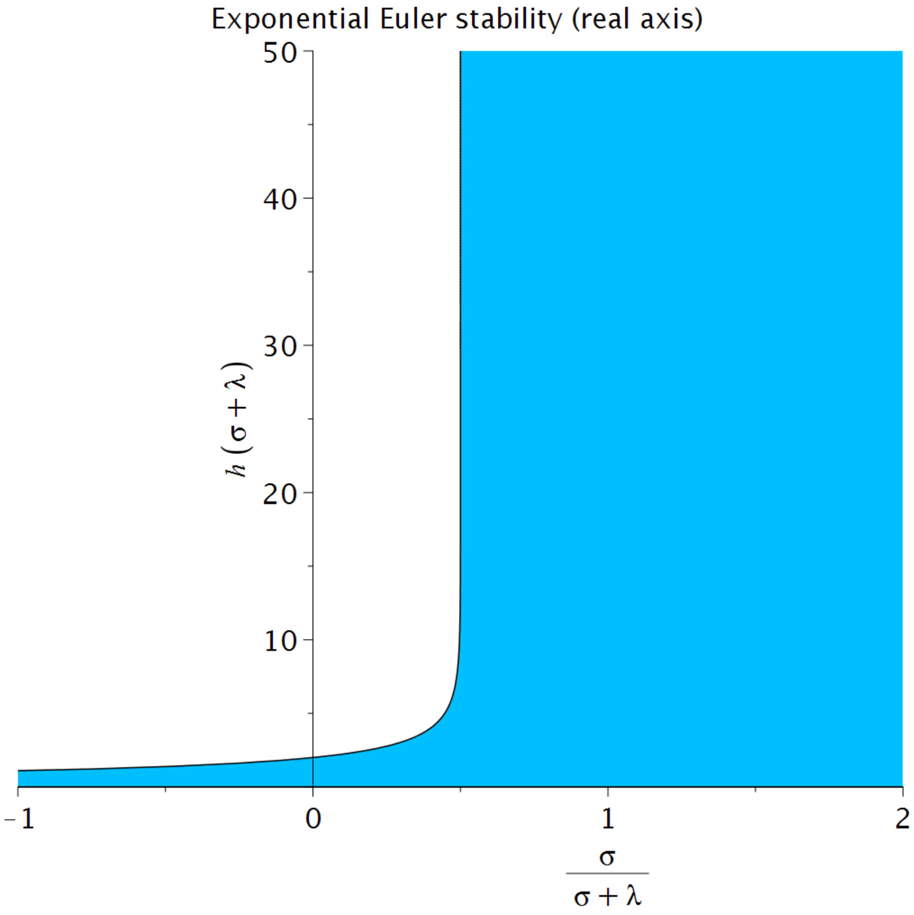
  with

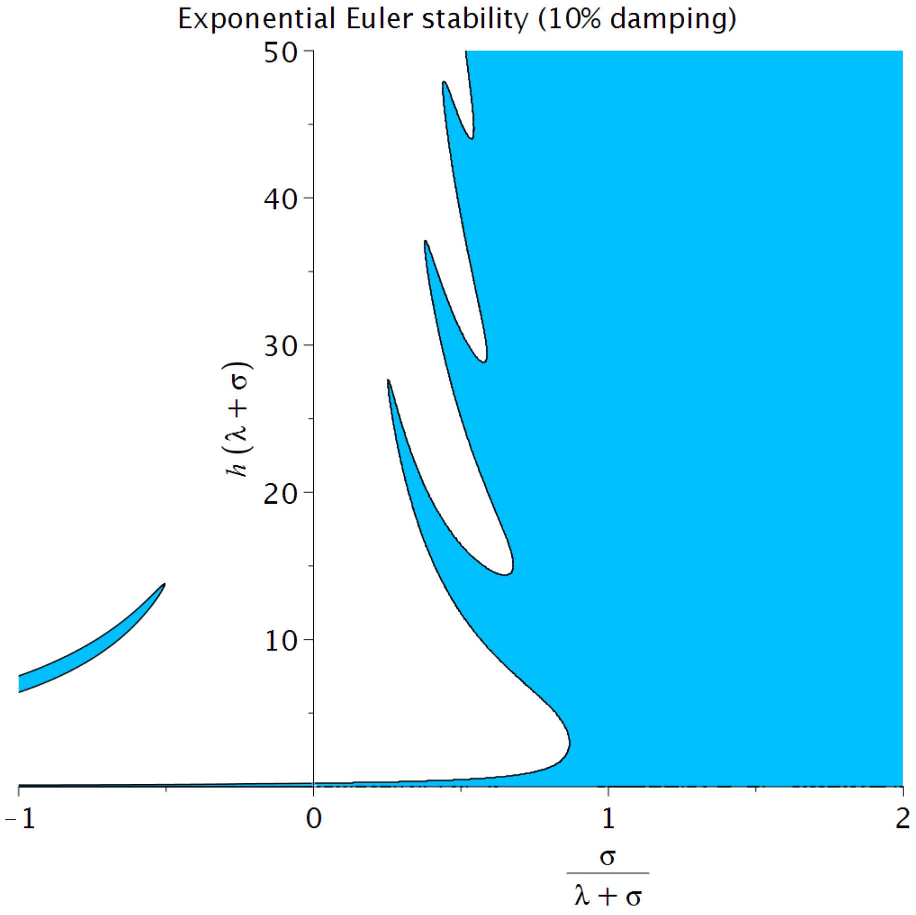
$$S = \begin{pmatrix} \sigma_{re} & \sigma_{im} \\ -\sigma_{im} & \sigma_{re} \end{pmatrix}, \qquad L = \begin{pmatrix} \lambda_{re} & \lambda_{im} \\ -\lambda_{im} & \lambda_{re} \end{pmatrix}.$$

- Plots assume that $S = c(S + L), c \in \boldsymbol{R}$
  (eigenvalues of S have the “correct angle” in the complex plane)

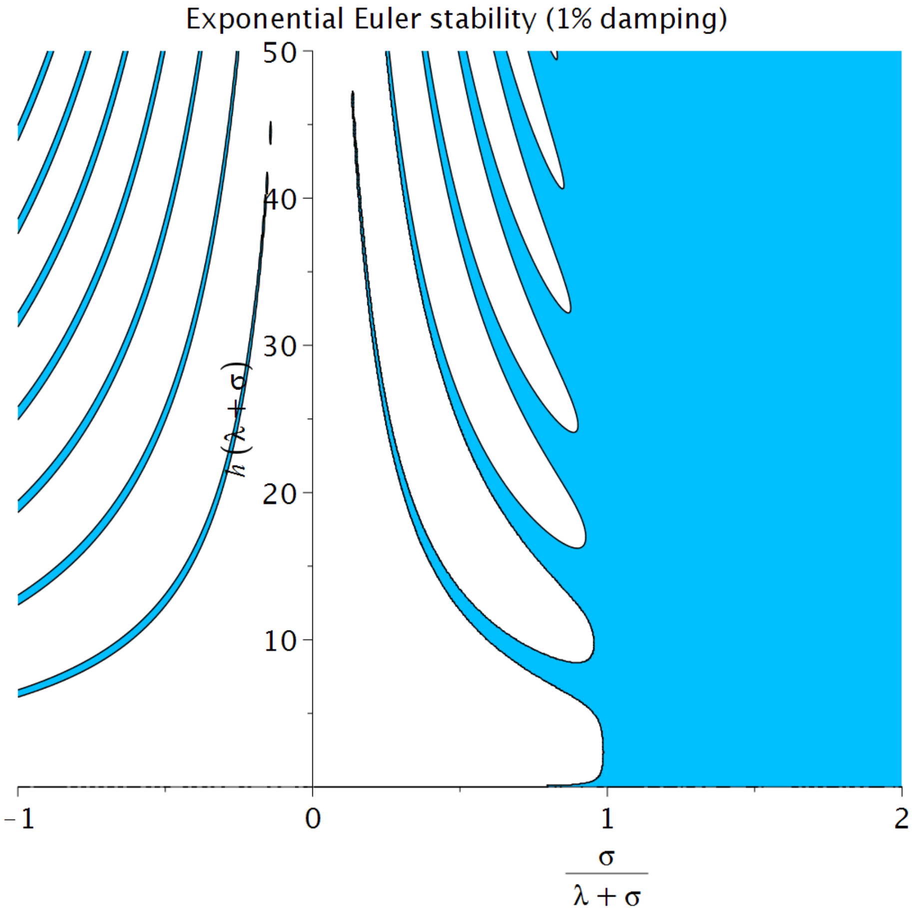
- Consider different “cuts” of the complex plane:
  - real axis: $\sigma_{im} = \lambda_{im} = 0$
  - 10% damping: $\sigma_{re} + \lambda_{re} = \frac{1}{9}(\sigma_{im}+\lambda_{im})$
  - 1% damping: $\sigma_{re} + \lambda_{re} = \frac{1}{99}(\sigma_{im}+\lambda_{im})$
  - imaginary axis: $\sigma_{re} = \lambda_{re} = 0$

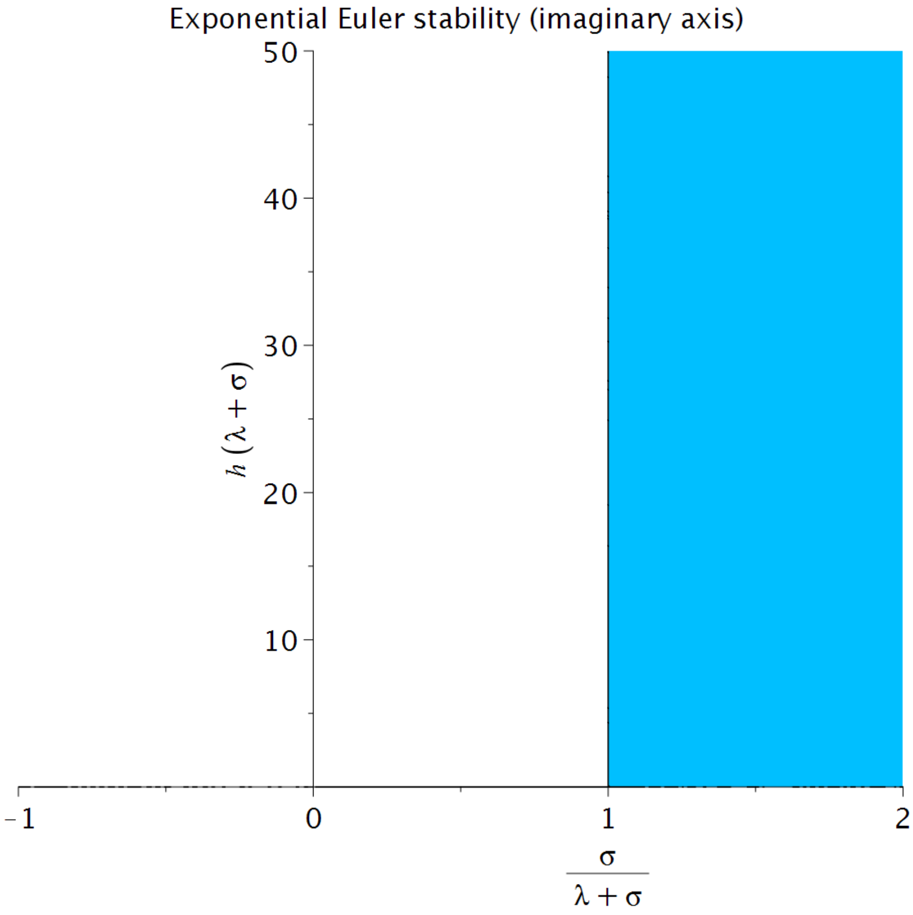
Even more stability regions (2)



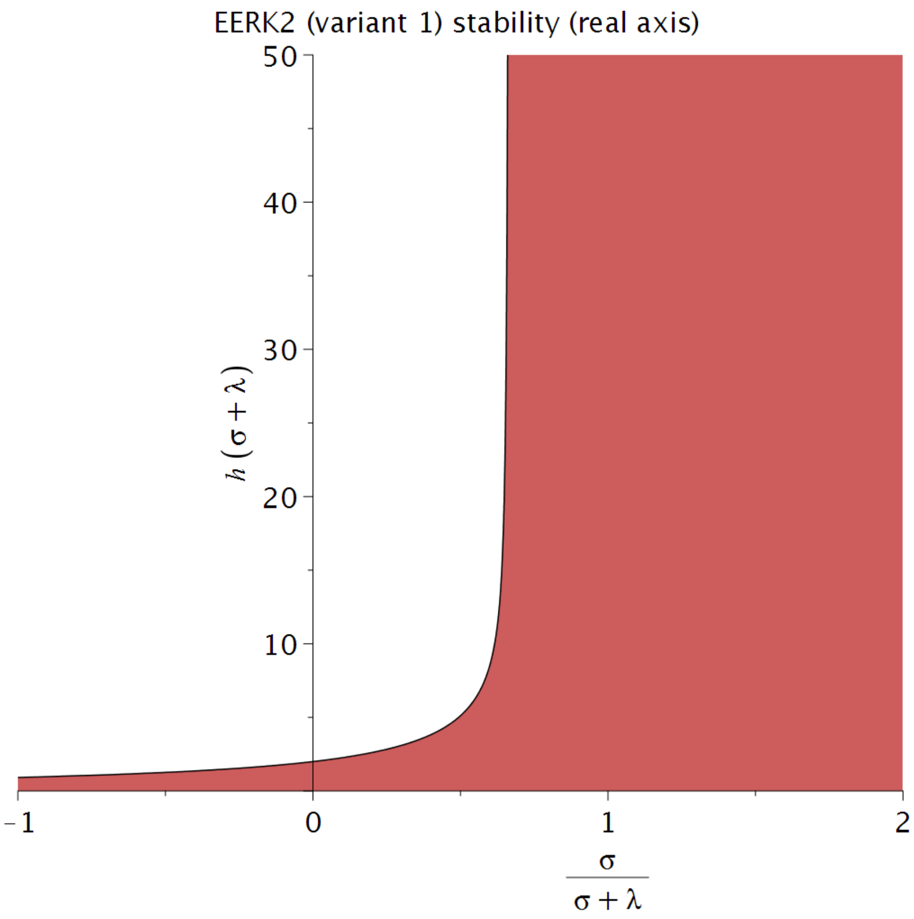


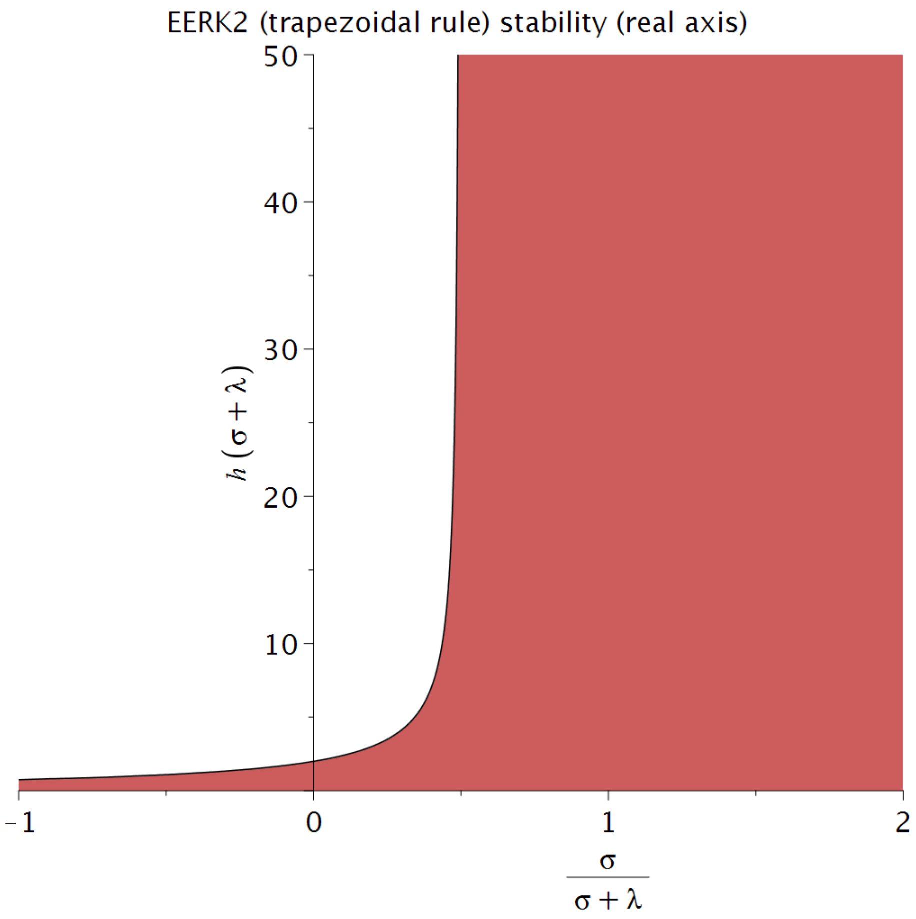
Even more stability regions (3)





Even more stability regions (4)





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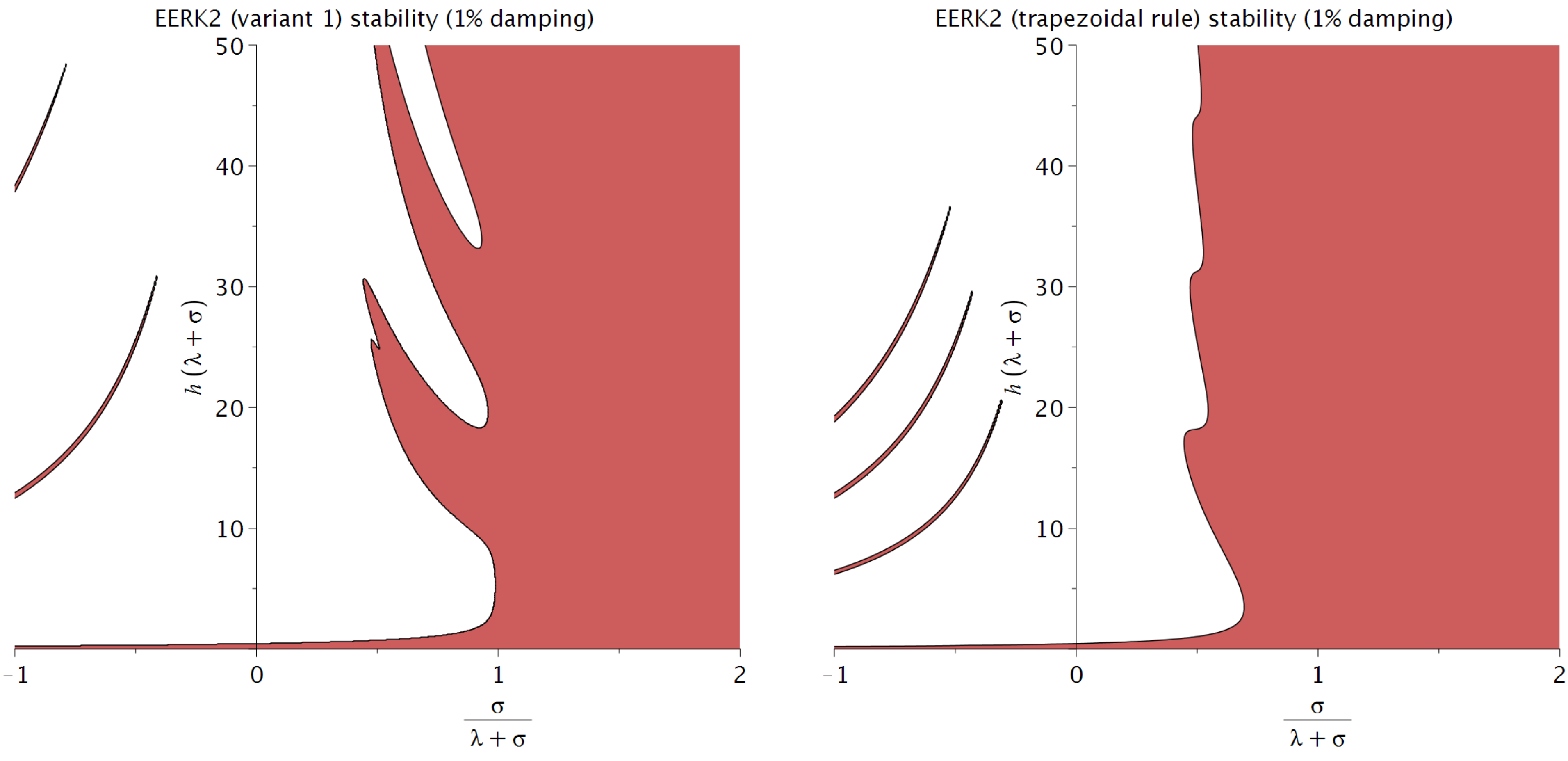

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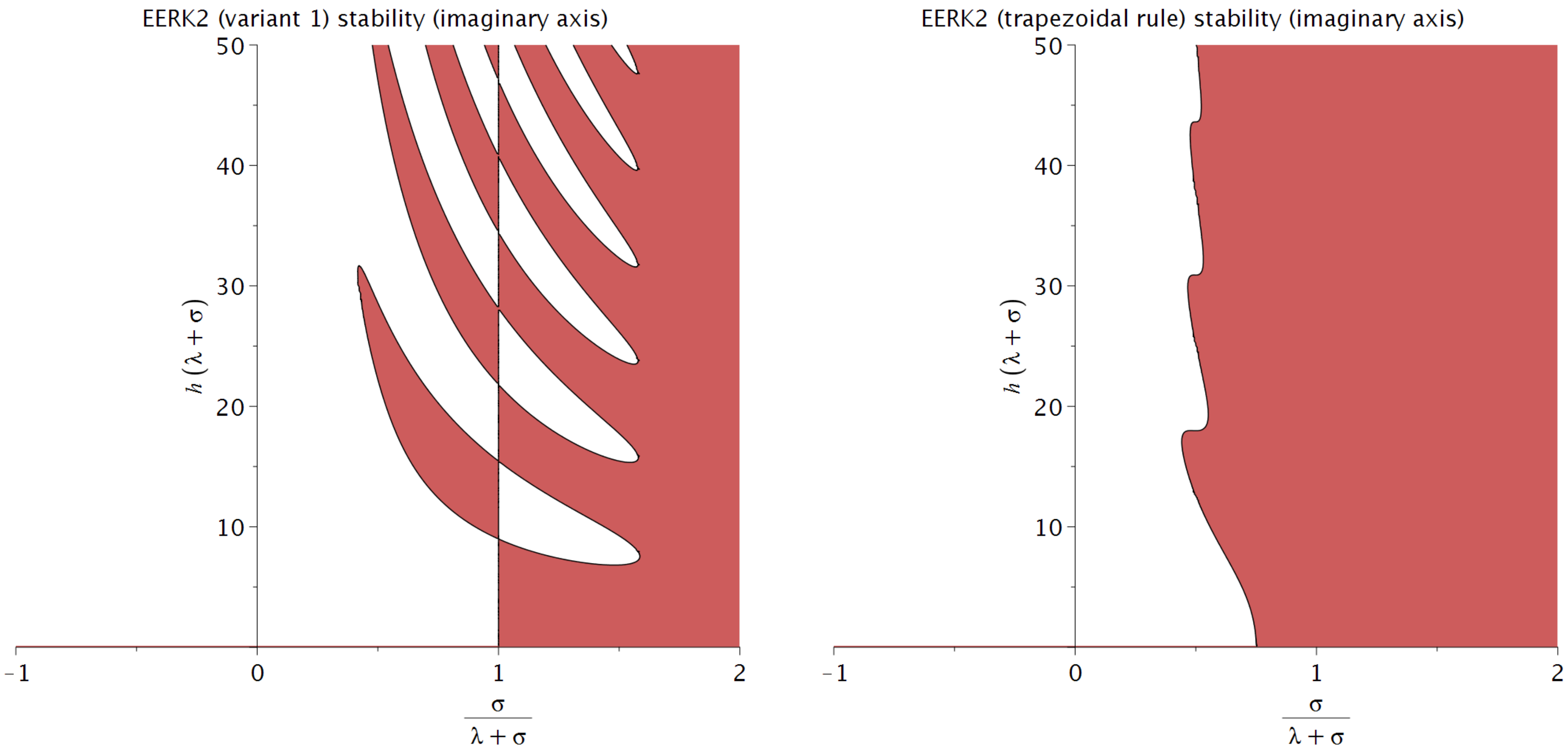

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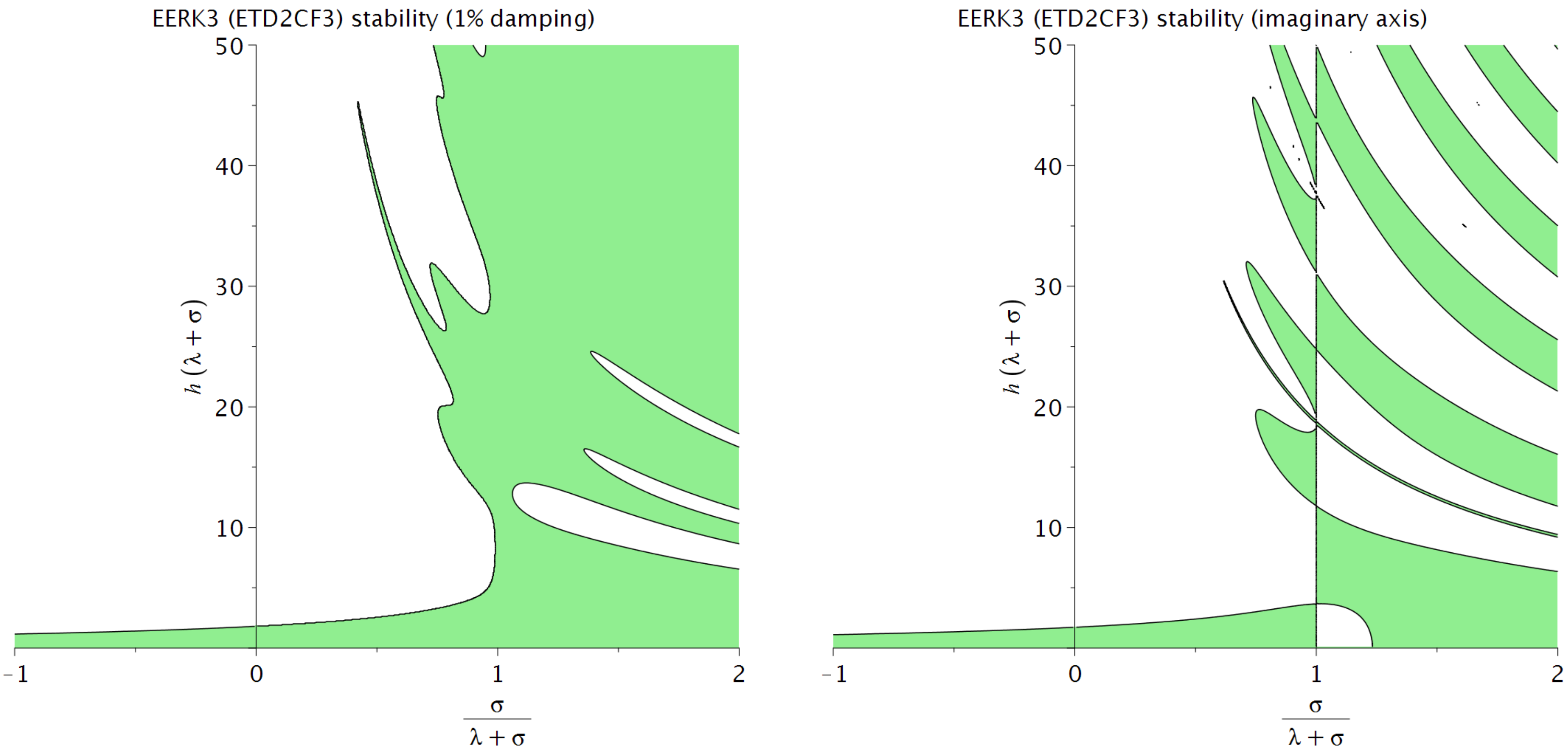
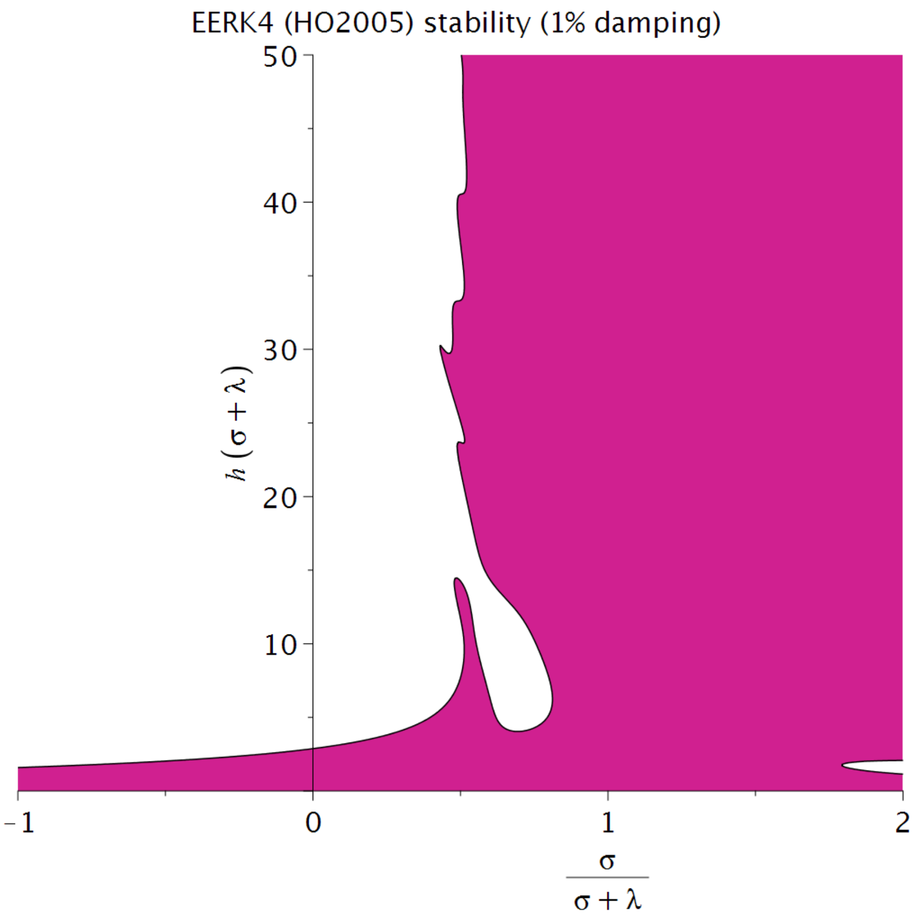
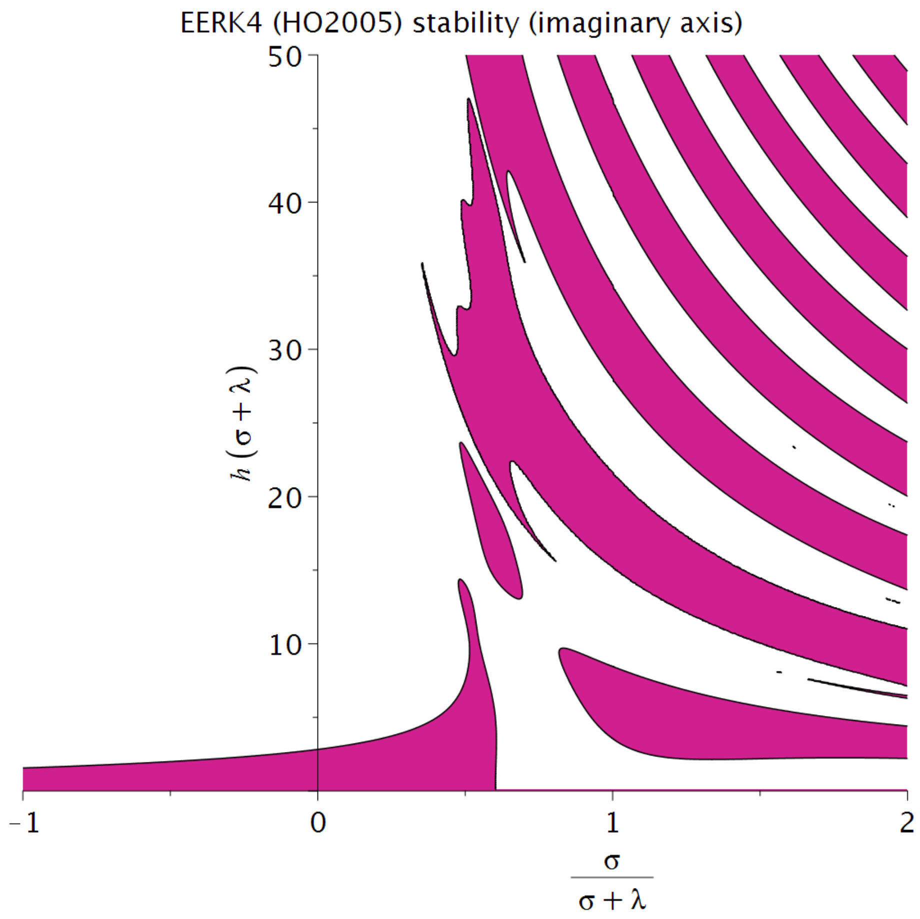
Even more stability regions (7)

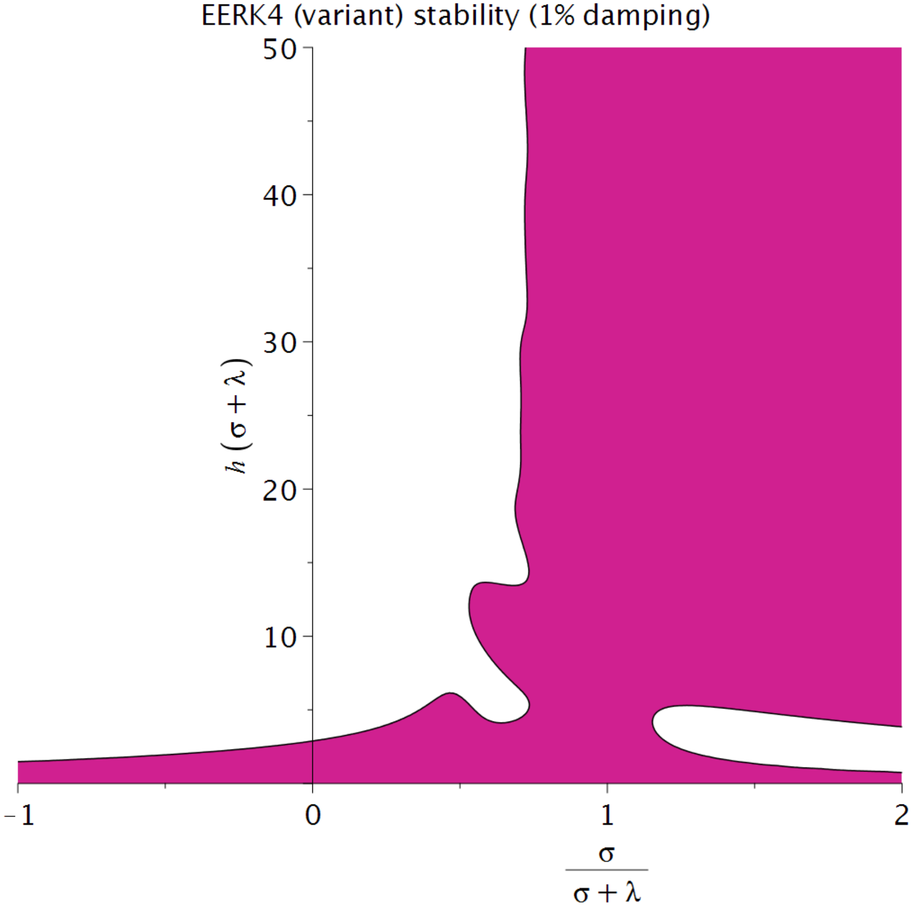


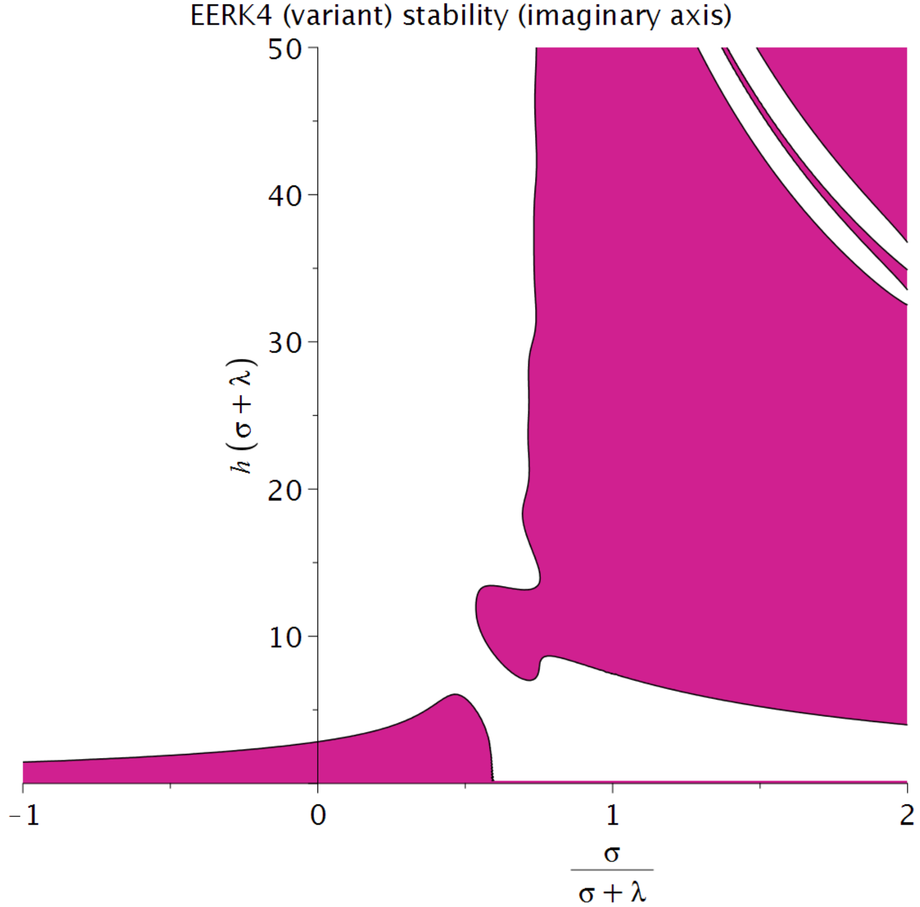
Even more stability regions (8)





Even more stability regions (9)



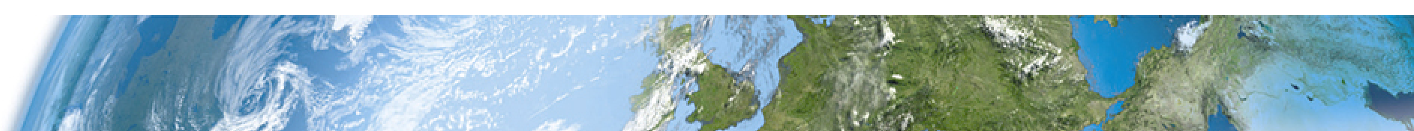


Even more stability regions (10)

- Observations:
  - Near the **real axis**:
    All methods stable when linear part dominates (roughly for $\frac{\sigma}{\sigma+\lambda} \in (0.5, 1.5)$)
    "Overshooting" is fine …

  - Near the **imaginary axis**:
    - Exponential Euler and EERK3 variants seem unusable
    - **EERK2** (trapezoidal rule) seems unconditionally stable for $\frac{\sigma}{\sigma+\lambda} \geq \frac{3}{4}$
    - EERK4 (HO2005) possibly unstable directly on the imaginary axis
    - EERK4 (variant) promising, but not perfect

- Open points:
  - What if σ and λ point in different directions in the complex plane?
  - "Optimal" choice for EERK4?



Conclusion

- Half-explicit exponential RK methods for stiff index-1 DAEs
    - Work fine in our numerical experiments

    → Interesting & flexible class of explicit methods
    - Open question: best way to define the linear stiff part for our application?
- Goal: **Paper** on the Stability Regions of Explicit Exponential RK methods
    - Not fully analyzed in the literature, yet...
      in particular the behavior near the imaginary axis
    - Stability depends on multiple parameters – difficult to visualize…
    - Possibly suggest a more stable EERK4 variant?
- Further reading:
  Kremser, Elena: “Towards Helicopter Simulation with an Index-1 Differential-Algebraic Equations System”, Master’s Thesis, 2017