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Task II.B.1 - Selection of Software for Controller Reduction[1]

Andras Varga[2]

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[2]Deutsches Zentrum für Luft und Raumfahrt, Institut für Robotik und Mechatronik, DLR Oberpfaffenhofen, Postfach 1116, D-82230 Wessling, Germany

Abstract

This report presents a short overview of methods suitable for controller reduction. A first class of methods considered are general purpose methods for reduction of unstable systems, as for example, absolute and relative error methods or frequency weighted methods, both in combination with modal separation or coprime factorization techniques. Special frequency weighted controller reduction methods able to preserve closed-loop stability and even closed-loop performance are also discussed. A selection of user callable and supporting routines to be implemented for controller reduction is proposed. The new routines will be included in the SLICOT library.

Contents

1 Introduction

High quality numerical software implementing LQG and H_∞-synthesis procedures has been recently developed by NICONET and included in SLICOT. However, these sophisticated controller design approaches lead often to high order controllers, with orders comparable to those of the plants. From a practical point of view simple controllers are preferred over complex ones. The main advantages of simpler controllers are their lower computational complexity allowing higher sampling rates in real-time implementation, and an easier maintenance (e.g. easier bug fix). There are two main approaches to design lower order controllers [1]: the *direct* approach, where the parameters of a fixed low order controller are computed by some optimization or other procedure, and the *indirect* approach where a high-order controller is first found and then model reduction techniques are used to simplify it. In our discussion we focus on controller reduction techniques to support the indirect approach to low order controller design.

Since controllers, seen as systems, can be unstable, methods for reduction of unstable systems can be in principle used also for controller reduction. For example, the modal separation approach [21] in conjunction with balancing related techniques [11, 9, 3, 5] as well as the coprime factorization based approach [18] can be used to compute lower order controllers using the model reduction software available in SLICOT. The main difficulty using these model reduction based approaches is that the presence of the plant in the control loop is completely ignored and the resulting reduced controllers can lead to unsatisfactory closed-loop performance and even to the loss of closed-loop stability.

It follows that the controller reduction problems must be handled distinctly from the open-loop model reduction problems because of the presence of the plant. What is finally important in controller reduction is the preservation of the overall closed-loop performance achieved with the original controller. In this short survey we discuss two method classes appropriate for controller reduction: frequency weighting methods [7, 4] and methods using a fractional representation of the controller [8]. A particular case of frequency weighted reduction is the class of relative error methods [12]. Both method classes can address the controller reduction problem trying to preserve the closed-loop stability, the closed-loop performance, and the closed-loop transfer function. Comprehensive presentations of controller reduction methods and the reasons behind different approaches can be found in the textbooks [2, 25].

The basis for standardization of the controller reduction routines in SLICOT will form a small set of routines available in the RASP-MODRED library [17] for relative error methods and frequency weighted reduction. However, for some methods of potential interest for practical applications (e.g., the frequency-weighted reduction using Enns's approach [4]), no reliable Fortran software exists. Thus the implementation of several new routines is necessary. The basis of new implementations will form the already existing routines in the Model Reduction Toolbox prepared within Task II.A.

2 Overview of Absolute Error Methods

Consider the n-th order original state-space model $G := (A, B, C, D)$ with the *transfer-function matrix* (TFM) $G(\lambda) = C(\lambda I - A)^{-1}B + D$, and let $G_r := (A_r, B_r, C_r, D_r)$ be an r-th order approximation of the original model ($r < n$), with the TFM $G_r = C_r(\lambda I - A_r)^{-1}B_r + D_r$. Absolute (or additive) error model reduction tries to minimize the absolute approximation error

$$\|G - G_r\|_\infty. \tag{1}$$

A large class of model reduction methods can be interpreted as performing a similarity transformation Z yielding

$$\left[\begin{array}{c|c} Z^{-1}AZ & Z^{-1}B \\ \hline CZ & D \end{array}\right] := \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \tag{2}$$

and then defining the reduced model G_r as the leading diagonal system

$$(A_r, B_r, C_r, D_r) = (A_{11},\, B_1,\, C_1,\, D).$$

When writing

$$Z := [\,T\;U\,], \quad Z^{-1} := \left[\begin{array}{c} L \\ V \end{array}\right],$$

then $\Pi = TL$ is a projector on T along L and $LT = I_r$. Thus the reduced system is given by

$$(A_r, B_r, C_r, D_r) = (LAT,\, LB,\, CT,\, D).$$

Partitioned forms as in (2) can be used to construct a so-called *singular perturbation approximation* (SPA). The matrices of the reduced model in this case are given by

$$\begin{array}{rcl} A_r & = & A_{11} + A_{12}(\gamma I - A_{22})^{-1}A_{21}, \\ B_r & = & B_1 + A_{12}(\gamma I - A_{22})^{-1}B_2, \\ C_r & = & C_1 + C_2(\gamma I - A_{22})^{-1}A_{21}, \\ D_r & = & D + C_2(\gamma I - A_{22})^{-1}B_2. \end{array} \tag{3}$$

where $\gamma = 0$ for a continuous-time system and $\gamma = 1$ for a discrete-time system. Note that SPA formulas preserve the DC-gains of stable original systems.

The emphasis on improving the accuracy of computations led to so-called algorithms with *enhanced accuracy.* In many model reduction methods, the matrices L and T are determined from two positive semi-definite matrices P and Q, called generically *gramians.* The gramians can be always determined in Cholesky factorized forms $P = S^TS$ and $Q = R^TR$, where S and R are upper-triangular matrices. The computation of L and T can be done from the singular value decomposition

$$SR^T = \left[\begin{array}{cc} U_1 & U_2 \end{array}\right] \mathrm{diag}(\Sigma_1, \Sigma_2) \left[\begin{array}{cc} V_1 & V_2 \end{array}\right]^T,$$

where

$$\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_r), \quad \Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \ldots, \sigma_n),$$

and $\sigma_1 \geq \ldots \geq \sigma_r > \sigma_{r+1} \geq \ldots \geq \sigma_n \geq 0$.

The so-called *square-root* (**SR**) methods determine L and T as [14]

$$L = \Sigma_1^{-1/2}V_1^TR, \qquad T = S^TU_1\Sigma_1^{-1/2}.$$

If r is the order of a minimal realization of G then the gramians corresponding to the resulting realization are diagonal and equal. In this case the minimal realization is called *balanced*. The **SR** approach is usually very accurate for well-equilibrated systems. However if the original system is highly unbalanced, potential accuracy losses can be induced in the reduced model if either L or T is ill-conditioned.

In order to avoid ill-conditioned projections, a *balancing-free* (**BF**) approach has been proposed in [13] in which always well-conditioned matrices L and T can be determined. These matrices are computed from orthogonal matrices whose columns span orthogonal bases for the right and left eigenspaces of the product PQ corresponding to the first r largest eigenvalues $\sigma_1^2, \ldots, \sigma_r^2$. Because of the need to compute explicitly P and Q as well as their product, this approach is usually less accurate for moderately ill-balanced systems than the **SR** approach.

A *balancing-free square-root* (**BFSR**) algorithm which combines the advantages of the **BF** and **SR** approaches has been introduced in [16]. L and T are determined as

$$L = (Y^T X)^{-1} Y^T, \qquad T = X,$$

where X and Y are $n \times r$ matrices with orthogonal columns computed from two QR decompositions

$$S^T U_1 = XW, \quad R^T V_1 = YZ$$

with W and Z non-singular and upper-triangular. The accuracy of the **BFSR** algorithm is usually better than either of **SR** or **BF** approaches.

The SPA formulas can be used directly on a balanced minimal order realization of the original system computed with the **SR** method. A **BFSR** method to compute SPAs has been proposed in [15]. The matrices L and T are computed such that the system (LAT, LB, CT, D) is minimal and the product of corresponding gramians has a block-diagonal structure which allows the application of the SPA formulas.

Provided the Cholesky factors R and S are known, the computation of matrices L and T can be done by using exclusively numerically stable algorithms. Even the computation of the singular value decomposition of SR^T can be done without forming this product. Thus the effectiveness of the **SR** or **BFSR** techniques depends entirely on the accuracy of the computed Cholesky factors of the gramians.

In the *balance & truncate* (B&T) method [11] P and Q are the controllability and observability gramians satisfying a pair of continuous- or discrete-time Lyapunov equations

$$\left\{ \begin{array}{rcl} AP + PA^T + BB^T & = & 0 \\ A^TQ + QA + C^TC & = & 0 \end{array} \right. \quad (c)\ , \qquad \left\{ \begin{array}{rcl} APA^T + BB^T & = & P \\ A^TQA + C^TC & = & Q \end{array} \right. \quad (d)\ .$$

These equations can be solved directly for the Cholesky factors of the gramians by using numerically reliable algorithms proposed in [6]. The **BFSR** version of the B&T method is described in [16]. Its **SR** version [14] can be used to compute balanced minimal representations. Such representations are also useful for computing reduced order models by using the SPA formulas [9] or the *Hankel-norm approximation* (HNA) method [5]. A **BFSR** version of the SPA method is described in [15].

The attractive feature of all balancing related methods is the availability of an *a priori* error bound. The key result is

$$\|G - G_r\|_\infty \leq 2\mathrm{tr}\,\Sigma_2$$

for all these methods. Note that the actual error may be considerably less than the above error bound, so that this formula can be seen generally as a guide to choose the appropriate order of the reduced system.

Since controllers, seen as systems, can be unstable, the B&T, SPA and HNA are suitable for controller reduction only in combination with modal approaches or coprime factorizations techniques. If K is the transfer-function matrix of the controller, than the controller reduction can be performed by separating first the stable and unstable parts of K and reducing only the stable dynamics of the controller. Then the unstable part is included unmodified in the resulting reduced controller. The following is a simple procedure for this computation:

1. Decompose additively K as $K = K_1 + K_2$, such that K_1 has only stable poles and K_2 has only unstable poles.
2. Determine K_{1r}, a reduced order approximation of the stable part K_1.
3. Assemble the reduced model K_r as $K_r = K_{1r} + K_2$.

Note that for the model reduction at step 2 any of methods available for stable systems can be used. In particular, the frequency weighting model reduction methods discussed in section 4 can be used to perform controller reduction as well. An advantage of this approach is that it preserves the number of unstable poles of the controller, a condition often required to guarantee closed-loop stability when using reduced controllers.

The second approach is based on computing a stable rational *coprime factorization* (CF) of K. The following procedure can be used to compute an r-th order approximation K_r of an n-th order (not necessarily stable) controller K:

1. Compute a left CF of the TFM K in the form $K = M^{-1}N$, where M, N are stable and proper rational TFMs.
2. Approximate the stable system of order n $[\,N\ M\,]$ with $[\,N_r\ M_r\,]$ of order r.
3. Form the r-th order approximation $K_r = M_r^{-1}N_r$.

The usefulness of this approach relies on the assumption that the McMillan degree of $K = M^{-1}N$ is generally that of $[\,N\ M\,]$ and similarly for $K_r = M_r^{-1}N_r$ and $[\,N_r\ M_r\,]$. In this way the reduction of the McMillan degree of $[\,N\ M\,]$ through approximation by $[\,N_r\ M_r\,]$ is equivalent to reduction of the McMillan degree of K by K_r, which is our objective. The factorization methods proposed in [18, 20] fulfill the above assumptions, and thus can be employed to implement the above procedure.

The coprime factorization approach used in conjunction with the B&T method fits well in the general projection formulation introduced in Section 2. The gramians necessary to compute the projection are the gramians of the system $[\,N\ M\,]$. Still, the truncation matrices L and T, determined by using either the **SR** or **BFSR** methods, can be directly applied to the matrices of the original system. The already mentioned factorization algorithms are based on a numerically reliable Schur technique for pole assignment and can be used to compute two types of CFs: the CF with prescribed stability degree [18] and the CF with inner denominator [20]. For the class of observer based controllers, fractional representations are readily available, thus controller reduction can be performed immediately on the factors. This aspect will be discussed in more detail in Section 5.

The modal separation or coprime factorization approaches combined with balancing related techniques (e.g., B&T, SPA or HNA) are not always suited for controller reduction. The main problem is that there is no guarantee for closed-loop stability or for preserving closed-loop performance. Therefore, alternative techniques are necessary to be considered. This justifies the main goal of Task II.B to implement special software focusing on the controller reduction aspect.

Robust numerical software for additive error model reduction methods is available in the recently developed Model Reduction Toolbox for SLICOT[1] [22] and covers both the reduction of stable as well as the reduction of unstable systems. These routines will serve also as basis for the controller reduction software to be implemented in SLICOT.

3 Relative Error Methods

Relative (or multiplicative) error reduction methods have several properties which recommend them for controller reduction. While additive error methods compute good approximations in terms of peek errors (e.g., H_∞-norm), relative error methods have good approximation properties over the whole frequency range. This is why it is expected that relative error methods in combination with modal separation techniques are better suited for controller approximation than absolute error methods.

The *balanced stochastic truncation* (BST) method [3] is a relative error method which tries to minimize $\|\Delta_r\|_\infty$, where Δ_r is the relative error defined implicitly by $G_r = (I - \Delta_r)G$. In the BST method the gramian Q satisfies a Riccati equation, while the gramian P still satisfies a Lyapunov equation. Although the determination with high accuracy of the Cholesky factor of Q is computationally involved, it is however necessary to guarantee the effectiveness of the **BFSR** approach. Iterative refinement techniques are described for this purpose in [23].

Both the **SR** and **BFSR** versions of the BST algorithm are implemented in the following routines available in the RASP-MODRED library:

SRST	computes reduced order models using the **SR** BST method [12]
SRBFS	computes reduced order models using the **BFSR** BST method [23]

Both SRST and SRBFS are applicable to both continuous- and discrete-time systems. For each routine a parameter α can be used as a weight between the absolute and relative errors. For $\alpha > 0$, the BST method is performed on a modified system with the transfer-function matrix $[\,G\ \alpha I\,]$. A zero value of α means a pure relative error minimization. Large positive values of α produce approximations which minimize the absolute approximation error (1). When $\alpha \to \infty$,

[1]available at `ftp://wgs.esat.kuleuven.ac.be/pub/WGS/REPORTS/nic1999-8.ps.Z`

the BST method produces identical results with the B&T method.

4 Frequency-Weighted Model Reduction

The methods for *frequency-weighted model reduction* (FWMR) try to minimize a weighted error of the form

$$\|W_o(G - G_r)W_i\|_\infty,$$

where W_o and W_i are suitable output and input weighting TFMs, respectively. The presence of weights reflects the desire that the approximation process be more accurate at certain frequencies where W_o and/or W_i have larger singular values. Note that for a square invertible G, taking $W_o = G^{-1}$ and $W_i = I$, the relative error model reduction problem discussed in the previous section can be formulated as a FWMR problem. Many controller reduction problems can also be formulated appropriately as FWMR problems [1].

Let K be the TFM of the controller and let K_r an r-th order approximation of it. On basis of stability margins considerations it is desirable that

$$\|(I + GK)^{-1}G(K - K_r)\|_\infty < 1 \tag{4}$$

or

$$\|(K - K_r)G(I + KG)^{-1}\|_\infty < 1. \tag{5}$$

This leads to the following FWMR problem: find K_r with the same number of unstable poles as K such that the left side of (4) or (5) is minimized. In this case, the matrices $W_o = (I+GK)^{-1}G$ and $W_i = G(I + KG)^{-1}$ act in (4) or (5) as one-sided weighting matrices, respectively.

Performance preserving considerations lead also to FWMR problems. Let G_c and G_{cr} be the closed-loop TFMs for K and K_r, respectively, given by

$$G_c = GK(I + GK)^{-1} = I - (I + GK)^{-1},$$

$$G_{cr} = GK_r(I + GK_r)^{-1} = I - (I + GK_r)^{-1}.$$

Neglecting second order terms in $K - K_r$ we have

$$G_c - G_{cr} \approx (I + GK)^{-1}G(K - K_r)(I + KG)^{-1} \tag{6}$$

and this leads to a frequency weighted approximation problem with double-sided weights $W_o = (I + GK)^{-1}G$ and $W_i = (I + KG)^{-1}$.

Since most results on FWMR are available only for the continuous-time systems, we will address only methods for continuous-time reductions. The reduction of discrete-time systems can be done by using bilinear transformations. Two basic approaches can be used to solve FWMR problems. The first approach which we discuss is the *frequency-weighted balance and truncate* (FWB&T) method proposed originally by Enns in [4]. This approach can be easily embedded in the general **BFSR** formulation of Section 2. Provided G and the two weights W_o and W_i are all stable TFMs, then the frequency-weighted controllability and observability gramians P and Q, respectively, can be computed as follows. Let $W_o = (A_o, B_o, C_o, D_o)$ and

$W_i = (A_i, B_i, C_i, D_i)$ be minimal realizations of the weighting matrices and consider the following realizations of GW_i and W_oG

$$GW_i = \left[\begin{array}{c|c} \overline{A}_i & \overline{B}_i \\ \hline \overline{C}_i & \overline{D}_i \end{array}\right] =: \left[\begin{array}{cc|c} A & BC_i & BD_i \\ 0 & A_i & B_i \\ \hline C & DC_i & DD_i \end{array}\right], \quad W_oG = \left[\begin{array}{c|c} \overline{A}_o & \overline{B}_o \\ \hline \overline{C}_o & \overline{D}_o \end{array}\right] =: \left[\begin{array}{cc|c} A & 0 & B \\ B_oC & A_o & B_0D \\ \hline D_oC & C_o & D_oD \end{array}\right]$$

Let

$$\overline{P} = \left[\begin{array}{cc} P & P_{12} \\ P_{12}^T & P_{22} \end{array}\right], \quad \overline{Q} = \left[\begin{array}{cc} Q & Q_{12} \\ Q_{12}^T & Q_{22} \end{array}\right],$$

be the controllability gramian of GW_i and the observability gramian of W_oG, which contain in the leading positions the corresponding frequency-weighted gramians P and Q. The projection matrices L and T for the **BF**, **SR** and **BFSR** approaches can be computed directly from the Cholesky factorizations $P = S^TS$ and $Q = R^TR$, which are readily available as the leading Cholesky factors of $\overline{P}$ and $\overline{Q}$. The main drawbacks of Enns' method are the lack of an easily computable *a priori* error bound and the lack of stability guarantee of the reduced model in case of two-sided weighting unless either $W_o = I$ or $W_i = I$. For supplementary details on this approach see [1, 24].

Improvements of Enns' approach have been proposed by several authors. An improved Enns' method has been recently proposed by Wang et el. [24] and yields stable models also in case of double-sided weighting. An easily computable *a priori* error bound for the weighted approximation error has been also derived. For a continuous-time system the two gramians P and Q are determined from the solution of two Lyapunov equations

$$\begin{array}{rcl} AP + PA^T + \overline{B}\,\overline{B}^T & = & 0 \\ QA + A^TQ + \overline{C}^T\overline{C} & = & 0, \end{array}$$

where $\overline{B}$ and $\overline{C}$ are fictitious input and output matrices computed according to formulas given in [24]. Under some mild conditions it is proven in [24] that

$$\|W_o(G - G_r)W_i\|_\infty \leq k \operatorname{tr} \Sigma_2,$$

where $k = 2\|W_oL\|_\infty\|KW_i\|_\infty$ with L and K matrices resulting from the computation of $\overline{B}$ and $\overline{C}$.

In the second approach to FWMR we assume that G is stable and W_o, W_i are invertible, having only unstable poles and zeros. The technique proposed by Latham and Anderson in [7] to solve the FWMR problem has the following main steps:

1. Compute G_1 as the n-th order stable projection of W_oGW_i.
2. Compute G_{1r} as an r-th order approximation of G_1 by using one of methods for stable systems.
3. Compute G_r as the r-th order stable projection of $W_o^{-1}G_{1r}W_i^{-1}$.

The RASP-MODRED library provides some tools to perform FWMR. Two special routines based on algorithms proposed in [19] are available to compute efficiently the stable projections in the above procedure:

SFRLW	constructs for either $W_o^{-1}G$ or W_oG an n-th order state-space realization of its stable projection by using the explicit formulas derived in [19].
SFRRW	constructs for either GW_i^{-1} or GW_i an n-th order state-space realization of its stable projection by using the explicit formulas derived in [19].

5 Controller Reduction Using Fractional Representations

Feedback controllers resulting from LQG designs have a special structure which allows the direct use of model reduction techniques on appropriate coprime factorized representations. Let F and L be the state-feedback gain and the Kalman-gain of a full order state estimator, respectively. The overall feedback controller is given by

$$K(\lambda) = F(\lambda I - A - BF - LC)^{-1}L.$$

This form of the controller allows to write immediately a left coprime factorization representation of the controller as

$$K = M^{-1}N,$$

where the factors result from

$$[\,M - I \;\; N\,] = F(\lambda I - A - LC)^{-1}[\,B\ L\,].$$

By approximating $[\,M\ N\,]$ by some lower order approximation $[\,M_r\ N_r\,]$ leads to a reduced order controller $K_r = M_r^{-1}N_r$. A right coprime approximation based formulation is also possible (see [1] for details). Since the above approach essentially performs an order reduction of a full order state estimator with the state-feedback gain as output matrix, this technique can be also extended to controllers based on minimal order observers. To implement this approach practically all necessary software is available in SLICOT.

The coprime factorization approach can be combined with frequency weighting in order to guarantee the closed-loop stability, by using $\widetilde{K} = [\,M - I\ N\,]$ as the "controller" and $\widetilde{G} = [\,I\ G^T\,]^T$ as the "plant". Then, a FWMR can be performed along the lines of the formulation based on (5). Note that a particularly simple expression for the corresponding input weighting W_i is obtained in case of using full-order observer based controller (see [10] for further details).

6 Preliminary Selection of Controller Reduction Routines

In this section we present a preliminary selection of routines to be considered for standardization and implementation for the Controller Reduction Toolbox for SLICOT. For some routines software implementations are available in the RASP-MODRED library of DLR. These routines will serve as starting points for standardization for SLICOT. However, for some of the proposed routines there are no Fortran implementations available and therefore new implementations are necessary.

6.1 General purpose routines

These routines are intended to perform a FWMR with general weights (one- or double-sided) and for relative error methods. They complement and extend the collection of already available general purpose model reduction routines available in SLICOT.

AB09HD	stochastic B&T and SPA approaches using the BFSR approach [23] (based on SRST and SRBFS routines)
AB09ID	frequency-weighted B&T or SPA method using Enns' method [4]
AB09JD	frequency-weighted B&T or SPA method using improved Enns' method [24]
AB09KD	frequency-weighted Hankel-norm method using the Latham-Anderson approach [7]

6.2 Special controller reduction routines

These routines are intended to address explicitly the controller reduction problem using a FWMR framework with special one- or double-sided weights.

SB16AD	controller reduction using frequency-weighted B&T or SPA methods of Enns [4] for the three problems defined by (4), (5) and (6)
SB16BD	state-feedback/full-order estimator based controller reduction using coprime factorization with B&T or SPA methods [1]

6.3 Auxiliary tools

To implement the frequency-weighted Hankel-norm approximation method [7] the stable projections can be computed by using the explicit formulas derived in [19]. To implement this formulas routines to solve both continuous-time and discrete-time Sylvester equation with the coefficient matrix in real Schur form are also necessary. Note that presently no routines to solve discrete-time Sylvester equations are available in SLICOT. The following auxiliary and basic routines are necessary to implement frequency weighted model reduction:

AB07ND	constructs the matrices of an inverse system (based on SYSINV routine from RASP)
AB09KX	constructs for either $W_o^{-1}G$ or W_oG an n-th order state-space realization of its stable projection by using the explicit formulas derived in [19] (based on SFRLW routine).
AB09KY	constructs for either GW_i^{-1} or GW_i an n-th order state-space realization of its stable projection by using the explicit formulas derived in [19] (based on SFRRW routine).
SB04PD	solves continuous- and discrete-time Sylvester equations with both matrices in real Schur form (based on SYLVS routine from RASP)

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