Asymptotically stable limit cycles generation by using nullspace decomposition and energy regulation

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***Abstract*— In this paper we address the problem of generating asymptotically stable limit cycles for a fully actuated multibody mechanical system through a feedback control law. Using the concept of conditional stability the limit cycle can be designed for a lower dimensional dynamical system describing how the original one evolves on a chosen submanifold and the corresponding velocity space. Moreover, the controller can be split up in two parts that can be independently designed and analyzed in order to reach the constraint submanifold and then produce the oscillation. Even if designed assuming a lower dimensional system, the limit cycle implies a periodic motion for the whole system.**

I. INTRODUCTION

As shown in [1], [2] walking and running can be effectively described as periodic tasks. In these cases it is more important to stay on a prescribed orbit in the state space, rather than following the exact position in time along the desired curve. For these applications tracking a trajectory might not be the best solution, as already addressed in [3], [2]. Moreover in the latter the need of controlling the energy of the system to a desired value was already recognized. In this paper we solve the problem of generating a stable limit cycle for the system using directly the information on its energy level.

Similar approaches to the problem of orbital stabilization have been already shown in [4], [5], [6]. In [**?**], [5] the authors extend the potential field controller adding power-continuous terms, while in [6] the concepts of virtual constraint and feedback linearization are used to obtain a closed loop system that generates its own periodic stable motion. In this paper we formulate the problem based on the nullspace decomposition introduced in [7] and used for nullspace compliance control in [8]. In this way we think that several advantages can be achieved. Compared to [6] we take advantage of the passivity property of the system and do not completely alter the original dynamics of the system through feedback linearization. Moreover, we completely separate the problem of producing the limit cycle from the virtual constraints, instead of modifying the latter for achieving the first. Nevertheless it should be also mentioned that in [6] the more complicated problem of controlling an underactuated system is considered, which here we do not take into account yet. In [4] a passive control action is designed which allows to decouple the motion along a vector field from the remaining motion. The system is then forced to follow an integral curve of this vector field via a passive control law. In case of a closed integral field, the system thus converges to a closed orbit in the configuration space. In [5] additionally a non-passive control action was proposed to achieve regulation of the final velocity along the vector field. In contrast to [4], [5], we aim at achieving a stable limit cycle in the state space, which is achieved by regulating a virtual energy function in a one-dimensional submanifold of the configuration space. This virtual energy function consists of the physical kinetic energy and a virtual potential energy, which represents an additional design element in the controller. In future works, we plan to utilize the freedom in choosing this potential for achieving energy efficient motion in mechanical systems with compliant actuation.

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II. MAIN IDEA AND SYSTEM DESCRIPTION

Consider the 1-DOF system

$$\ddot{q} + d\tilde{H}\left(q,\dot{q}\right)\dot{q} + \omega^2 q = 0 \ , \tag{1}$$

where $d > 0$, $\tilde{H}\left(q,\dot{q}\right) = H\left(q,\dot{q}\right) - H_d$ and $H_d > 0$ is the desired value of the Hamiltonian, defined as $H\left(q,\dot{q}\right) = \frac{1}{2}\left(\dot{q}^2 + \omega^2 q^2\right)$. The term $d\tilde{H}\left(q,\dot{q}\right)\dot{q}$ forces the system to reach always H_d, obtaining a limit cycle defined by the set $\mathcal{L}_d = \{q,\dot{q} \mid H\left(q,\dot{q}\right) = H_d\}$.

While for a 1-DOF system $\mathcal{L}_d$ is a closed orbit in the state space (corresponding to a limit cycle), this is not true for a n-DOF system. The idea is then to force the system to evolve on a 1-dimensional submanifold of the configuration space and produce there a limit cycle, as sketched in Fig. 1.

Consider a fully actuated n-DOF system, with dynamic equation

$$\boldsymbol{M}\left(\boldsymbol{q}\right)\ddot{\boldsymbol{q}} + \boldsymbol{C}\left(\boldsymbol{q},\dot{\boldsymbol{q}}\right)\dot{\boldsymbol{q}} + \boldsymbol{g}\left(\boldsymbol{q}\right) = \boldsymbol{\tau} \ , \tag{2}$$

where $\boldsymbol{q},\dot{\boldsymbol{q}} \in \mathbb{R}^n$ is the state of the system, $\boldsymbol{\tau} \in \mathbb{R}^n$ is the input, $\boldsymbol{M}\left(\boldsymbol{q}\right) \in \mathbb{R}^{n\times n}$ is the mass matrix, $\boldsymbol{C}\left(\boldsymbol{q},\dot{\boldsymbol{q}}\right) \in \mathbb{R}^{n\times n}$ is the Coriolis matrix and $\boldsymbol{g}\left(\boldsymbol{q}\right) \in \mathbb{R}^n$ is the gravity vector. Let us assume that $\boldsymbol{x}\left(\boldsymbol{q}\right) = \boldsymbol{0}$ defines a 1 - dimensional submanifold, where $\boldsymbol{x} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\boldsymbol{J}\left(\boldsymbol{q}\right) \in \mathbb{R}^{(n-1)\times n}$ is the full rank Jacobian matrix of the mapping.

Omitting the dependences, the system can be written in the form

$$\begin{aligned}
\dot{\boldsymbol{q}} &= \boldsymbol{J}^{+_M}\dot{\boldsymbol{x}} + \boldsymbol{Z}^T v \\
\ddot{\boldsymbol{x}} &= \boldsymbol{\Lambda}_x^{-1}\left(-\boldsymbol{\Gamma}_x\dot{\boldsymbol{x}} - \boldsymbol{\Gamma}_{xn}v - \boldsymbol{J}^{+_M T}\boldsymbol{g} + \boldsymbol{\tau}_x\right) \\
\dot{v} &= \Lambda_n^{-1}\left(\boldsymbol{\Gamma}_{xn}^T\dot{\boldsymbol{x}} - \Gamma_n v - \boldsymbol{Z}\boldsymbol{g} + \tau_n\right)
\end{aligned} \tag{3}$$

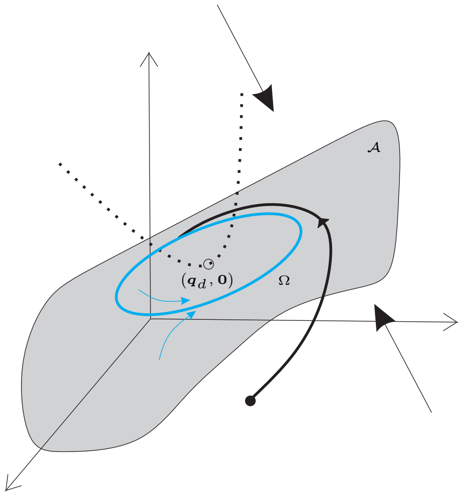

Fig. 1. Conceptual illustration of the main idea of the paper.

where $\boldsymbol{J}^{+M}$ and $\boldsymbol{Z}$ relate the old state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ to the new state $(\boldsymbol{q}, \dot{\boldsymbol{x}}, v)$, while

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_x & \mathbf{0} \\ \mathbf{0} & \Lambda_n \end{bmatrix} \quad \boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}_x & \boldsymbol{\Gamma}_{xn} \\ -\boldsymbol{\Gamma}_{xn}^T & \Gamma_n \end{bmatrix} \tag{4}$$

are the inertia matrix and the Coriolis matrix after the change of variables.

III. Controller and Stability Analysis

In [9] we proposed the control law

$$\boldsymbol{\tau} = \boldsymbol{g} + \boldsymbol{J}_N^T \left(\begin{bmatrix} -\boldsymbol{D}_x & \boldsymbol{\Gamma}_{xn} \\ -\boldsymbol{\Gamma}_{xn}^T & -d_n \tilde{H} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}} \\ v \end{bmatrix} - \begin{bmatrix} \boldsymbol{K}_x \boldsymbol{x} \\ \boldsymbol{Z} \frac{\partial U}{\partial \boldsymbol{q}}^T \end{bmatrix} \right) , \tag{5}$$

where $\boldsymbol{D}_x$, $\boldsymbol{K}_x$ are constant positive definite matrices, $d_n > 0$ and $U(\boldsymbol{q})$ is a virtual potential energy. With the previous controller we obtain the closed loop system

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{J}^{+M} \dot{\boldsymbol{x}} + \boldsymbol{Z}^T v \\ \ddot{\boldsymbol{x}} &= \boldsymbol{\Lambda}_x^{-1} \left(\boldsymbol{\Gamma}_x \dot{\boldsymbol{x}} + \boldsymbol{D}_x \dot{\boldsymbol{x}} + \boldsymbol{K}_x \boldsymbol{x} \right) \\ \dot{v} &= \Lambda_n^{-1} \left(\Gamma_n v + d_n \tilde{H} v + \boldsymbol{Z} \frac{\partial U}{\partial \boldsymbol{q}}^T \right) \end{aligned} \tag{6}$$

which can be proved to have an asymptotically stable limit cycle. The proof of this result is based on

Theorem 1 (Asymptotic stability): Let Ω be an invariant set for $\dot{\boldsymbol{\chi}} = \boldsymbol{f}(\boldsymbol{\chi})$, where $\boldsymbol{\chi} \in \mathcal{X} \subset \mathbb{R}^m$ and $\boldsymbol{f} : \mathcal{X} \to \mathbb{R}^m$ is a Lipschitz continuous function, and let $V(\boldsymbol{\chi})$ be a C^1 function defined in $B_\nu(\Omega) \subset \mathcal{X}$ such that $V(\boldsymbol{\chi}) \geq 0 \ \forall \boldsymbol{\chi} \in B_\nu(\Omega)$, $V(\Omega) = 0$ and $\dot{V}(\boldsymbol{\chi}) \leq 0 \ \forall \boldsymbol{\chi} \in B_\nu(\Omega)$. If Ω is asymptotically stable conditionally to the largest positively invariant set $\mathcal{A}$ within $\mathcal{M} = \left\{ \boldsymbol{\chi} \in B_\nu(\Omega) \mid \dot{V}(\boldsymbol{\chi}) = 0 \right\}$, then Ω is asymptotically stable.

The function

$$V_x = \frac{1}{2} \dot{\boldsymbol{x}}^T \boldsymbol{\Lambda}_x \dot{\boldsymbol{x}} + \frac{1}{2} \boldsymbol{x}^T \boldsymbol{K}_x \boldsymbol{x} \ , \tag{7}$$

is a C^1 positive semidefinite function with negative semidefinite derivative for the system (6). The set $\mathcal{A} = \{(\boldsymbol{q}, \dot{\boldsymbol{x}}, v) \mid \boldsymbol{x}(\boldsymbol{q}) = \mathbf{0}, \ \dot{\boldsymbol{x}} = \mathbf{0}\}$ is the largest invariant set within $\mathcal{M}$, since it is an invariant set and $\boldsymbol{x}(\boldsymbol{q}) = \mathbf{0}$ is a necessary condition for an invariant set within $\mathcal{M}$, i.e. if $\boldsymbol{x}(\boldsymbol{q}) \neq \mathbf{0}$ we leave $\mathcal{M}$.

To prove that $\Omega = \{(\boldsymbol{q}, v) \mid \boldsymbol{x}(\boldsymbol{q}) = \mathbf{0}, \ H(\boldsymbol{q}, v) = H_d\}$ is asymptotically stable conditionally to $\mathcal{A}$, we can consider the Lyapunov function

$$V_n(\boldsymbol{q}, v) = \frac{1}{2} \left(\frac{1}{2} \Lambda_n(\boldsymbol{q}) v^2 + U(\boldsymbol{q}) - H_d \right)^2 , \tag{8}$$

with $\boldsymbol{x}(\boldsymbol{q}) = \mathbf{0}$. We then conclude that an asymptotically stable limit cycle for the whole system is obtained.

IV. Conclusions

We have addressed the problem of generating asymptotically stable limit cycles, for multibody mechanical systems. To this end we have generalized the results for the stability of equilibrium points with positive semidefinite functions from [10], in order to study the stability of limit cycles. The main result of the paper is that with this approach we can force the system to evolve on a submanifold and the corresponding velocity space where a limit cycle is designed, which can be proven to be an asymptotically stable invariant set for the whole system.

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