ABOUT COMBINING TISSERAND GRAPH GRAVITY-ASSIST SEQUENCING WITH LOW-THRUST TRAJECTORY OPTIMIZATION

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ABSTRACT

Gravity-assist maneuvers have the potential to be mission enablers, due to "free energy" they provide. The efficiency of low-thrust propulsion is further one means of improving mission payload mass. Combining both for a given mission possibly improves overall mission performance, which makes it desirable to investigate low-thrust gravity-assist missions. For means of investigating a broad range of mission options, the System Analysis Space Segment department of DLR is working on methods of combining the optimization of low-thrust trajectories and gravity-assist sequences with the help of the Tisserand Criterion and shape-based trajectory models. The hurdles faced by violations of Tisserand Criterion premises are shortly discussed along with their repercussions on planning a gravity-assist sequence for a low-thrust mission. A methodology, based on benchmarking the results with non-gravity-assist trajectories is presented in this paper, grounded on solution populations combining the optimization of the trajectory and the selection of the next gravity-assist partner. Furthermore it is shown how the solution space can be reduced with the help of constraints originating in the maximum possible Δv gain and the gravity-assist partner pool.

Index Terms— Gravity-assist, Tisserand Graphs, Low-thrust Optimization

1. INTRODUCTION

Exploration missions in our solar system are becoming more and more ambitious and therefore often rely on gravity-assist maneuvers as source for “free energy” [1] and thus enabler of a given mission. Previous missions like *Voyager*, *Cassini*, *Messenger* and *New Horizons* all relied on gravity assists for accomplishing their missions [2] and the low-thrust mission *Dawn* conducted a gravity-assist at Mars, although it was not mission critical [3].

The potential for fuel mass savings of low-thrust propulsion due to its large specific impulse (typically some 1000 s) and the energy benefits of gravity-assist maneuvers makes combining low-thrust and gravity-assist to optimize mission trajectories one of the research topics in the System Analysis Space Segment (SARA) department of the German Aerospace Center (DLR).

Usually for gravity-assist trajectories, the actual gravity-assist partner sequence is set by a mission analyst and not part of the optimization of a low-thrust mission scenario. Various approximation methods are used for the actual trajectory calculations, whereas the sequence of the gravity-assist partners is usually simply fed into the respective methodology [4 and 5].

Sequencing of gravity-assist partners for impulsive missions is often obtained via so called Tisserand Graphs, a graphical method developed independently by different research groups [6, 7, 8] and based on Tisserand’s Criterion, an energy-based function of orbit parameters.

Tisserand’s Criterion can be described as [8]:

$$\frac{R_{\mathrm{pl}}}{a} + 2\sqrt{\frac{a(1-e^2)}{R_{\mathrm{pl}}}}\cos i = const, \qquad (1)$$

where R_{pl} is the solar distance of the gravity-assist partner planet, a the semi-major axis of a comet’s heliocentric orbit, e its eccentricity and i its inclination (note: often the semi-major axis is scaled with the planetary distance, therefore R_{pl} does not show up in some descriptions of the Tisserand Criterion). This equation remains approx. constant before and after an encounter with a planetary body. Initially developed by Tisserand to identify comets that have been subjected to an orbit change by Jupiter, it also provides a constraint on a possible outcome in terms of heliocentric orbit after a gravity-assist encounter of a spacecraft.

Due to its simple and efficient approach, the Tisserand Criterion resp. Tisserand Graphs have been investigated by DLR as method for sequencing gravity-assists for low-thrust missions, which is presented in this paper.

2. TISSERAND CRITERION AND GRAPHS

An example for Tisserand Graphs is provided in Fig. 1. Tisserand Graphs, as graphical representations of the Tisserand Criterion, show which heliocentric orbits (given by the orbital period - a function of the semi-major axis -

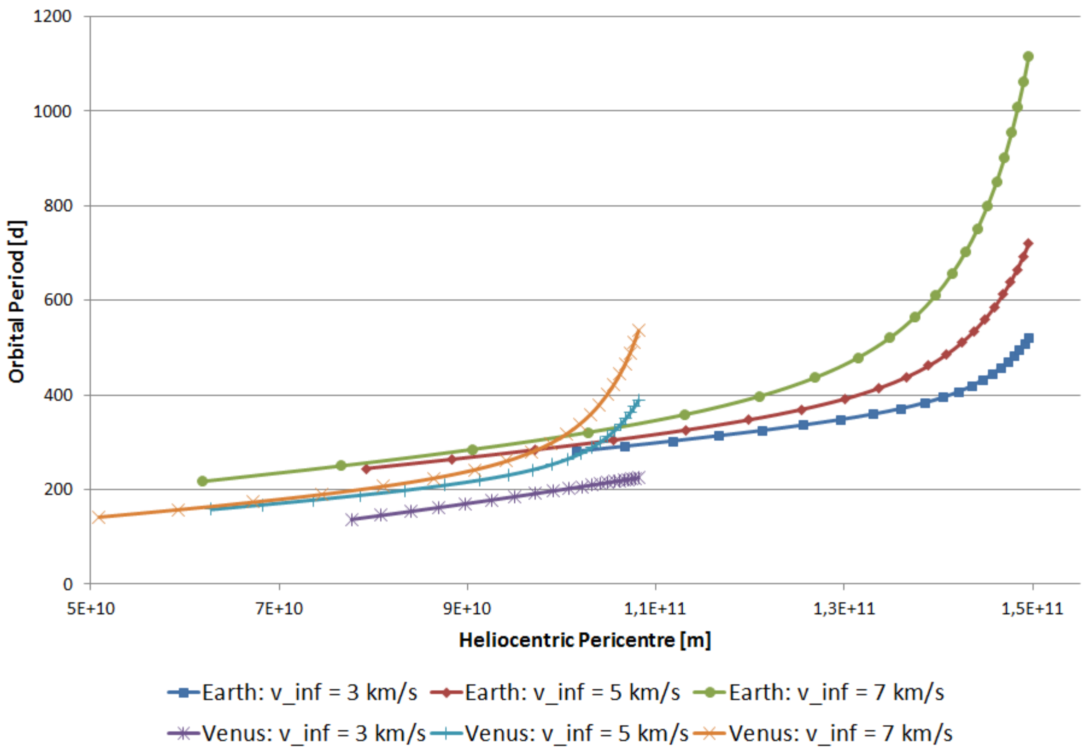

Figure 1: Example of Tisserand Graphs for Earth (right, note the maximum possible heliocentric pericentre being approx. 1AU for the spacecraft) and Venus (left) for various hyperbolic excess velocities (planetcentric). Orbital period (proportional to the semi-major axis, just like the specific orbit energy) as function of the heliocentric pericentre of the spacecraft is given.

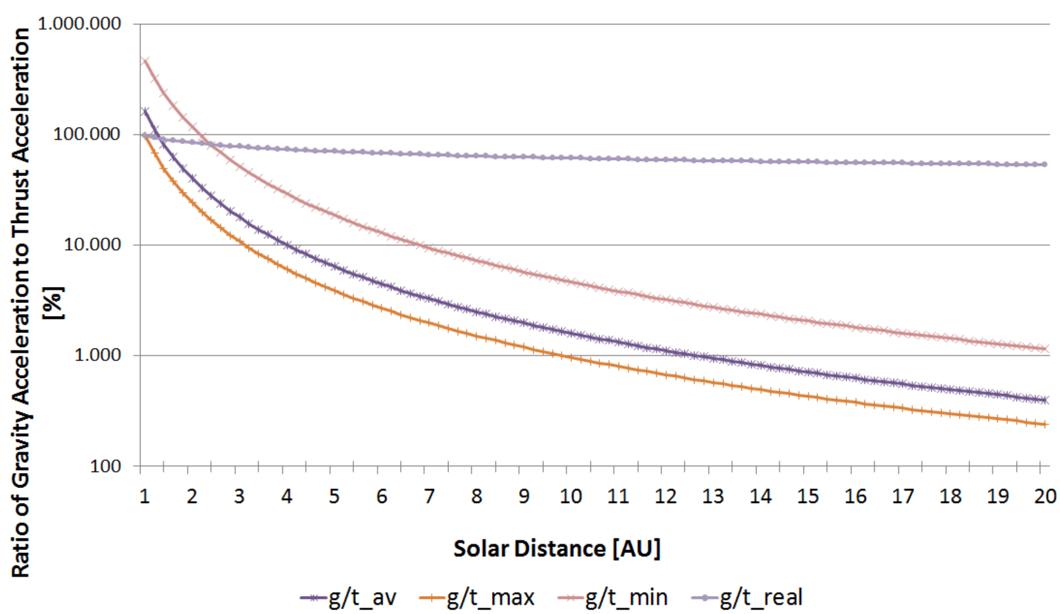

Figure 2: Ratio (logarithmic) of gravity to thrust acceleration as function of solar distance of a sample spacecraft for three cases (av: average thrust of *Dawn*, 55 mN; max: maximum thrust of *Dawn*, 91 mN; min: minimum thrust of *Dawn*, 19 mN; real: realistic thrust drop-off due to power reduction, no cut-out considered).

over the pericentre) are possible at which planetcentric relative energy between spacecraft and planet (given by the hyperbolic excess velocity v_inf in Fig. 1). These graphs are a visual representation of the constraint stated by Eq. (1).

With this Tisserand Graphs can be used for planning gravity-assist sequences by mapping all possible – from an energy point of view – orbit a spacecraft (or any other small body) can obtain after a close encounter with a planetary body. This mapping does not include phasing considerations and therefore presents no sufficient, but a necessary condition. Possible mission paths can be analyzed and evaluated a priori. A spacecraft in a given orbit around the Sun encountering a planet will have a change in its orbital parameters depending on the turning of the velocity vector during approach. This is represented by a shift along the respective graph (depending on the turning angle, which is constraint e.g. by the minimum distance to the planet).

2.1. Premises and violations in application

For the derivation of the Tisserand Criterion, the major premise is the *restricted, circular three body system*, which implies that only gravity is acting on the bodies of that system. Introducing thrust into the situation violates that premise [9].

Application of the Tisserand Criterion however always includes violations of the premises of the restricted, circular three body system, primarily because the real solar system has eccentric planetary orbits and more than three bodies involved. To assess the significance of the diversion from a gravity-only situation, the effect of these previously mentioned violations on the outcome of the Tisserand Criterion were estimated in comparison to the error caused by the introduction of thrust.

The results are only briefly summarized, a more thorough evaluation and explanation can be found in a previous paper[10].

2.1.1. Non-constant spacecraft mass

The n-body problem assumes that the celestial body masses are all constant over time, which is not true for a thrusting spacecraft using fuel (it would be true for a sailing spacecraft). This eventually yields the equations of the forces to:

$$\frac{d}{dt}(m \cdot \vec{v}) = \dot{\vec{v}} + \frac{\dot{m}}{m} \cdot \vec{v} = \sum_i \frac{\vec{F}_{g,i}}{m} + \vec{t}, \qquad (3)$$

where m denotes the spacecraft mass, v its velocity, t the time and F a force, t is the thrust acceleration and index g denoting gravity forces. For a constant mass, the second part of the centre term becomes zero and only gravity forces occur on the right side. The consequence of the mass flow is also the thrust of the space craft. It is noted that the total mass of the spacecraft for typical cases of low-thrust is usually very large compared to a fuel mass flow of some mg/s, therefore $\dot{\vec{v}}$ is still the dominating part of the equation.

To determine the thrust's impact in this equation, the ratio of (the Sun's) gravity and the thrust force has been estimated for a realistic mission scenario.

Based on *Dawn* [11] a minimum thrust case of 19 mN, maximum thrust of 91 mN and a numerical average of 55 mN was assumed for this estimate. Furthermore a realistic case has been considered, where due to reduction of solar electric power with increasing solar distance, the thrust is realistically reduced as well. Since the larger solar distance also reduces the solar array temperature, the drop-off was however modelled with a factor of $1/r^{1.8}$, with r as the solar distance, as proposed in [12]. No power-cut-out was assumed, when the power would drop below a minimum necessary power for the engine.

The results are shown in Fig. 2 as ratio of the thrust (t) and gravity (g) acceleration over solar distance r. Even for the largest thrust (t_max), a ratio of 100 occurs for this worst case at a solar distance of 20 AU. For the realistic mission, a factor of 80,000 is the result. It is therefore plausible to state that the non-constant spacecraft mass and the resulting thrust force are no significant diversion from the three-body premises regarding the acting forces. Only libration points would result in different situations and since the naturally occurring forces cancel each other out at these points (although less clearly in non-circular orbits) eventually only the thrust acceleration would remain. While these portions of a trajectory are small compared to its overall length, a spacecraft might still pass through them. But as the Tisserand Criterion is an energy quantity as opposed to forces, this temporary dominance of the thrust force is not assumed as critical.

2.1.2. Non-circular orbits

One major aspect of the derivation of the Tisserand Criterion are the eccentric orbits, which violate the circular orbit premise. As can be seen in Eq. (1) by the presence of the solar distance of the planet (R_{pl}), a change in that position due to eccentric orbits would change the Tisserand Criterion's value, either.

To estimate these effects, simulations have been conducted in an ideal circular system and a correctly eccentric orbital system with identical "sample" spacecraft. The gravitational effect on their orbital parameters caused by a close encounter with the respective planet has been determined and the values of the respective Tisserand Parameters according to Eq. (1) have been compared.

Evaluating the results of these simulations did not reveal a correlation of the difference of the Tisserand Parameter and the respective planet's orbital eccentricity, but even though the number of simulations was relatively small (150 different sample spacecraft), deviations between both values of the Tisserand Parameter reached up to 25% (the difference of both values divided by the Tisserand Criterion value for the elliptical case in percent). Ratios above 10% have been reached several times. Due to the random sampling method applied, even larger values cannot be ruled out.

2.1.3. Large distance between spacecraft and planet

One further assumption during the derivation of the Tisserand Criterion is that the distance between the spacecraft (or comet) and the planet is large. This is assumed to rid the equation leading to Eq. (1) of a term that includes the distance of the spacecraft to the planet. Depending on the exact trajectory this assumption can be violated (resp. during a close encounter leading to a gravity-assist the Tisserand Criterion's value will change due to that). To determine when the term containing the distance between planet and spacecraft is of the same order of magnitude (i.e. significance) as the term containing the solar distance, sample calculations have been conducted.

For this, the distance between spacecraft and planet was assumed to be the radius of the planet's Sphere of Influence. In the calculations the eccentricity of the spacecraft and its semi-major axis have been varied to determine what effect the term containing the distance between spacecraft and planet would have on the result of the Tisserand Criterion. The largest difference between the two cases (1.5%) occurred for Jupiter and an eccentricity of 0.99 at a semi-major axes of 1.5 times of Jupiter's solar distance. For all other planets, due to their significantly smaller masses, the errors were below 1%. Therefore the violation of this premise for the derivation of the Tisserand Criterion does not result in a very noticeable error.

2.2. Effect of low-thrust and correction term

As described before the effect of the thrust acceleration on the forces equation is not significant. However as the Tisserand Criterion is an energy property, it has to be investigated what the energy effect of the thrusting means for the scenario of a low-thrust mission.

Assuming for a given moment the spacecraft is on a trajectory that can be described by a Keplerian orbit (with altering properties for each time step, caused by the changes due to thrust), the orbital energy of this time specific orbit is:

$$\xi(t) = -\frac{\mu}{2\,a(t)}, \qquad (4)$$

which is the orbital energy of an elliptical orbit [1], but with an added dependency on time.

For instance, a mission from Earth's solar distance to Jupiter's, i.e. from 1 AU to ca. 5 AU, would need a specific energy change of:

$$|\xi_2 - \xi_1| = | -\frac{\mu}{2 \cdot 5\ \text{AU}} + \frac{\mu}{2 \cdot 1\ \text{AU}} | \qquad (5)$$

$$|\xi_2 - \xi_1| = \left|\frac{1}{5}\xi_1 - \xi_1\right| = | -\frac{4}{5}\xi_1 |$$

This change has to be achieved by the thrusting of the spacecraft and of course depends on the exact mission. In this example 80% of the specific energy needs to be created by the thrusting of the spacecraft to complete the mission. This exceeds the errors previously explored and is not negligible.

While missions with less demanding energy changes are possible and thinkable, energy changes of only 10% to 20% (i.e. within the error magnitude of the non-circular planetary orbits, as described above) are likely not large enough to warrant low-thrust propulsion with gravity-assists in the first place. The reason to use low-thrust propulsion is to enable highly challenging missions (and combine them with gravity assists). Therefore these demanding missions should be selected as benchmark.

As the low-thrust's contribution to orbital energy is significant it cannot be argued that the Tisserand Criterion can be used without modification for gravity-assist sequencing of low-thrust missions.

To derive a correction term for missions that deviate from the "gravity only" premise, i.e. involve thrusting, the thrust acceleration has to be introduced in the equations of motion. Transforming the equations similarly as without the thrust acceleration eventually yields a modified Jacobi Integral:

$$C_j^* = C_j + 2\int \vec{V} \cdot \vec{T}\, dt. \qquad (6)$$

Finishing the reformulation finally leads to Eq. (7):

$$\frac{1}{2\,a_2} + \sqrt{a_2\,(1 - {e_2}^2)}\cos i_2 \qquad (7)$$

$$= \frac{1}{2\,a_1} + \sqrt{a_1\,(1 - {e_1}^2)}\cos i_1 + 2\int_{t_1}^{t_2} \vec{V} \cdot \vec{T}\, dt$$

This equation accounts for the orbital energy introduced by the thrust (note: no assumption has been made for the

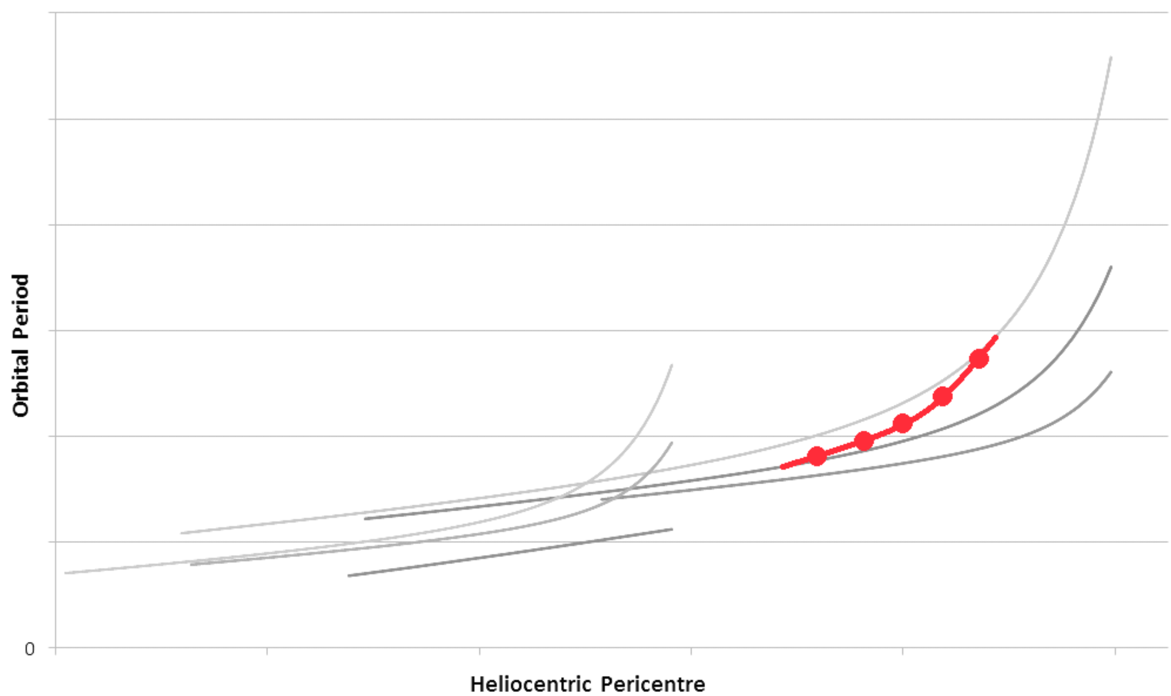

Figure 3: Example graphs (grey) for orbital period as function of heliocentric pericenter and a possible transfer between the respective energy conditions (as represented by the graphs) via a thrusting scheme (red, dotted line).

magnitude of the thrust). In this case condition (2) is equal to condition (1) plus the energy of the thrusting between the two instances. Simplifications based on assumptions can be made, but in any case the thrust energy part of this term is no state quantity. Therefore for applying the correction term the exact trajectory, including the velocity history, has to be known. More details on this subject can be found in [10].

3. APPLICATION ON LOW-THRUST MISSIONS

It has been shown that a correction term exists that includes the energy effects of thrusting and therefore eliminates the problem of violating the Tisserand's Criterion's gravity-only premise.

This correction term also changes the shape of the graphs as depicted in Fig. 1, depending hugely on the actual integral part of the velocity and thrust acceleration. In difference to the original graphs, which are the result of state variables, the modified graphs could not be used universally, but would be trajectory specific for a given mission or flight path.

So while theoretically the modified Tisserand Criterion can be used for low-thrust missions, practically the lack of a priori information of a mission scenario reduces the usefulness of it.

However the thrust and the resulting "open" result of the trajectory allows a mission to be designed between orbital states that would not be connected via Tisserand Graphs originally. That means the thrusting parts of a trajectory allow a transition of one state to another apart from following the graphs. This is illustrated in Fig. 3 in a sketch (no actual numerical values have been used), where the red, dotted line connects two states of different Tisserand Graphs (grey), which would otherwise not be connected.

3.1. Optimization variables and constraints

Depending on the nature of the trajectory model various variables exist that are used for optimization. Unknowns are for example Launch Date, Flight Time, the hyperbolic excess velocity vector, and the pericentre distance to the gravity-assist partner (or e.g. the turn angle of the velocity vector). Similarly the thrust history describing the thrust direction (in two angles for a 3D case) and the thrust magnitude are relevant.

The exact variables depend on the trajectory model used for the optimization, e.g. a propagation method applying an integration of the equations of motions will use the thrust history as variables and receive a result for the Flight Time. Other methods, e.g. shape-based methods (which work for low-thrust missions analogously to a Lambert's problem solution for impulsive missions), apply Flight Time, the number of revolutions around the system centre and the

Launch Date as variables and receive values for the thrust and position history [13].

For a gravity-assist mission that is to be optimized regarding the sequence of gravity-assist partners also the gravity-assist partner becomes a variable.

These variables are subject to constraints, depending on the mission but also physical restrictions. E.g. a flyby's pericentre distance cannot be arbitrarily set but must observe the planet's radius (or other limits like radiation belts of Jupiter). The hyperbolic excess velocity might be constrained by a maximum allowable velocity depending on scientific observations.

Furthermore the number of variables as listed before is depending on each other. For example in case of shape-based methods, the number of revolutions and flight time cannot arbitrarily set. A too large number of revolutions requires a large velocity change as a long distance has to be covered in a given time. A pairing exists which is optimal and the change in the discrete value of numbers for revolution can have a profound effect on the suitability of the flight time. The same is true for e.g. the variable of a gravity-assist partner. This variable creates a strong sensibility for a trajectory regarding its usefulness. A limited amount of compatible variable values are created by this. For example a flight time suitable for a gravity-assist at Mars would likely be very unsuitable for a gravity-assist at Jupiter. So a change of the gravity-assist partner can have a profound effect on the quality of a solution candidate, i.e. the solution's fitness will be highly sensitive to this variable.

Furthermore the number of gravity-assist partners changes the number of variables involved in the optimization as certain variables are only needed for a gravity-assist, e.g. the hyperbolic excess velocity or the turning angle. If the overall trajectory is divided into legs between the individual gravity-assist partners, this effect is even enhanced because in that case for each leg new flight times and so on need to be set.

3.2. Repercussions on optimization algorithms

Theoretically two types of searches for a solution space are possible. One which is directed, meaning a forward progressing by systematically varying the variables involved in the problem and thus creating branches of a search tree. For example: Setting up the launch conditions of the mission, from there vary the possible partner according to a pool of partners. For each partner an interval of hyperbolic excess velocities (in Tisserand Graphs this would equal the individual graphs) is investigated. For each of these again a set of turning angles is investigated, creating more and more branches. In between each partner has the potential to be the target body and thus end the branch with the arrival of the spacecraft. This creates two challenges:

1) Large amount of solutions, i.e. large computation demand, and
2) unknown amount of variables for one solution until end of the branch is reached.

Especially the latter means that solutions cannot be put in populations of similar solutions beforehand.

Typical search algorithms for low-thrust are evolutionary algorithms (of various kinds) or e.g. simple algorithms like multi-start [14].

Evolutionary algorithms are depending on the possibility to recombine (and mutate) properties of two or more solution candidates [15], i.e. on similarly structured solutions. Solutions with different amounts of optimization variables are unsuitable, because e.g. a value of one solution's variable cannot be used to modify another solution if it does not contain that variable, due to a smaller number of gravity-assists. Only solution candidates with the same structure can be successfully recombined. Therefore it is advisable to create populations of similarly structured solutions (i.e. with the same amount of gravity-assists).

This leads to a second type of search, where populations with a set value of the number of gravity-assists, are created and for each encounter randomly a gravity-assist partner is selected (in difference to a forwardly progressing search).

Another aspect hinders the optimization via evolutionary algorithms (resp. algorithms which reuse previous solutions' information in general): The in parts large sensitivity of the solution to certain variables of the optimization as variables are interlinked as described in the previous section.

This means that for a recombination and mutation of the optimization variables it is likely that good combinations of variables are actually separated and lead to less suitable results. While even for non-gravity-assist trajectories interdependencies exist between variables, the introduction of the gravity-assist partners as variable creates one discrete variable, which adds a strong sensitivity to the overall solution.

Furthermore the interdependency restricts also the recombination of variables. The encounters divide the trajectory into segments, but these segments need to be matched. Arrival dates need to match departure dates resp. flight times for example. Thus the search cannot be randomly organized.

This has repercussions on the suitability of a search algorithm. As evolutionary algorithms depend on using information of other solution candidates for new ones and thus sorting out "good" properties from "bad" ones, the fact that a reasonable exchange, recombination or modification of variables is restricted reduces the convergence properties of evolutionary algorithms.

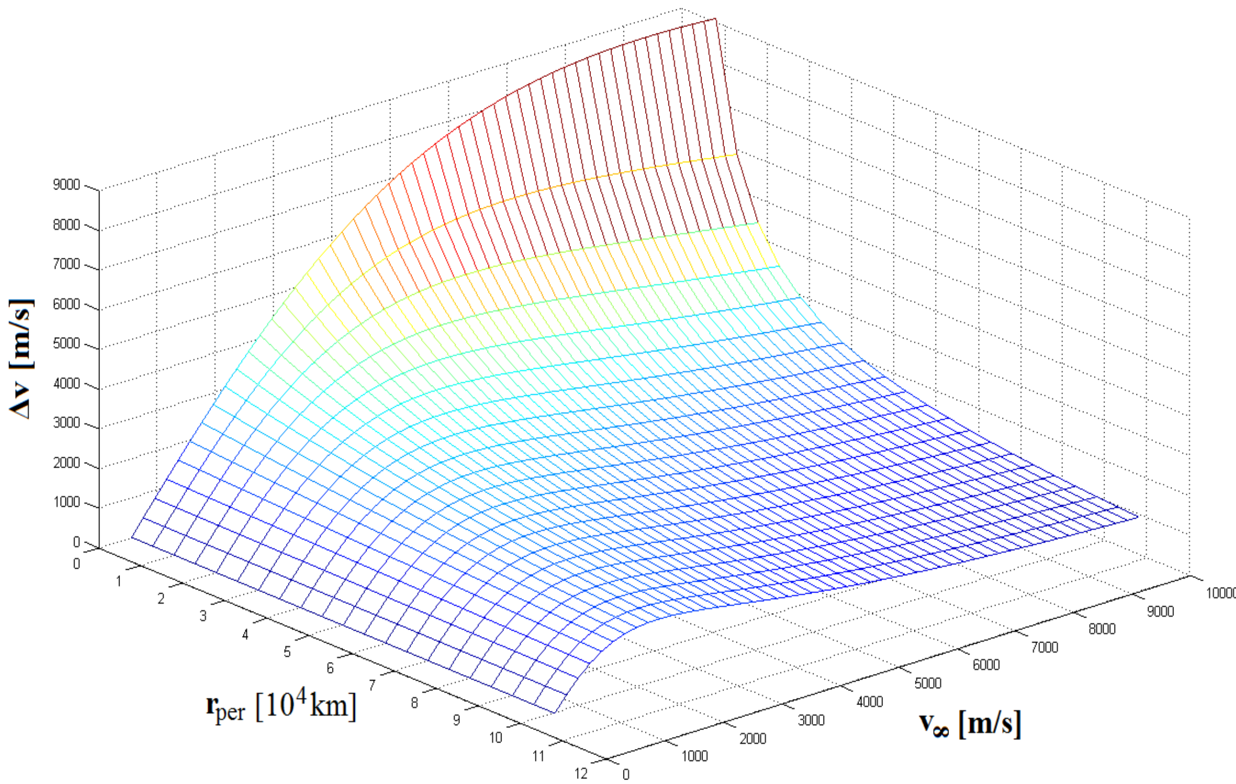

Figure 4: The Δ*v* gain of a gravity-assist maneuver as function of hyperbolic pericenter distance r_{per} and hyperbolic excess velocity v_∞ for the example of an Earth fly-by (μ_{pl} = 3.98 · 1014 m³/s²).

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3.3. Constraining solutions via Δv gain

To increase the convergence speed of a solution space search, it is possible to reduce the solution space size. This can be done by e.g. excluding solutions which are obviously outside a reasonable mission scenario, e.g. by restricting the flight time of a mission to Jupiter to an interval with a lower bound, which is assumed as possible. Or by accepting solutions only with a certain “quality” of gravity-assists.

Starting from the geometry of the velocity vectors involved in a gravity-assist planetcentric description, one can set a formula for the Δ*v* gain as [1]:

$$\Delta v = 2\, v_\infty \sin\left(\frac{\delta}{2}\right), \tag{8}$$

where v_∞ denotes the hyperbolic excess velocity and δ the turning angle between incoming and outgoing velocity vector. This can be rewritten as [1]:

$$\frac{\Delta v}{v_s} = \frac{2\, v_\infty}{v_s\left(1+\left(\frac{v_\infty}{v_s}\right)^2\left(\frac{r_{per}}{r_s}\right)\right)}, \tag{9}$$

where v_s is the circular orbit speed at the gravity-assist partner’s surface, r_s the partner’s radius and r_{per} the pericenter distance of the fly-by hyperbola. The definition of the circular orbital speed is:

$$v_s = \sqrt{\frac{\mu_{pl}}{r_s}}, \tag{10}$$

where μ_{pl} is the gravity-assist partner’s (planet’s) gravity parameter. Combining Eq. (10) and Eq. (9), the latter can be rewritten as:

$$\Delta v = \frac{2\, v_\infty}{1+v_\infty^2 \cdot \frac{r_{per}}{\mu_{pl}}}. \tag{11}$$

Eq. (11) is important, because it directly links the boundary condition between the heliocentric and planetcentric coordinate systems, i.e. v_∞, with the planetcentric trajectory via the planetcentric pericenter distance (of the hyperbolic trajectory) r_{per} and the heliocentric energy gain Δv.

It is also clear that for a given gravity-assist partner (and thus a constant μ_{pl}) the former two are the only influences that can change the Δv. Both are basically random and depend on the trajectory leading to the gravity-assist partner, i.e. when and where the spacecraft and planet meet. The function of Eq. (11) is depicted in Fig. 4 for Earth as a flyby-body and it can already be seen that there are extreme values present. It is visible that for a decrease in the pericenter distance, the Δv gain increases, with the maximum for a distance of 0, which is unrealistic, because this would place the pericentre within the partner of the gravity-assist.

For the hyperbolic excess velocity the maximum is different. Taking a look at the limit values for a varying v_∞ and a given r_{per}, reveals that for v_∞ approaching zero and one approaching infinity, the Δv magnitude becomes zero:

$$\lim_{v_\infty \to 0} \Delta v = \frac{2\, v_\infty}{1 + \frac{{v_\infty}^2 \cdot r_{per}}{\mu_{pl}}} = 0 \tag{12}$$

$$\lim_{v_\infty \to \infty} \Delta v = \frac{2\, v_\infty}{1 + \frac{{v_\infty}^2 \cdot r_{per}}{\mu_{pl}}} = 0 \tag{13}$$

Eq. (11) is a rational function and continuous (as it is a combination of two continuous functions, where the denominator cannot become 0 for $v_\infty \geq 0$) in $[0,\infty)$ and differentiable on the same interval. Therefore it follows that according to Rolle's Theorem [16] there has to be an extreme value present, which is also clearly visible in Fig. 4.

Creating the derivative of Eq. (11) for v_∞ via the product rule, provides:

$$\frac{\partial \Delta v_{GA}}{\partial v_\infty} = \frac{2}{1 + {v_\infty}^2 \cdot k} - \frac{4{v_\infty}^2 \cdot k}{(1 + {v_\infty}^2 \cdot k)^2}, \tag{14}$$

where k is a constant, consisting of the term $\frac{r_{per}}{\mu_{pl}}$. To find the position of the extreme, $v_{\infty,ex}$, the first derivate has to be equal 0:

$$\frac{2}{1 + {v_\infty}^2 \cdot k} - \frac{4{v_\infty}^2 \cdot k}{(1 + {v_\infty}^2 \cdot k)^2} = 0 \tag{15}$$

$$\Leftrightarrow \frac{2}{1 + {v_\infty}^2 \cdot k}\left(1 - \frac{2{v_\infty}^2 \cdot k}{1 + {v_\infty}^2 \cdot k}\right) = 0$$

$$\Leftrightarrow v_{\infty,ex} = \sqrt{\frac{1}{k}} = \sqrt{\frac{\mu_{pl}}{r_{per}}}.$$

To determine whether or not this really is an extreme value and of which kind, the second of Δv derivate has to be investigated. The second derivative over ∂v_∞ can be written as:

$$\frac{\partial^2 \Delta v_{GA}}{\partial {v_\infty}^2} = \frac{4 v_\infty \cdot k}{(1 + {v_\infty}^2 \cdot k)^3}({v_\infty}^2 \cdot k - 3), \tag{16}$$

Inserting $v_{\infty,ex}$, yields:

$$\frac{\partial^2 \Delta v_{GA}}{\partial v_\infty^2}\left(v_{\infty,ex} = \sqrt{1/k}\right) = -\sqrt{k} < 0, \tag{17}$$

This shows that $v_{\infty,ex}$ is a maximum, which matches the graph in Fig. 2.

Inserting the result of Eq. (15) into Eq. (11) reveals the maximum possible Δv for a given r_{per} to be:

$$\Delta v_{max} = \frac{1}{\sqrt{k}} = \sqrt{\frac{\mu_{pl}}{r_{per}}}. \tag{18}$$

This is equals the circular orbit velocity for the pericenter distance of the hyperbola and also equals $v_{\infty,ex}$. It also means that knowledge of just r_{per} leads to knowledge of the possible Δv gain.

The influence on Δv due to a change of the pericenter distance r_{per} is expected. The closer the spacecraft is to the gravity-assist partner during the hyperbolic pericenter passage, the stronger its influence and therefore effect on the turning angle δ resp. the Δv-gain. However the pericenter distance is restricted by physical properties (e.g. the gravity-assist partner's body radius) and also system properties of the spacecraft. Radiation or atmosphere effects need to be regarded when designing the fly-by trajectory. Therefore the possible maximum is influenced by the system and physical constraints.

The influence on Δv due to a change of the velocity at infinity v_∞ is less straightforward.

The case of $\Delta v = v_\infty$ can only occur for an equilateral triangle, i.e. when the turn angle δ is 60°. The same result occurs, when Eq. (8) is transformed and for v_∞ the value $v_{\infty,ex} = \Delta v_{max}$ is inserted. In this case Eq. (8) becomes:

$$2 \arcsin\left(\frac{1}{2}\right) = \delta, \tag{19}$$

which is true for a turn angle δ of 60°. Another expression for the turn-angle δ is [17]:

$$\frac{1}{e} = \sin\left(\frac{\delta}{2}\right). \tag{20}$$

This means the maximum Δv occurs exactly and always for an eccentricity of 2.

Taking a look at Eq. (8), it becomes clear that for small values of δ, the Δv-gain is also small, because of the sinus-part of the term. Subsequently, a larger δ, in the area of 180° would be preferable from this point of view, to gain the most from this part (for δ = 180°, the sinus becomes 1 as only δ/2 occurs in the term). From a theoretical point of view, this is to be expected, although realistically it is not possible as the radius of the gravity-assist partner is larger than 0.

Figure 5 illustrates that for δ to increase, v_∞ has to decrease for keeping the same Δv. The change of direction, due to the influence of the planet's gravity, increases the more time is spent in its vicinity. Therefore a small approach velocity is beneficial for the turn angle δ.

However, it is evident from Eq. (8) that regarding the Δv, a decreasing v_∞ directly affects the linear part of the term. Both effects balance each other at an angle of 60°, which is why there a maximum is found.

The movement of the spacecraft and the planet are not coupled until the spacecraft enters the sphere of influence of the gravity-assist partner – the v_∞'s direction and magnitude are basically random, depending on the location and time spacecraft and gravity-assist partner meet (and if the trajectory is not timed correctly, these two do not meet at all). But the v_∞ is the boundary condition that hands over the properties of the spacecraft's trajectory between the heliocentric and planetcentric coordinate systems and vice versa in case of the outgoing velocity.

While Eq. (8) shows that a large v_∞ benefits the gain from the gravity-assist – in the linear part of the equation – it is also clear that a small turn angle δ reduces this effect. The explanation for this relation is analogue to the effect on the turn angle δ: a large v_∞ means only a relatively short amount of time (relative to a smaller value of v_∞) in the planet's vicnity, i.e. less time close to the gravity-assist body and therefore subject to its gravitational influence. Consequently the amount of turning (i.e. the value for δ) is limited and only the turning of the v_∞-vector results in the occurrence of a Δv gain [1].

Generally speaking a large magnitude of v_∞ is not desirable because it:

- usually needs effort – in terms of energy – to be achieved, and
- the Δv gain decreases after passing $v_{\infty,\mathrm{ex}}$.

This relation can and should be exploited when searching for optimal gravity-assist sequences for a given mission. The selection of to be investigated trajectories could be reduced. If the described effect is used for gravity-assist sequence optimization, the benefit is its independence of the spacecraft's system properties. The effect depends only on the trajectory itself and occurs for low-thrust and impulsive missions alike.

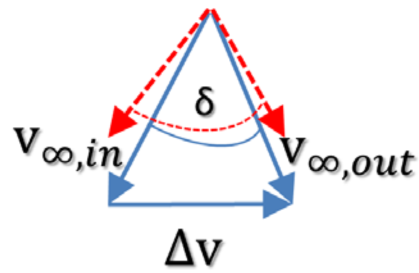

Figure 5: Change of the turn angle δ for decreasing values of v_∞.

If the transfer trajectory and the point of encounter are known, this provides a value for v_∞ and the planetcentric pericenter distance of the spacecraft. This in turn provides the theoretically possible maximum Δv resp. $v_{\infty,\mathrm{ex}}$, the velocity, where the maximum Δv occurs, based on Eq. (15) and (18). Any trajectory which has a v_∞ that is too far distanced from $v_{\infty,\mathrm{ex}}$ could be discarded, especially if it is too large, because in this case more effort than necessary has been made to achieve this trajectory, even though it does not provide an optimal Δv.

Basically the diagram in Fig. 4 can be used to "advise" an optimization process on which trajectories are worthwhile to be investigated and which are not. Trajectories resulting in a v_∞ outside a certain interval around $v_{\infty,\mathrm{ex}}$ could be excluded from further evaluation, with little effort, simply by determining the v_∞ related to this trajectory.

Assuming that a large Δv-gain by gravity-assists reduces the propellant mass necessary for a mission and by trying to achieve this Δv-gain with as little gravity-assist partners as possible (by targeting the area around the maximum Δv-gain for each of them) a shorter flight time should be achieved. Therefore using the Δv-v_∞ relation could improve the results, especially if propellant mass is to be optimized (with a limit in flight time), which relates directly to the achievable Δv-gain.

3.4. Constraining solutions via partner pool

During the optimization of the sequence, possible next partners have to be selected. Theoretically every planet (or e.g. moon) can be selected, but practically it is unlikely that jumping inwards and outwards within the system is leading to usable solutions. For example going from Earth to Saturn by selecting Mars and Jupiter as partners and then go back and select e.g. Venus would likely mean large flight times and unnecessary Δv increases for the mission. Therefore it is reasonable to restrict the list of possible "next" partners when selecting the sequence, depending on the exact mission, e.g. by limiting a “back propagation”.

3.5. Optimization methodology

In the previous sections the variables involved with the optimization of a gravity-assist trajectory for low-thrust missions and their structure has been explained along with their repercussions regarding algorithms and search strategies. While initially a strategy which would have progressed forward in steps and investigate the suitability of one further gravity-assist (by benchmarking it with a no-gravity-assist trajectory), was favored by the author, this approach has been dropped. As the number of variables varies with the number of gravity-assist encounters, it was decided to create populations with similar numbers of gravity-assist to be able to recombine certain aspects of the solution with each other for finding new solutions.

Furthermore it has been discussed how in difference to impulsive propulsion missions, the Tisserand Criterion is not usable to obtain a priori knowledge about a complete low-thrust mission as the thrust arcs cannot be described with state variables (see Eq. (6), resp. (7)) and thus depend on the actual trajectory flown. This means that “mapping” of the complete mission is not possible with Tisserand Graphs. Also two major ways of restricting the search space have been pointed out.

For the optimization it is assumed that the mission is determined by all encounters (including the launch and arrival bodies) and consists of trajectories in between these encounters. There are variables globally describing the mission, e.g. the planet of an encounter, the overall flight time, the launch date, and there are variables describing locally one encounter, e.g. the hyperbolic excess velocity, the turning angle and the time of flight for this segment. Global variables also affect local ones. E.g. a change in the overall flight time affects the segments’ flight times as their sum cannot exceed the overall flight time.

The following steps are proposed as method for finding a low-thrust gravity assist-sequence, assuming the mission constraints are known (e.g. target body or launch window):

- i) optimize a no-gravity-assist trajectory as benchmark for the mission
- ii) create a population of solution candidates for each value of the number of gravity-assists from 1 to maximum allowed number, which then have the same amount of variables within a population with random sequences (random start all variables)
- iii) optimize the trajectory segments between each encounter
- iv) model the gravity-assist effect under application of the Tisserand Criterion and using e.g. the turning angle as (local, see above) optimization variable (possibly constrained by e.g. the allowable minimum distance to the encountered body, minimum being the planet’s radius; otherwise the minimum distance can be used as variable resulting in a turning angle) and the hyperbolic excess velocity
- v) optimize the solution candidates within a population via a search algorithm, recombination of mission global variables (e.g. Launch Date, gravity-assist partners) is possible, mission local variables need to be defined in dependence to global variables (e.g. the launch date of the second trajectory segment depends on the arrival date of the first one)

--- *repeat until stopping criterion is reached* --

- vi) compare solution with non-gravity-assist benchmark for determining final solution

Step ii) resp. v) can be constrained by the suitability of the gravity-assist partner pool, as described in Section 3.4 and step iv) can be constrained by the Δv-gain as described in Section 3.3.

The exact variables depend on the model used for the trajectory. Currently the author plans to apply shape-based methods as mentioned before due to their fast computation and thus the inherent ability of evaluating a large amount of solutions quickly (as opposed to solving equations of motion for each solution candidate).

For modelling the gravity-assist the Tisserand Criterion can be used. First of all it is reasonable to assume that for the relatively small part of the trajectory where the encounter is placed in, a no-thrust assumption is tolerable. Furthermore even if thrust is assumed for this duration, the short amount of time in comparison to the overall mission would render the integral in Eq. (7) small as well. Thus the gravity-assist is modeled by shifting along the graph line as described before.

4. OUTLOOK AND CONCLUSION

This paper thoroughly discussed the Tisserand Criterion’s application on low-thrust mission design and showed that the thrust deviation from the gravity-only situation leads to a correction term for the Tisserand Criterion. This term is not consisting of state variables and therefore an a priori evaluation of a mission sequence is not possible – as opposed to impulsive missions.

Therefore a search strategy addressing this has been proposed along with further constraints on the search space to speed up the search.

Currently this methodology is implemented into software code (using C++) and will then be tested on its usefulness. Furthermore it is to be investigated whether or not evolutionary algorithms can be used for this search or if only multi-start search (i.e. randomly changing the variables) is applicable. Also the suitability of the before mentioned constraints will be tested.

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