Orbital stabilization of mechanical systems through semidefinite Lyapunov functions

Gianluca Garofalo, Christian Ott and Alin Albu-Schäffer

***Abstract*— In this paper we address the problem of generating asymptotically stable limit cycles for a fully actuated multibody mechanical system through a feedback control law. Using the concept of conditional stability the limit cycle can be designed for a lower dimensional dynamical system describing how the original one evolves on a chosen submanifold and the corresponding velocity space. Moreover, the controller can be split up in two parts that can be independently designed and analyzed in order to reach the constraint submanifold and then produce the oscillation. Even if designed assuming a lower dimensional system, the limit cycle implies a periodic motion for the whole system.**

I. INTRODUCTION

As shown in [1], [2], [3] walking and running can be effectively described as periodic tasks. In these cases it is more important to stay on a prescribed orbit in the state space, rather than following the exact position in time along the desired curve. For these applications tracking a trajectory might not be the best solution, as already addressed in [4], [3]. Moreover in the latter the need of controlling the energy of the system to a desired value was already recognized. In this paper we solve the problem of generating a stable limit cycle for the system using directly the information on its energy level.

Similar approaches to the problem of orbital stabilization have been already shown in [5], [6], [7], [8]. In [5], [6], [7] the authors extend the potential field controller adding power-continuous terms, while in [8] the concepts of virtual constraint and feedback linearization are used to obtain a closed loop system that generates its own periodic stable motion. In this paper we formulate the problem based on the null space decomposition introduced in [9] and used for nullspace compliance control in [10], [11]. In this way we think that several advantages can be achieved. Compared to [8] we take advantage of the passivity property of the system and do not completely alter the original dynamics of the system through feedback linearization. Moreover, we completely separate the problem of producing the limit cycle from the virtual constraints, instead of modifying the latter for achieving the first. The input torques are split in the ones necessary for producing the limit cycle and the ones necessary for satisfying the virtual constraints. These, in turn, are responsible for the subspace in which the system will oscillate. Nevertheless it should be also mentioned that in [8] the more complicated problem of controlling an underactuated system is considered, which here we do not take into account yet. It can also be shown that for a one-dimensional nullspace, as used in this paper, the power-conserving nullspace decoupling from [11] is equivalent to the nominal control law from [5], [6]. However, the approach from [11] can also be applied in case of a higher-dimensional nullspace. In [5], [6] a passive control action is designed which allows to decouple the motion along a vector field from the remaining motion. The system is then forced to follow an integral curve of this vector field via a passive control law. In case of a closed integral field, the system thus converges to a closed orbit in the configuration space. In [7] additionally a non-passive control action was proposed to achieve regulation of the final velocity along the vector field. In contrast to [5], [6], [7], we aim at achieving a stable limit cycle in the state space, which is achieved by regulating a virtual energy function in a one-dimensional submanifold of the configuration space. This virtual energy function consists of the physical kinetic energy and a virtual potential energy, which represents an additional design element in the controller. In future works, we plan to utilize the freedom in choosing this potential for achieving energy efficient motion in mechanical systems with compliant actuation.

The paper is organized as follows. In Section II a simple nonlinear oscillator is presented, which produces limit cycles based on an energy argument. Section III and Section IV are the main contribution of the paper. There we explain our main idea for producing a desired limit cycle for the whole system. To this end we use a nullspace decomposition and extend known results for conditional stability of equilibrium points to generic invariant sets, i.e. also for limit cycles. These results will allow us to derive in Section V the proposed controller, where also the main advantage of a conditional stability analysis will become clear: the two parts of the control law can be independently derived and analyzed, as sketched in Fig. 1. Finally, simulation results will be presented in Section VI to validate the proposed approach, followed by the final discussion and outline of future work in Section VII.

II. FROM CENTER TO LIMIT CYCLE

Consider the 1-DOF system

$$\ddot{q} + \omega^2 q = 0 \ , \tag{1}$$

with equilibrium point $q^* = 0$, $\dot{q}^* = 0$. This type of equilibrium point is also called centre [12], since for each initial condition $q = q_0$, $\dot{q} = \dot{q}_0$ the resulting trajectory will be a closed orbit around the equilibrium point, that is the

G. Garofalo, C. Ott and A. Albu-Schäffer are with the Institute of Robotics and Mechatronics, German Aerospace Center (DLR), Wessling, Germany. {gianluca.garofalo, christian.ott, alin.albu-schaeffer}@dlr.de

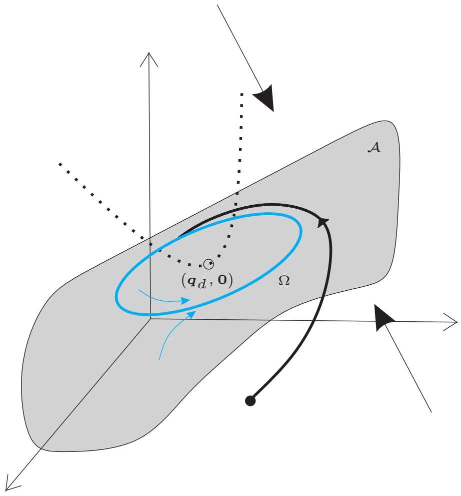

Fig. 1. Conceptual illustration of the main idea of the paper. We choose to produce an oscillation on a constraint submanifold. To this end we force the system to reach a subset of the state space, where we produce a limit cycle.

level set $\mathcal{L}_0$ of the Hamiltonian $H(q,\dot{q}) = \frac{1}{2}\left(\dot{q}^2 + \omega^2 q^2\right)$, defined as $\mathcal{L}_0 = \{q, \dot{q} \mid H(q,\dot{q}) = H(q_0, \dot{q}_0)\}$.

The difference between these closed orbits and limit cycles is that they are not isolated. If we force the system to reach always a desired value of the Hamiltonian, then we will obtain a limit cycle. This is possible by modifying the system as

$$\ddot{q} + d\tilde{H}(q,\dot{q})\,\dot{q} + \omega^2 q = 0 \ , \tag{2}$$

where $d > 0$, $\tilde{H}(q,\dot{q}) = H(q,\dot{q}) - H_d$ and $H_d > 0$ is the desired value of the Hamiltonian[1].

Proof: We consider the Lyapunov function $V(q,\dot{q}) = \frac{1}{2}\tilde{H}^2(q,\dot{q})$, with derivative $\dot{V}(q,\dot{q}) = \tilde{H}(q,\dot{q})\,\dot{\tilde{H}}(q,\dot{q}) = -d\tilde{H}^2(q,\dot{q})\,\dot{q}^2 \leq 0$. Applying LaSalle's invariance theorem, where $\Omega = \{q, \dot{q} \mid H(q,\dot{q}) > 0\}$ is an invariant set which excludes the equilibrium point $(q^*, \dot{q}^*)$, $\mathcal{M} = \{q, \dot{q} \mid q \neq 0, \dot{q} = 0 \; or \; H(q,\dot{q}) = H_d\}$ is the set $\mathcal{M} \subset \Omega$ where $\dot{V} = 0$ and $\mathcal{L}_d = \{q, \dot{q} \mid H(q,\dot{q}) = H_d\}$ is the only invariant set in $\mathcal{M}$, we can conclude that every solution starting in Ω approaches $\mathcal{L}_d$ as $t \to \infty$. ■

From the proof it follows that the result is actually more general, meaning that even for a multibody system we can conclude that $H(\boldsymbol{q},\dot{\boldsymbol{q}}) \to H_d$ as $t \to \infty$. In fact, for a multibody mechanical system, with kinetic energy T, potential energy U, non conservative forces $\boldsymbol{\tau}$, generalized coordinates $\boldsymbol{q}$ and corresponding generalized velocities $\dot{\boldsymbol{q}}$, we know that

$$\dot{H}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{T} + \dot{U} = \dot{\boldsymbol{q}}^T \boldsymbol{\tau} \ . \tag{3}$$

The choice $\boldsymbol{\tau} = -\tilde{H}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\boldsymbol{D}\dot{\boldsymbol{q}}$, with $\boldsymbol{D}$ positive definite, will always ensure $\tilde{H}(\boldsymbol{q},\dot{\boldsymbol{q}}) \to 0$. The difference is that for a 1-DOF system the set $\mathcal{L}_d = \{q, \dot{q} \mid H(q,\dot{q}) = H_d\}$ is a closed orbit in the state space (corresponding to a limit cycle), which is not true for a n-DOF system.

[1]The condition $H_d > 0$ excludes that the system will stop even if $T = 0$, since those configurations do not correspond to equilibrium points. In other words if $T = 0$ then $U = H_d$ and there will be a non zero acceleration due to the conservative forces $-\nabla U \neq \boldsymbol{0}$.

III. SYSTEM MODEL AND COORDINATE TRANSFORMATION

Consider a fully actuated n-DOF system, with dynamic equation

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \ , \tag{4}$$

where $\boldsymbol{q}, \dot{\boldsymbol{q}} \in \mathbb{R}^n$ is the state of the system, $\boldsymbol{\tau} \in \mathbb{R}^n$ is the input, $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the mass matrix, $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) \in \mathbb{R}^{n\times n}$ is the Coriolis matrix and $\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^n$ is the gravity vector.

We want to constrain the system to evolve on a submanifold and the corresponding velocity space, where we will produce a limit cycle. To this end we consider the function

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{q}) \ , \tag{5}$$

where $\boldsymbol{x} : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $\boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{0}$ defines the 1 - dimensional constraint submanifold of the configuration space and $\boldsymbol{J}(\boldsymbol{q}) \in \mathbb{R}^{(n-1)\times n}$ is the Jacobian matrix of the mapping (5), which is assumed to be a full rank matrix. Accordingly we can write the dynamics of the system with a new set of coordinates, as in [9], [10], [11]. We first compute a null space base matrix[2] $\boldsymbol{Z}(\boldsymbol{q}) \in \mathbb{R}^{1\times n}$ which allows us to obtain the directions orthogonal to the submanifold, then we use $\boldsymbol{Z}(\boldsymbol{q})$ to compute a dynamically consistent[3] null space projector $\boldsymbol{N}(\boldsymbol{q})$, which will be part of the extended Jacobian matrix $\boldsymbol{J}_N(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$, such that

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ v \end{bmatrix} = \boldsymbol{J}_N(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \begin{bmatrix} \boldsymbol{J}(\boldsymbol{q}) \\ \boldsymbol{N}(\boldsymbol{q}) \end{bmatrix} \dot{\boldsymbol{q}} \ , \tag{6}$$

where $\boldsymbol{N}(\boldsymbol{q}) = \left(\boldsymbol{Z}(\boldsymbol{q})\,\boldsymbol{M}(\boldsymbol{q})\,\boldsymbol{Z}^T(\boldsymbol{q})\right)^{-1} \boldsymbol{Z}(\boldsymbol{q})\,\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{1\times n}$ and v is an additional null space velocity. One can show that by this choice the extended Jacobian $\boldsymbol{J}_N(\boldsymbol{q})$ is non singular and the inverse is given by

$$\boldsymbol{J}_N^{-1}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{J}^{+_M}(\boldsymbol{q}) & \boldsymbol{Z}^T(\boldsymbol{q}) \end{bmatrix} \ , \tag{7}$$

where $\boldsymbol{J}^{+_M}(\boldsymbol{q})$ denotes the dynamically consistent weighted pseudo inverse defined as

$$\boldsymbol{J}^{+_M}(\boldsymbol{q}) = \boldsymbol{M}^{-1}(\boldsymbol{q})\,\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{M}^{-1}(\boldsymbol{q})\,\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1} . \tag{8}$$

The joint velocity can thus be computed from the Cartesian velocity and the null space velocity via

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{+_M}(\boldsymbol{q})\,\dot{\boldsymbol{x}} + \boldsymbol{Z}^T(\boldsymbol{q})\,v \ , \tag{9}$$

where, unlike the usual null space projector (see e.g. [13]) defined as $\boldsymbol{I} - \boldsymbol{J}^{+_M}(\boldsymbol{q})\,\boldsymbol{J}(\boldsymbol{q})$, the matrix $\boldsymbol{Z}^T(\boldsymbol{q})$ has only one independent column.

[2]I.e. it fulfils the condition $\boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{Z}^T(\boldsymbol{q}) = \boldsymbol{0}$.
[3]I.e. it fulfils the condition $\boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{M}^{-1}(\boldsymbol{q})\,\boldsymbol{N}^T(\boldsymbol{q}) = \boldsymbol{0}$.

From (6) and (9) it is straightforward to rewrite (4) in the extended velocity coordinates as

$$\begin{aligned}\mathbf{\Lambda}(\boldsymbol{q})\begin{bmatrix}\ddot{\boldsymbol{x}}\\ \dot{v}\end{bmatrix}+\mathbf{\Gamma}(\boldsymbol{q},\dot{\boldsymbol{q}})\begin{bmatrix}\dot{\boldsymbol{x}}\\ v\end{bmatrix}&=\\ \boldsymbol{J}_N^{-T}(\boldsymbol{q})\left(-\boldsymbol{g}(\boldsymbol{q})+\boldsymbol{\tau}\right)&\ ,\end{aligned}\tag{10}$$

with the matrices $\mathbf{\Lambda}(\boldsymbol{q})$ and $\mathbf{\Gamma}(\boldsymbol{q},\dot{\boldsymbol{q}})$ given by[4]

$$\begin{aligned}\mathbf{\Lambda}(\boldsymbol{q})&=\boldsymbol{J}_N^{-T}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}_N^{-1}(\boldsymbol{q})\\ &=\begin{bmatrix}\mathbf{\Lambda}_x(\boldsymbol{q}) & \mathbf{0}\\ \mathbf{0} & \Lambda_n(\boldsymbol{q})\end{bmatrix}\\ \mathbf{\Lambda}_x(\boldsymbol{q})&=\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\\ \Lambda_n(\boldsymbol{q})&=\boldsymbol{Z}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{Z}^T(\boldsymbol{q})\end{aligned}$$

and (omitting the dependences)

$$\begin{aligned}\mathbf{\Gamma}(\boldsymbol{q},\dot{\boldsymbol{q}})&=\begin{bmatrix}\mathbf{\Gamma}_x(\boldsymbol{q},\dot{\boldsymbol{q}}) & \mathbf{\Gamma}_{xn}(\boldsymbol{q},\dot{\boldsymbol{q}})\\ \mathbf{\Gamma}_{nx}(\boldsymbol{q},\dot{\boldsymbol{q}}) & \Gamma_n(\boldsymbol{q},\dot{\boldsymbol{q}})\end{bmatrix}\\ \mathbf{\Gamma}_x&=\mathbf{\Lambda}_x\left(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{C}-\dot{\boldsymbol{J}}\right)\boldsymbol{J}^{+_M}\\ \mathbf{\Gamma}_{xn}&=\mathbf{\Lambda}_x\left(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{C}-\dot{\boldsymbol{J}}\right)\boldsymbol{Z}^T\\ \mathbf{\Gamma}_{nx}&=-\mathbf{\Gamma}_{xn}^T\\ \Gamma_n&=\Lambda_n\left(\boldsymbol{N}\boldsymbol{M}^{-1}\boldsymbol{C}-\dot{\boldsymbol{N}}\right)\boldsymbol{Z}^T\end{aligned}$$

From now on we will use the variables $(\boldsymbol{q},\dot{\boldsymbol{x}},v)$ as the state variables instead of $(\boldsymbol{q},\dot{\boldsymbol{q}})$. The relation between these two sets of coordinates is given by (6) and (9).

Accordingly we can split our control input $\boldsymbol{\tau}$ in the components $\boldsymbol{\tau}_x$ and τ_n along the two spaces

$$\boldsymbol{\tau}=\boldsymbol{J}_N^T(\boldsymbol{q})\begin{bmatrix}\boldsymbol{\tau}_x\\ \tau_n\end{bmatrix}=\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{\tau}_x+\boldsymbol{N}^T(\boldsymbol{q})\tau_n\ ,\tag{11}$$

so that (omitting the dependences) the complete system can be written in the form

$$\dot{\boldsymbol{q}}=\boldsymbol{J}^{+_M}\dot{\boldsymbol{x}}+\boldsymbol{Z}^T v\tag{12}$$

$$\ddot{\boldsymbol{x}}=\mathbf{\Lambda}_x^{-1}\left(-\mathbf{\Gamma}_x\dot{\boldsymbol{x}}-\mathbf{\Gamma}_{xn}v-\boldsymbol{J}^{+_M T}\boldsymbol{g}+\boldsymbol{\tau}_x\right)\tag{13}$$

$$\dot{v}=\Lambda_n^{-1}\left(\mathbf{\Gamma}_{xn}^T\dot{\boldsymbol{x}}-\Gamma_n v-\boldsymbol{Z}\boldsymbol{g}+\tau_n\right)\tag{14}$$

We will design the input $\boldsymbol{\tau}_x$ to keep the system on the submanifold and the corresponding velocity space, while the input τ_n will be responsible for producing an oscillation on the submanifold itself.

IV. STABILITY WITH SEMIDEFINITE FUNCTIONS

Here we present the theorems used to conclude the asymptotic stability of the limit cycle. They extend the results for equilibrium points that can be found in [14], [15].

Please refer to the Appendix for some useful definitions.

Consider the system

$$\dot{\boldsymbol{\chi}}=\boldsymbol{f}(\boldsymbol{\chi})\ ,\tag{15}$$

where $\boldsymbol{\chi}\in\mathcal{X}\subset\mathbb{R}^m$ and $\boldsymbol{f}:\mathcal{X}\to\mathbb{R}^m$ is a Lipschitz continuous function, so that a unique solution exists. We denote with $\boldsymbol{\chi}(t;\boldsymbol{\chi}_0)$ the solution of (15) starting at $\boldsymbol{\chi}_0$ and evaluated at the time instant t, with $\boldsymbol{\chi}(0;\boldsymbol{\chi}_0)=\boldsymbol{\chi}_0$.

Theorem 1 (Stability): Let Ω be an invariant set for (15), and let $V(\boldsymbol{\chi})$ be a C^1 function defined in $B_\nu(\Omega)\subset\mathcal{X}$ such that $V(\boldsymbol{\chi})\geq 0\ \forall\boldsymbol{\chi}\in B_\nu(\Omega)$, $V(\Omega)=0$ and $\dot{V}(\boldsymbol{\chi})\leq 0\ \forall\boldsymbol{\chi}\in B_\nu(\Omega)$. If Ω is asymptotically stable conditionally to $\mathcal{A}=\{\boldsymbol{\chi}\in B_\nu(\Omega)\ |\ V(\boldsymbol{\chi})=0\}$, then Ω is stable.

Proof: Suppose by contradiction that Ω is unstable. Then exist $\epsilon>0$, a sequence $(\boldsymbol{\chi}_{0n})_{n\in\mathbb{N}}\subset B_\epsilon(\Omega)$, $\lim_{n\to\infty}d(\boldsymbol{\chi}_{0n},\Omega)=0$, and a sequence $(t_n)_{n\in\mathbb{N}}\subset\mathbb{R}^+$ in such a way that

$$\begin{cases}d(\boldsymbol{\chi}(t;\boldsymbol{\chi}_{0n}),\Omega)<\epsilon & 0\leq t<t_n\\ d(\boldsymbol{\chi}(t_n;\boldsymbol{\chi}_{0n}),\Omega)=\epsilon & \forall n\in\mathbb{N}\end{cases}\tag{16}$$

Since $S_\epsilon(\Omega)$ is compact, $\boldsymbol{y}_n=\boldsymbol{\chi}(t_n;\boldsymbol{\chi}_{0n})\to\boldsymbol{y}\in S_\epsilon(\Omega)$ as $n\to\infty$. Moreover because of the continuity of the solutions of (15), $t_n\to\infty$ as $n\to\infty$.

Now we show that $V(\boldsymbol{\chi}(-t;\boldsymbol{y}))=0$. Let $\tau<0$ and $N\in\mathbb{N}$ be such that $0<t_n+\tau<t_n,\ \forall n\geq N$. Because V is not increasing along the solutions of (15), we have that

$$0\leq V(\boldsymbol{\chi}(t_n+\tau;\boldsymbol{\chi}_{0n}))\leq V(\boldsymbol{\chi}_{0n})\ .\tag{17}$$

From $\lim_{n\to\infty}d(\boldsymbol{\chi}_{0n},\Omega)=0$, $V(\Omega)=0$ and the continuity of V, it follows

$$\begin{aligned}V(\boldsymbol{\chi}(\tau;\boldsymbol{y}))&=\lim_{n\to\infty}V(\boldsymbol{\chi}(\tau;\boldsymbol{\chi}(t_n;\boldsymbol{\chi}_{0n})))\\ &=\lim_{n\to\infty}V(\boldsymbol{\chi}(t_n+\tau;\boldsymbol{\chi}_{0n}))=0\ .\end{aligned}\tag{18}$$

It remains to prove that $\boldsymbol{\chi}(-t;\boldsymbol{y})\in\mathcal{A}$ and $d(\boldsymbol{y},\Omega)=\epsilon$ cannot hold if Ω is asymptotically stable conditionally to $\mathcal{A}$. Since Ω is asymptotically stable conditionally to $\mathcal{A}$, $\exists\ T=T(\epsilon)>0\ |\ d(\boldsymbol{\chi}(T;\boldsymbol{\chi}_0),\Omega)\leq\frac{\epsilon}{2}$, with $\boldsymbol{\chi}_0\in\mathcal{A}$. If we choose $\boldsymbol{\chi}_0=\boldsymbol{\chi}(-T;\boldsymbol{y})\in\mathcal{A}$, then

$$\frac{\epsilon}{2}\geq d(\boldsymbol{\chi}(T;\boldsymbol{\chi}_0),\Omega)=d(\boldsymbol{\chi}(0;\boldsymbol{y}),\Omega)=d(\boldsymbol{y},\Omega)=\epsilon\ .$$

Since this is a contradiction, we conclude that Ω must be stable. ∎

Thanks to the properties of positive semidefinite functions it is possible to ensure not just the stability but even the asymptotic stability of an invariant set Ω. To this end let us first recall two results.

Lemma 1 ([14]): Let V be a nonnegative function defined in $B_\nu(\Omega)\subset\mathcal{X}$. Suppose that $\dot{V}(\boldsymbol{\chi})\leq 0\ \forall\boldsymbol{\chi}\in B_\nu(\Omega)$, then $\mathcal{A}=\{\boldsymbol{\chi}\in B_\nu(\Omega)\ |\ V(\boldsymbol{\chi})=0\}$ is a positively invariant set and $\mathcal{A}\subset\mathcal{M}=\left\{\boldsymbol{\chi}\in B_\nu(\Omega)\ |\ \dot{V}(\boldsymbol{\chi})=0\right\}$.

Lemma 2 ([12]): If a solution $\boldsymbol{\chi}(t;\boldsymbol{\chi}_0)$ of (15) is bounded and belongs to $\mathcal{X}$ for $t\geq 0$, then its positive limit set Ł$^+$ is a nonempty, compact, invariant set. Moreover, $\boldsymbol{\chi}(t;\boldsymbol{\chi}_0)$ approaches Ł$^+$ as $t\to\infty$

Theorem 2 (Asymptotic stability): Let Ω be an invariant set for (15), and let $V(\boldsymbol{\chi})$ be a C^1 function defined in $B_\nu(\Omega)\subset\mathcal{X}$ such that $V(\boldsymbol{\chi})\geq 0\ \forall\boldsymbol{\chi}\in B_\nu(\Omega)$, $V(\Omega)=0$ and $\dot{V}(\boldsymbol{\chi})\leq 0\ \forall\boldsymbol{\chi}\in B_\nu(\Omega)$. If Ω is asymptotically stable conditionally to the largest positively invariant set $\mathcal{M}^*$ within

[4]Notice that using a dynamically consistent null space projector the matrix $\mathbf{\Lambda}(\boldsymbol{q})$ is block diagonal.

$\mathcal{M} = \left\{\boldsymbol{\chi} \in B_\nu(\Omega) \mid \dot{V}(\boldsymbol{\chi}) = 0\right\}$, then Ω is asymptotically stable.

Proof: In order to prove asymptotic stability we have to show stability and attractiveness.

From Lemma 1 it follows that $\mathcal{A}$ is a positively invariant set and $\mathcal{A} \subset \mathcal{M}$, so since Ω is conditionally stable to $\mathcal{M}^*$ and $V(\Omega) = 0$ i.e. $\Omega \subset \mathcal{A}$, then it must be conditionally stable to $\mathcal{A}$, hence by Theorem 1 Ω is stable.

We will prove the attractiveness by contradiction. Since Ω is stable then $\forall \epsilon > 0\ \exists\ \delta = \delta(\epsilon) > 0$ such that $\forall \boldsymbol{\chi}_0 \in B_\delta(\Omega) \Rightarrow \boldsymbol{\chi}(t; \boldsymbol{\chi}_0) \in B_\epsilon(\Omega)$, $\forall t \geq 0$. This means that using also Lemma 2, then Ł$^+$ is a positively invariant set and Ł$^+ \in B_\epsilon(\Omega) \cap \mathcal{M}^*$. Now let us assume by contradiction that Ł$^+$ is not Ω. Since Ω is asymptotically stable conditionally to $\mathcal{M}^*$, then $\lim_{t\to\infty} d(\boldsymbol{\chi}(t; \boldsymbol{\chi}_0), \Omega) = 0$ if $\boldsymbol{\chi}_0 \in B_\epsilon(\Omega) \cap \mathcal{M}^*$. Choosing $\boldsymbol{\chi}_0 = \boldsymbol{y} \in$ Ł$^+ \neq \Omega$ we reach a contradiction. ■

V. CONTROLLER DESIGN

In this section we will derive the control input for satisfying the $n-1$ constraints and generating at the same time a limit cycle for the "remaining dynamics". To this end the results from Section II for the 1-DOF case will be used to enforce the behaviour of the system on the submanifold of dimension 1 defined by the constraints and the corresponding velocity space. The result will be an asymptotically stable limit cycle for the whole system.

A. Positive semidefinite function for the constraint space

Let $\boldsymbol{K}_x, \boldsymbol{D}_x \in \mathbb{R}^{(n-1)\times(n-1)}$ be two positive definite matrices. Consider the C^1 positive semidefinite function

$$V_x(\boldsymbol{q}, \dot{\boldsymbol{x}}, v) = \frac{1}{2}\dot{\boldsymbol{x}}^T \boldsymbol{\Lambda}_x(\boldsymbol{q})\dot{\boldsymbol{x}} + \frac{1}{2}\boldsymbol{x}^T(\boldsymbol{q})\boldsymbol{K}_x\boldsymbol{x}(\boldsymbol{q}) \ . \quad (19)$$

By choosing

$$\boldsymbol{\tau}_x = \boldsymbol{\Gamma}_{xn}(\boldsymbol{q}, \dot{\boldsymbol{x}}, v)v + \boldsymbol{J}^{+_M T}(\boldsymbol{q})\boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{D}_x\dot{\boldsymbol{x}} - \boldsymbol{K}_x\boldsymbol{x} \ , \quad (20)$$

its derivative results in $\dot{V}_x(\boldsymbol{q}, \dot{\boldsymbol{x}}, v) = -\dot{\boldsymbol{x}}^T\boldsymbol{D}_x\dot{\boldsymbol{x}} \leq 0$ and (13) becomes

$$\boldsymbol{\Lambda}_x(\boldsymbol{q})\ddot{\boldsymbol{x}} + \boldsymbol{\Gamma}_x(\boldsymbol{q}, \dot{\boldsymbol{x}}, v)\dot{\boldsymbol{x}} + \boldsymbol{D}_x\dot{\boldsymbol{x}} + \boldsymbol{K}_x\boldsymbol{x} = \boldsymbol{0} \ . \quad (21)$$

The role of this control law is to force the system to evolve onto the desired submanifold and the corresponding velocity space.

B. Energy control for the oscillation on the submanifold

Let us consider $d_n > 0$ and a C^1 function $U(\boldsymbol{q})$ positive definite on the submanifold, such that $U(\boldsymbol{q}_d) = 0$, $\boldsymbol{x}(\boldsymbol{q}_d) = \boldsymbol{0}$ and where $(\boldsymbol{q}_d, \boldsymbol{0})$ defines the equilibrium around which we will create a limit cycle[5].

Assume that the system is forced to evolve on the constraint submanifold and the corresponding velocity space, where the system reduces to

$$\begin{aligned} &\dot{\boldsymbol{q}} = \boldsymbol{Z}^T(\boldsymbol{q})v \\ &\dot{v} = \Lambda_n^{-1}(\boldsymbol{q})\left(-\Gamma_n(\boldsymbol{q}, v)v - \boldsymbol{Z}(\boldsymbol{q})\boldsymbol{g}(\boldsymbol{q}) + \tau_n\right) \quad (22) \\ \text{s.t.}\ \ &\boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{0} \end{aligned}$$

so that the function

$$H(\boldsymbol{q}, v) = \frac{1}{2}\Lambda_n(\boldsymbol{q})v^2 + U(\boldsymbol{q}) \ , \quad (23)$$

with $\boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{0}$, is a C^1 positive definite function for the system. Choosing

$$\tau_n = \boldsymbol{Z}(\boldsymbol{q})\boldsymbol{g}(\boldsymbol{q}) - d_n\tilde{H}(\boldsymbol{q}, v)v - \boldsymbol{Z}(\boldsymbol{q})\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}^T \ , \quad (24)$$

where $\tilde{H}(\boldsymbol{q}, v) = H(\boldsymbol{q}, v) - H_d$ and $H_d > 0$, the derivative results in

$$\dot{H}(\boldsymbol{q}, v) = -d_n\tilde{H}(\boldsymbol{q}, v)v^2 \quad (25)$$

and (14) becomes

$$\begin{aligned} \Lambda_n(\boldsymbol{q})\dot{v} + \Gamma_n(\boldsymbol{q}, v)v + d_n\tilde{H}(\boldsymbol{q}, v)v + \\ +\boldsymbol{Z}(\boldsymbol{q})\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}^T = 0 \ . \end{aligned} \quad (26)$$

From Section II it follows that using the Lyapunov function $V_n(\boldsymbol{q}, v) = \frac{1}{2}\tilde{H}^2(\boldsymbol{q}, v)$, with $\boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{0}$, it is possible to prove the asymptotic stability for the restricted system of the limit cycle defined by $\Omega = \{(\boldsymbol{q}, v) \mid \boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{0},\ H(\boldsymbol{q}, v) = H_d\}$.

Remark 1: The problem of forcing the system to evolve along the constraint submanifold and the corresponding velocity space can be reformulated as

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \\ = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{\tau}_x + \boldsymbol{N}^T(\boldsymbol{q})\tau_n \quad (27) \\ \text{s.t.}\ \ \boldsymbol{\Lambda}_x(\boldsymbol{q})\ddot{\boldsymbol{x}} + \boldsymbol{\Gamma}_x(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{x}} + \boldsymbol{D}_x\dot{\boldsymbol{x}} + \boldsymbol{K}_x\boldsymbol{x} = \boldsymbol{0} \quad (28) \end{aligned}$$

While substituting the dynamic equation (27) in the constraint (28) gives input $\boldsymbol{\tau}_x$ needed to keep the system on the constraint submanifold and the corresponding velocity space, as in (20), premultiplying (27) by $\boldsymbol{Z}(\boldsymbol{q})$ and using $\dot{\boldsymbol{q}} = \boldsymbol{Z}^T(\boldsymbol{q})v$ gives the dynamic of the system when evolving along them[6,7], as in (22).

In other words requiring to have dynamic matrices in (28) derived from the ones in (27) results in the control law that we have proposed. On the other hand, if in (28) we choose a constant inertia matrix and accordingly $\boldsymbol{\Gamma}_x(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{0}$, then the feedback linearization approach is obtained [8]. The result is not surprising at all since, in this way, we completely change the dynamic that describes how the system is pushed on the constraint submanifold and the corresponding velocity space.

[5]Obviously the choice of $U(\boldsymbol{q})$ will have an influence on the shape of the limit cycle, as we will show in Section VI.

[6]It is possible to show that $\Lambda_n\left(\boldsymbol{N}\boldsymbol{M}^{-1}\boldsymbol{C} - \dot{\boldsymbol{N}}\right)\boldsymbol{Z}^T = \boldsymbol{Z}\left(\boldsymbol{C}\boldsymbol{Z}^T + \boldsymbol{M}\dot{\boldsymbol{Z}}^T\right)$

[7]Using a null space base matrix instead of the usual null space projector allows us to obtain directly a minimum set of equation describing the dynamics in the null space, i.e. one equation instead of n in our specific case.

C. Complete controller

First of all we notice that, since (12) - (14) is a complete description of the system, compensating for the gravity in both spaces is equivalent to compensate for it in (4) or (10).

Without altering the stability analysis that will follow, we can add in (24) the term $-\mathbf{\Gamma}_{xn}^T(\boldsymbol{q}, \dot{\boldsymbol{x}}, v)\,\dot{\boldsymbol{x}}$, since it has no influence when the system has reached the constraint submanifold and the corresponding velocity space. In this way we compensate for the coupling terms in the Coriolis matrix with a power conserving term[8]

$$\boldsymbol{\tau}_d = \boldsymbol{J}_N^T \begin{bmatrix} \mathbf{0} & \mathbf{\Gamma}_{xn} \\ -\mathbf{\Gamma}_{xn}^T & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}} \\ v \end{bmatrix}, \tag{29}$$

where $\boldsymbol{\tau}_d^T \dot{\boldsymbol{q}} = 0$. However, note that $\boldsymbol{\tau}_d$ is not decoupling the dynamics in the two spaces, because the remaining blocks of the Coriolis matrix are still function of the whole state.

Using (11), (20) and the modified (24), the complete controller can be written as

$$\boldsymbol{\tau} = \boldsymbol{g} + \boldsymbol{J}_N^T \left(\begin{bmatrix} -\boldsymbol{D}_x & \mathbf{\Gamma}_{xn} \\ -\mathbf{\Gamma}_{xn}^T & -d_n \tilde{H} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}} \\ v \end{bmatrix} - \begin{bmatrix} \boldsymbol{K}_x \boldsymbol{x} \\ \boldsymbol{Z} \frac{\partial U}{\partial \boldsymbol{q}}^T \end{bmatrix} \right), \tag{30}$$

where, for easiness, we have omitted the dependences.

D. Stability analysis

Here we prove that the closed loop system

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{J}^{+_M}(\boldsymbol{q})\,\dot{\boldsymbol{x}} + \boldsymbol{Z}^T(\boldsymbol{q})\,v \\ \ddot{\boldsymbol{x}} &= -\boldsymbol{\Lambda}_x^{-1}(\boldsymbol{q})\left(\mathbf{\Gamma}_x(\boldsymbol{q}, \dot{\boldsymbol{x}}, v)\,\dot{\boldsymbol{x}} + \boldsymbol{D}_x \dot{\boldsymbol{x}} + \boldsymbol{K}_x \boldsymbol{x}\right) \\ \dot{v} &= -\Lambda_n^{-1}(\boldsymbol{q}) \Big(\Gamma_n(\boldsymbol{q}, \dot{\boldsymbol{x}}, v)\, v + \\ &\quad + d_n \tilde{H}(\boldsymbol{q}, \dot{\boldsymbol{x}}, v)\, v + \boldsymbol{Z}(\boldsymbol{q}) \frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}^T \Big) \end{aligned} \tag{31}$$

has an asymptotically stable limit cycle. For the stability analysis we will use Theorem 2. The function in (19) is a C^1 positive semidefinite function with negative semidefinite derivative for the system (31). The set $\mathcal{A} = \{(\boldsymbol{q}, \dot{\boldsymbol{x}}, v) \mid V_x(\boldsymbol{q}, \dot{\boldsymbol{x}}, v) = 0\}$ is given by $\mathcal{A} = \{(\boldsymbol{q}, \dot{\boldsymbol{x}}, v) \mid \boldsymbol{x}(\boldsymbol{q}) = \mathbf{0},\ \dot{\boldsymbol{x}} = \mathbf{0}\}$. As expected from Lemma 1 the set $\mathcal{A}$ is a positively invariant set and $\mathcal{A} \subset \mathcal{M} = \left\{(\boldsymbol{q}, \dot{\boldsymbol{x}}, v) \mid \dot{V}_x(\boldsymbol{q}, \dot{\boldsymbol{x}}, v) = 0\right\} = \{(\boldsymbol{q}, \dot{\boldsymbol{x}}, v) \mid \dot{\boldsymbol{x}} = \mathbf{0}\}$. Moreover $\mathcal{A}$ is the largest invariant set within $\mathcal{M}$, since it is an invariant set and $\boldsymbol{x}(\boldsymbol{q}) = \mathbf{0}$ is a necessary condition for an invariant set within $\mathcal{M}$, i.e. if $\boldsymbol{x}(\boldsymbol{q}) \neq \mathbf{0}$ we leave $\mathcal{M}$. If we prove that Ω is asymptotically stable conditionally to $\mathcal{A}$, then all the requirements of Theorem 2 are satisfied. This is exactly what we have done in Section V-B, so we conclude that we obtain an asymptotically stable limit cycle for the whole system.

Remark 2: Although the results of the semidefinite Lyapunov analysis are of local nature, the limit cycle is almost globally conditionally asymptotically stable. This was already shown in Section II and can alternatively proved using the results in [16], which additionally allows to conclude that the equilibrium in the origin is unstable.

[8]This is a generalization of the nominal control in [5], [6].

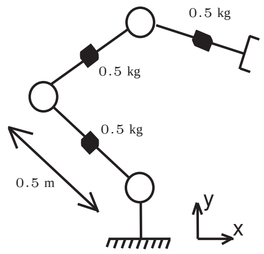

Fig. 2. Simulation model in the start configuration.

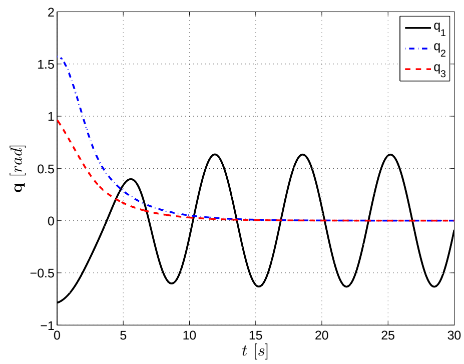

Fig. 3. While the last two joints reach the desired values $q_{d2} = 0$, $q_{d3} = 0$, the first one will oscillate around the equilibrium position $q_{d1} = 0$. The control parameters used in the simulation to push the system on the submanifold and the corresponding velocity space are: $\boldsymbol{K}_x = 0.25\boldsymbol{I}$, $\boldsymbol{D}_x = 0.75\boldsymbol{I}$.

VI. SIMULATION

In this section we apply the results shown so far to a simulated 3-DOF planar robot. Fig. 2 shows a sketch of the model in the starting configuration for the simulations. The computation of all the components for the dynamics and kinematics was performed symbolically in Maple and exported as a C-code function, which was later used in a Simulink model.

We have performed two types of simulations. In the first one we analyze the problem of generating the limit cycle in the joint space, while in the second in the Cartesian space.

A. Joint space

As first example we have chosen to make the arm oscillate along the vertical axis in a completely stretched configuration. To this end we define the constraint submanifold as

$$\begin{bmatrix} q_2 - q_{d2} \\ q_3 - q_{d3} \end{bmatrix} = \mathbf{0}, \tag{32}$$

setting $q_{d2} = q_{d3} = 0$ and as in [10], [11] we choose

$$U(\boldsymbol{q}) = \frac{1}{2} k_n \tilde{\boldsymbol{q}}^T \tilde{\boldsymbol{q}}. \tag{33}$$

In Fig. 3 it is possible to check that the system reaches the constraint submanifold and then starts to oscillate. Accordingly $H \to H_d$, as shown in Fig. 4.

Remark 3: The importance of the condition $\boldsymbol{x}(\boldsymbol{q}_d) = \mathbf{0}$ is already pointed out in [10], [11]. There the equivalent

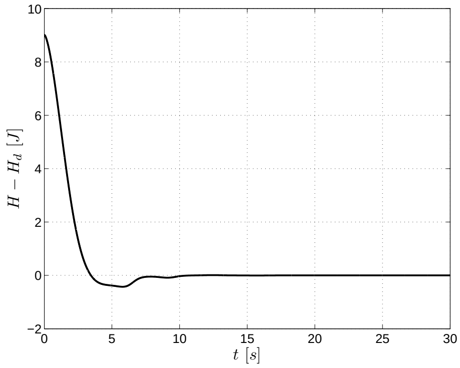

Fig. 4. Once the system reaches the constraint submanifold and the corresponding velocity space, H will converge to the desired value $H_d = 1J$. The control parameters used in the simulation to produce the limit cycle are: $k_n = 5$, $d_n = 8$.

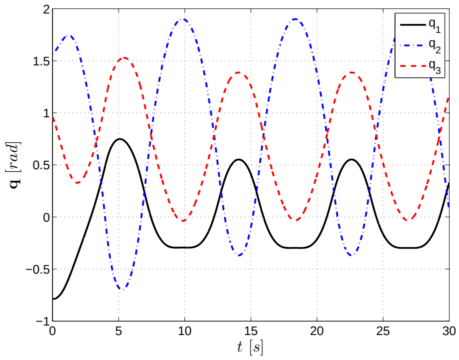

Fig. 5. Periodic behaviour of the joints in order to produce a vertical oscillation along a virtual wall. In this case $U(\boldsymbol{q}) = \frac{1}{2} k_n \tilde{\boldsymbol{q}}^T \tilde{\boldsymbol{q}}$.

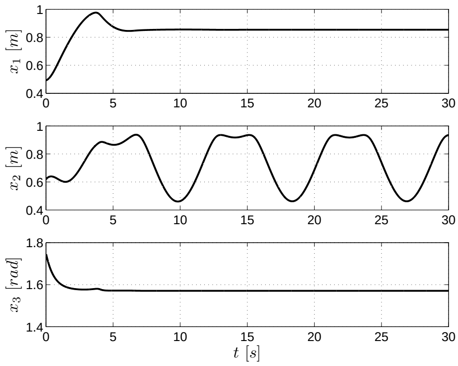

Fig. 6. End effector coordinates. In this case $U(\boldsymbol{q}) = \frac{1}{2} k_n \tilde{\boldsymbol{q}}^T \tilde{\boldsymbol{q}}$. The control parameters used in the simulation are: $\boldsymbol{K}_x = 4\boldsymbol{I}$, $\boldsymbol{D}_x = 3\boldsymbol{I}$, $k_n = 0.1$, $d_n = 8$.

condition $\boldsymbol{x}(\boldsymbol{q}_d) = \boldsymbol{x}_d$ is a clear consequence of requiring $\boldsymbol{q}_d$ to be a conditionally asymptotically stable equilibrium point to a set where $\boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{x}_d$ holds. In our case let us assume that $\boldsymbol{x}(\boldsymbol{q}_d) \neq \boldsymbol{0}$. Then $U(\boldsymbol{q})$ will have a non zero minimum on the submanifold. This offset has the effect of changing the desired value of the energy to H_d' and consequently two cases are possible: either $H_d' \leq 0$ and then the system will reach an equilibrium, or $H_d' > 0$ and then a limit cycle will be produced.

B. Cartesian space

In this example we suppose that we want to produce an oscillation along the vertical direction in the Cartesian space while keeping the horizontal one and the orientation constant. For this case the submanifold is chosen as

$$\begin{bmatrix} x_1(\boldsymbol{q}) - x_{d1} \\ x_3(\boldsymbol{q}) - x_{d3} \end{bmatrix} = \boldsymbol{0} \ , \tag{34}$$

where $x_1(\boldsymbol{q})$ is the horizontal position of the end effector, $x_2(\boldsymbol{q})$ the vertical, $x_3(\boldsymbol{q})$ the orientation with respect to the vertical axis and the desired values are $x_{d1} = 0.85$, $x_{d3} = \frac{\pi}{2}$.

In order to show the effect of a different choice for the potential function in the energy controller, we first use the one in (33) and then we consider

$$U(\boldsymbol{q}) = \frac{1}{2} k_n \tilde{x}_2^2(\boldsymbol{q}) \ , \tag{35}$$

where $\tilde{x}_2(\boldsymbol{q}) = x_2 - x_{d2}$, $x_{d2} = 0.85$. The results are shown in Fig. 5, Fig. 6, Fig. 7 and in Fig. 8, Fig. 9, respectively. Changing the potential function $U(\boldsymbol{q})$ we can change the shape of the limit cycle.

VII. CONCLUSIONS

We have addressed the problem of generating asymptotically stable limit cycles, for multibody mechanical systems. To this end first we have shown that in the special case of 1-DOF systems this type of solutions can be easily enforced using a velocity dependent term related to the Hamiltonian of the system. Secondly we have generalized the results for the stability of equilibrium points with positive semidefinite functions from [14], [15], in order to study the stability of limit cycles. The main result of the paper is that with this approach we can force the system to evolve on a submanifold and the corresponding velocity space where a limit cycle is designed, which can be proven to be an asymptotically stable invariant set for the whole system.

A possible scenario where to apply these concepts is bipedal robotics, where often the goal is to obtain periodic patterns. In this field another usual problem is underactuation, which is also the case when dealing with elastic actuators. In the last years this actuators are spreading more and more, because of the possibility to achieve higher performances and improve the efficiency of actuation. As already mentioned in the introduction, we believe that with an energy based approach we can exploit the benefits of such actuators, taking into account the energy stored in the springs. Moreover another issue in bipedal robotics are impacts. These will cause a periodic energy loss, that we think can fit well in our analysis. For this reasons we plan to extend the results to underactuated hybrid systems.

APPENDIX

Here we recall some fundamental definitions applied to an invariant set Ω of the system (15).

a) (Distance): $d(\boldsymbol{\chi}, \Omega) \triangleq \min_{\boldsymbol{y} \in \Omega} \|\boldsymbol{\chi} - \boldsymbol{y}\|$

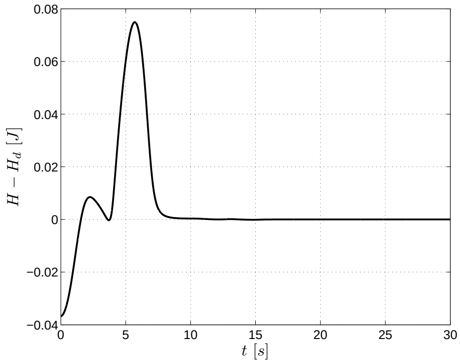

Fig. 7. Energy error for the first example in the Cartesian space. The system reaches the limit cycle where $H(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0.1J$.

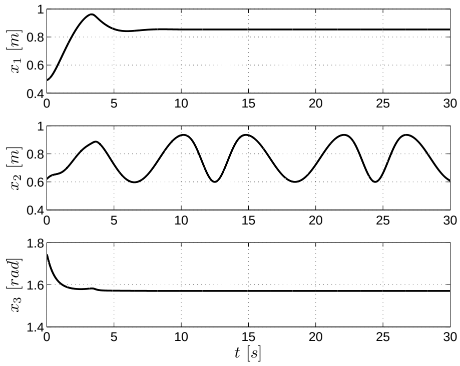

Fig. 8. End effector coordinates. In this case $U(\boldsymbol{q}) = \frac{1}{2} k_n \tilde{x}_2^2(\boldsymbol{q})$. The control parameters used in the simulation are: $\boldsymbol{K}_x = 4\boldsymbol{I}$, $\boldsymbol{D}_x = 3\boldsymbol{I}$, $k_n = 5$, $d_n = 8$.

b) (Open ball): $B_\epsilon(\Omega) \triangleq \{\boldsymbol{\chi} \in \mathcal{X} \mid d(\boldsymbol{\chi}, \Omega) < \epsilon\}$

c) (Closed ball): $\bar{B}_\epsilon(\Omega) \triangleq \{\boldsymbol{\chi} \in \mathcal{X} \mid d(\boldsymbol{\chi}, \Omega) \leq \epsilon\}$

d) (Sphere): $S_\epsilon(\Omega) \triangleq \{\boldsymbol{\chi} \in \mathcal{X} \mid d(\boldsymbol{\chi}, \Omega) = \epsilon\}$

e) (Stability): Ω is stable if $\forall \epsilon > 0\ \exists\ \delta = \delta(\epsilon) > 0$ such that $\forall \boldsymbol{\chi}_0 \in B_\delta(\Omega) \Rightarrow \boldsymbol{\chi}(t; \boldsymbol{\chi}_0) \in B_\epsilon(\Omega),\ \forall t \geq 0$

f) (Asymptotic stability): Ω is asymptotically stable if $\exists\ \delta > 0$ such that $\forall \boldsymbol{\chi}_0 \in B_\delta(\Omega) \Rightarrow \lim_{t\to\infty} d(\boldsymbol{\chi}(t; \boldsymbol{\chi}_0), \Omega) = 0$

g) (Conditional stability): Ω is conditionally stable to $\mathcal{A}$ if $\Omega \subset \mathcal{A}$ and $\forall \epsilon > 0\ \exists\ \delta = \delta(\epsilon) > 0$ such that $\forall \boldsymbol{\chi}_0 \in B_\delta(\Omega) \cap \mathcal{A} \Rightarrow \boldsymbol{\chi}(t; \boldsymbol{\chi}_0) \in B_\epsilon(\Omega),\ \forall t \geq 0$

h) (Conditional asymptotic stability): Ω is conditionally asymptotically stable to $\mathcal{A}$ if $\Omega \subset \mathcal{A}$ and $\exists\ \delta = \delta(\epsilon) > 0$ such that $\forall \boldsymbol{\chi}_0 \in B_\delta(\Omega) \cap \mathcal{A} \Rightarrow \lim_{t\to\infty} d(\boldsymbol{\chi}(t; \boldsymbol{\chi}_0), \Omega) = 0$

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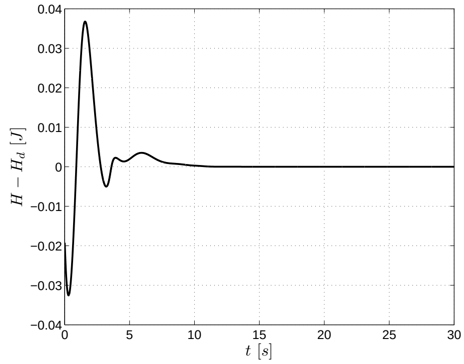

Fig. 9. Energy error for the second example in the Cartesian space. The system reaches the limit cycle where $H(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0.1J$.

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