IAC-15-C1.1.2

APPLICABILITY OF TISSERAND CRITERION FOR OPTIMIZATION OF GRAVITY-ASSIST SEQUENCES FOR LOW-THRUST MISSIONS

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Missions for space exploration are becoming more ambitious and gravity-assist maneuvers act as one enabler for them. The "free energy" associated with this technique is often vital for conducting a mission in the first place. Consequently, new methods for optimizing gravity-assist maneuvers and sequences are investigated and further developed – for missions involving impulsive and low-thrust propulsion alike. The System Analysis Space Segment department of the German Aerospace Center (DLR) in Bremen is currently conducting research to combine gravity-assist sequences with low-thrust optimization. One technique, which is prominently used to sequence gravity-assist maneuvers are Tisserand Graphs, based on the Tisserand Criterion, which states that a function of certain orbit parameters of a comet (for mission design purposes spacecraft) remains approximately constant even after a close encounter with a planetary body. However one condition for the validity of the Tisserand Criterion is that the only force acting on the spacecraft is gravity, which obviously would not be the case for a low-thrust mission. By investigating approximations and simplifications necessary for deriving the Tisserand Criterion, e.g. non-constant spacecraft mass, and the deviation they cause from the real situation in the solar system, this paper analyses how well suited the Tisserand Criterion is for use in low-thrust mission design. Furthermore a correction term is presented that allows inclusion of thrust into the criterion and thus reducing the accompanied error.

I. INTRODUCTION

The exploration of our solar system with the help of planetary probes is becoming more and more ambitious and therefore relying on gravity-assist maneuvers as source for "free energy" [1] and thus enabler of a given mission. Spacecraft like *Voyager*, *Cassini*, *Messenger* and *New Horizons* all relied on gravity assists to accomplish their missions [2]. Even the low-thrust mission *Dawn* has utilized this technique, although it would have been feasible without the gravity-assist maneuver at Mars [3].

The potential for fuel mass savings of low-thrust propulsion due to its large specific impulse (typically some 1000 s) and the energy benefits of gravity-assist maneuvers makes combining low-thrust and gravity-assist to optimize mission trajectories one of the research topics in the System Analysis Space Segment department of the German Aerospace Center (DLR).

I.I Low-thrust Optimization

Low-thrust missions require extensive optimization to achieve the largest effect due to the theoretically infinite solution space of the trajectory's control variables [4] whereas impulsive maneuvers usually have a solution space limited to three dimensions (two thrust angles and thrust magnitude) [5].

Due to this nature of the problem, simplifications have to be made and the fact that the function behind the optimization problem is unknown calls for heuristic solution approaches, e.g. evolutionary algorithms [5].

I.II Gravity Assist Sequencing

For optimization of gravity-assist trajectories the sequence is often already determined by a mission analyst and not part of the actual optimization.

Various approximation methods are used for the actual trajectory calculations, whereas the sequence of the gravity-assist partners is usually simply fed into the respective methodology [6 and 7]. For sequences not involving low-thrust trajectories, but impulsive ones, e.g. Strange and Longuski [8] and Labunsky et al. [9] have applied a methodology based on Tisserand's Criterion [10] – an energy-based function of orbit parameters used to determine gravity-assist effects on comets passing by Jupiter – to create graphs enabling the sequencing of gravity-assist partners.

Tisserand's Criterion can be described as [10]:

$$\frac{R_{pl}}{a} + 2\sqrt{\frac{a(1-e^2)}{R_{pl}}}\cos i = const, \tag{1}$$

where R_{pl} is the solar distance of the planet, a the semi-major axis of the comet's heliocentric orbit, e its eccentricity and i its inclination (note: often the semi-major axis is scaled with the planetary distance, therefore R_{pl} does not show up in some descriptions of the Tisserand Criterion). This equation is also called Tisserand Parameter and remains approx. constant before and after an encounter with a planetary body. It can be used to determine whether or not two observed objects, based on their orbital parameters, are in fact the same object before and after such an encounter. If the Tisserand Parameter is (nearly) identical for both objects, they are likely the same. At the same time it constrains the possible trajectories a body can have after a gravity-assist, as it also has to fulfil Equ. (1).

Figure 1 shows an example of a set of Tisserand graphs, which could possibly be used for mission design. It is a graphical representation of Tisserand's Criterion, showing which heliocentric orbits (given by the orbital period, which is a function of the semi-major axis, and the pericentre) are possible at which planetcentric relative energy between spacecraft and planet (given by the hyperbolic excess velocity v_inf). These graphs are a depiction of the constraint that is formed by Equ. (1).

However the premise for the original derivation of Tisserand's Criterion is the *restricted, circular three body system* and thus that gravity is the sole force acting on the small partner of a gravity-assist (and all other masses). This is violated by the intention of using low-thrust propulsion in combination with gravity-assist [11]. In difference to a mission with conventional thrust, where the trajectory is set once a maneuver has been executed (not taking into account perturbations and consequential correction maneuvers), the low-thrust is applied for long durations in comparison to the overall mission time and therefore has a continuous effect on the spacecraft trajectory.

With that in the mind, this paper explores the effect of including thrust in the Tisserand methodology for gravity-assist sequencing and compares the repercussions of diverting from the gravity-only premise with those of other violations of the circular three body system – which naturally occur due to the realistic nature of the solar system. In conclusion a recommendation for the application of the Tisserand Criterion for low-thrust missions is presented, along with a correction term that addresses the inclusion of thrust.

II. DERIVATION OF TISSERAND CRITERION AND VIOLATIONS OF ITS PREMISES

In order to understand the impact of low-thrust maneuvers on the Tisserand Criterion, other violations of the premises for the derivation of this conservation quantity are explored and compared regarding their respective impacts. If the impact of these additional violations is of the same order of magnitude as diverting from the gravity-only premise, this might indicate that the Tisserand Criterion can be used for low-thrust trajectories.

Tisserand's Criterion is derived under the premise of the restricted, circular three-body system. If x, y and z denote the axes of a rotating coordinate system, and U describes a so called pseudo-potential, the equations of motions can be written as [11]:

$$\ddot{x} - 2\,n\,\cdot\dot{y} = \frac{\partial U}{\partial x}, \tag{2a}$$

$$\ddot{y} + 2\,n\,\cdot\dot{x} = \frac{\partial U}{\partial y}, \tag{2b}$$

$$\ddot{z} = \frac{\partial U}{\partial z}. \tag{2c}$$

The pseudo-potential U contains first the elements of the centrifugal potential and then the gravity potential [11]:

$$U = \frac{n^2}{2}(x^2+y^2) + \frac{\mu_1}{R_1} + \frac{\mu_2}{R_2}, \tag{3}$$

where n is the angular velocity (on a circular orbit, it is also constant), μ the gravity parameter of the Sun (1) and the planet (2) and R their respective distances to the small mass, i.e. the spacecraft (or originally the comet).

These equations can be reformed to the Jacobi-Integral, a conservation quantity describing the relative energy of the three-body system motion, given in Equ. (4) [11]:

$$C_j = n^2(x^2+y^2) + 2\left(\frac{\mu_1}{R_1} + \frac{\mu_2}{R_2}\right) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \tag{4}$$

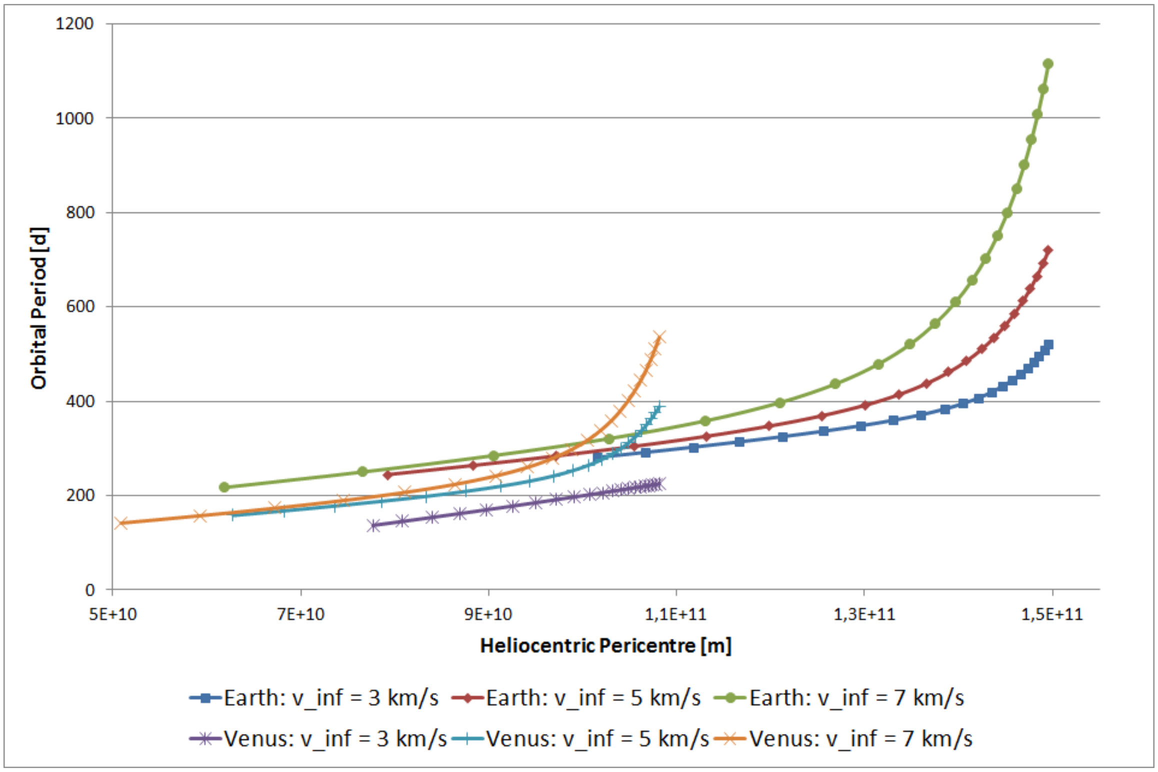

Figure 1: Example of Tisserand Graphs for Earth (right, note the maximum possible heliocentric pericentre being approx. 1AU for the spacecraft) and Venus (left) for various hyperbolic excess velocities (planetcentric). Orbital period (proportional to the semi-major axis, just like the specific orbit energy) as function of the heliocentric pericentre of the spacecraft is given.

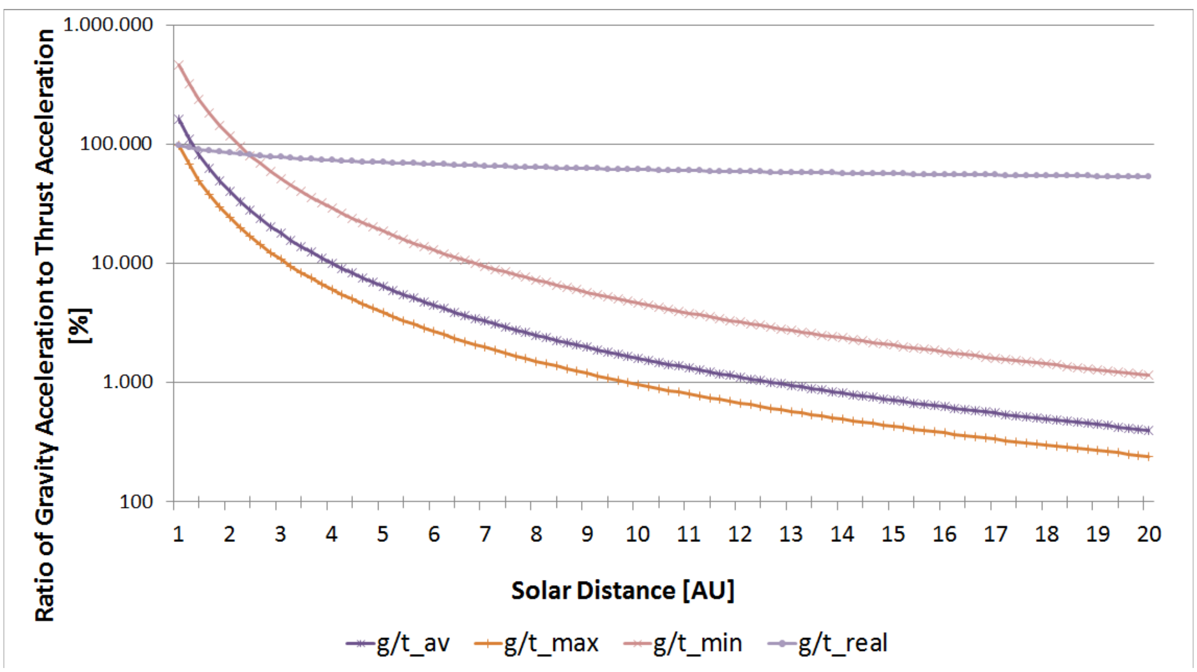

Figure 2: Ratio (logarithmic) of gravity to thrust acceleration as function of solar distance of a sample spacecraft for three cases (av: average thrust of *Dawn*, 55 mN; max: maximum thrust of *Dawn*, 91 mN; min: minimum thrust of *Dawn*, 19 mN; real: realistic thrust drop-off due to power reduction, no cut-out considered).

with the first term being again the centrifugal potential, the second part the gravity potential and the third part being the velocity, i.e. kinetic energy of the motion.

From this, the following assumptions are made to formulate the Tisserand Criterion and bring it into a form as analogue to Equ. (1).

- The angular velocity and semi-major axis of the planet are considered to be unity,
- the Sun's mass is considered to be dominating over the other two, therefore μ_1 is approx. 1,
- the barycentre is close to the Sun's centre (therefore R_1 is labelled R, the small masses distance to the barycentre) and due to the dominance of the Sun's mass the velocity part of Equ. (4) is replaced with the Vis Viva equation;
- the usage of the definition of the angular momentum of the orbit, introduces the cosine of the inclination into the equation as well as the semi-major axis and the eccentricity, replacing it with the centrifugal term,
- the distance of the spacecraft (or comet) to the planet, R_2, is assumed to be large (and μ_2 as small) and therefore can be neglected.

This culminates in the dimensionless formulation of [11]:

$$\frac{C_j}{2} = const = \frac{1}{2\,a} + \sqrt{a\,(1-e^2)}\cos i \qquad (5)$$

The difference between Equ. (1) and (5) is only the usage of dimensionless quantities, i.e. scaling of the semimajor axis with the solar distance of the planet.

II.I Non-constant Spacecraft Mass

The n-body problem assumes that the celestial body masses are all constant over time. For the Sun and planets this is true with adequate precision for formulating the equations of motion, however for a spacecraft, which is continuously thrusting, this assumption is violated.

A non-constant spacecraft mass changes the equations of the forces acting on the spacecraft in the following way:

$$\frac{d}{dt}(m\cdot\vec{v}) = m\cdot\dot{\vec{v}} + \dot{m}\cdot\vec{v} = \sum_i \vec{F}_i, \qquad (6)$$

where m denotes its mass, v its velocity, t the time and F a force. For a constant mass, the second part of the centre term becomes zero and only gravity forces occur on the right side. Introducing a non-constant mass could be dealt with by simply cancelling the mass out of the equation on both sides, but the mass flow remains. The consequence of the mass flow is also the thrust of the space craft, so Equ. (6) can be rewritten as:

$$\frac{d}{dt}(m\cdot\vec{v}) = \dot{\vec{v}} + \frac{\dot{m}}{m}\cdot\vec{v} = \sum_i \frac{\vec{F}_{g,i}}{m} + \vec{T}, \qquad (7)$$

where T is the thrust acceleration and index g denoting gravity forces. It should be noted that the total mass of the spacecraft for typical cases of low-thrust is usually very large compared to a fuel mass flow of some mg/s, therefore $\dot{\vec{v}}$ is still the dominating part of the equation.

In order to determine the impact of the thrust on the forces equation, it is investigated what the ratio between gravity force (of the Sun) and thrust force is for a realistic case.

Dawn is one primary example of a deep space mission using low-thrust propulsion. It has a minimum thrust of 19 mN and a maximum thrust of 91 mN (which leads to a numerical average of 55 mN) [12]. These are compared to the effects of gravity over an increase of solar distance (gravity being proportional to $1/r^2$, with r being the solar distance). Furthermore a realistic case is considered, where, due to the reduction of solar electric power, the thrust is realistically reduced as well. Since a larger solar distance however also reduces the temperature of the solar array and thus increases their efficiency, the power drop-off and thus thrust drop-off is modelled as proportional to $1/r^{1.8}$ [13]. The results are given in Fig. 2. It should be noted that for the realistic case, no power cut-out was assumed that would occur once the power drops below the minimum required power to operate the thruster.

As to be expected the worst case occurs for the maximum thrust of 91 mN, assumed as being constant over the whole calculation. But even at a solar distance of 20 AU (approx. Uranus' position) gravity exceeds thrust by more than factor 100, for the lowest constant thrust of 19 mN it is even factor 1000.

Considering the realistic case, where the thrust drops off due to reduction of available power, the situation becomes even more relaxed, because gravity outweighs thrust by a factor of 80,000 at 20 AU. Meaning even for a thrust 800 times larger than that of *Dawn* (and that would likely no longer be considered low thrust), gravity would still be 100 times larger than the thrust acceleration. It is therefore plausible to state that the non-constant spacecraft mass and the resulting thrust force are no significant diversion from the three-body premises regarding the acting forces. While this is not necessarily the case for nuclear powered spacecraft, those are not really at the immediate horizon of implementation, also the benefit of gravity-assist combined with nuclear power is not as clear, therefore this mission type is not considered.

It should be noted that one major exception has not been regarded yet: Libration or Lagrange Points. At these positions the gravity accelerations of planet(s) and Sun equal each other out so that only the thrust acceleration would remain. These areas are of course small in comparison to the whole trajectory, but it is still possible that the spacecraft might pass through them. However as the Tisserand Criterion is an energy quantity and not a forces quantity, this is not considered to be critical.

II.II Non-circular Orbits for Major Masses

As already the name gives away, the restricted, circular three-body system assumes circles as the orbit for the respecting masses. Referring to Equ. (1) it is apparent that a change in the solar distance of the planet, R_{pl}, caused by a non-circular orbit, does have an effect on the Tisserand Parameter's quantity.

To analyse the impact of the effect of the eccentricity, numerical simulations have been conducted, creating verified models of a circular and non-circular three body system respectively. Using both models a number of calculations have been conducted to compare the effects of planetary flybys by a sample comet/ spacecraft first in the circular and then in the non-circular model. The variable of the spacecraft has been its original semi-major axis, an encounter was considered once the spacecraft has been within the Sphere of Influence to create detectable changes in the spacecraft trajectory. Following these flyby calculations the Tisserand Parameter of the affected comet has been calculated and compared by determining the ratio of the two Tisserand Parameter sets in the following manner:

$$T_{\Delta} = \frac{|T_{circular} - T_{elliptical}|}{T_{elliptical}} \cdot 100\% \quad (8)$$

where T_{Δ} is the deviation of the Tisserand Parameter, and the indexes *circular* and *elliptical* denote the Tisserand Parameter values after the encounter in the circular resp. non-circular system model.

150 simulations were conducted, with the results given in Fig. 3. Each sample has been conducted with one set of parameters for the sample spacecraft, and the Sun as central body and one of the planets as the second body. For each sample it was assumed that the spacecraft was on a trajectory in the orbit plane of the planet, to eliminate effects due to inclination differences.

Investigating the deviations depicted, a correlation to the value of eccentricity is not visible. But it is apparent that the deviation can achieve large ratios (up to 25%) even in the relatively small set of samples taken. The observed changes in the Tisserand Parameter reach significant scales (>10%) several times, usually for the larger planets (Jupiter and Saturn) and due to the random sampling method used it is conceivable that even larger values of deviations are possible, depending on the sample spacecraft to be used.

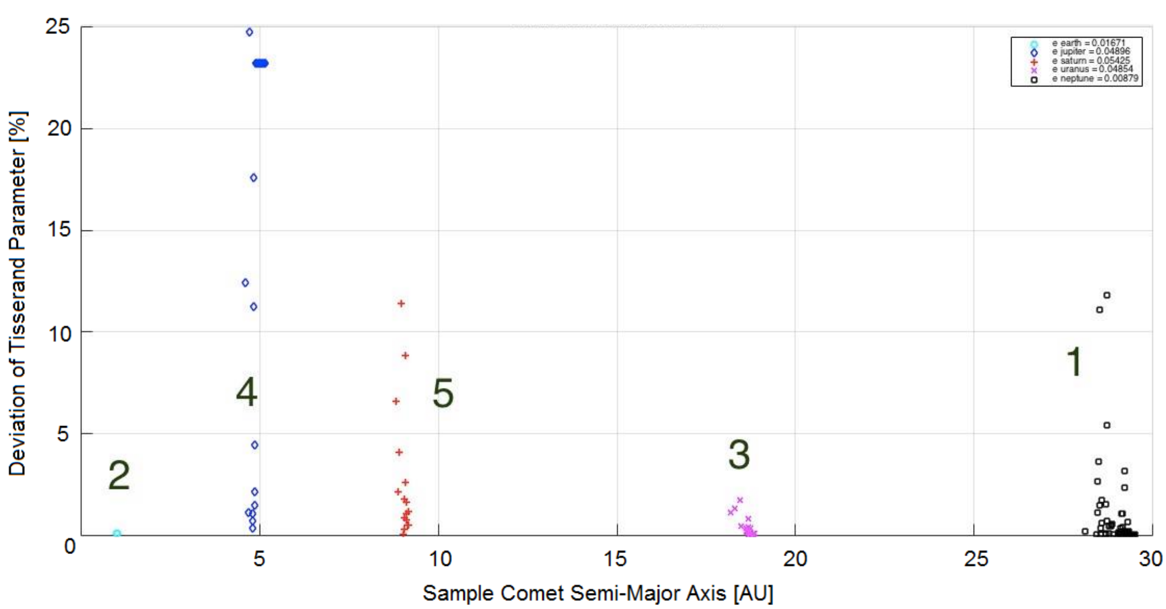

Figure 3: Deviations of the Tisserand Parameter over Sample Comet Semi-Major Axis, related to the Eccentricity of the respective planet.

II.III Small Distance Between Spacecraft and Planet

As described before, one assumption during the derivation of the Tisserand Criterion has been that the distance between spacecraft and planet is large and therefore the respective term can be neglected. Equ. (9) shows one intermediate step before coming to Equ. (5), still incorporating the term containing the gravity parameter of the planet.

$$C_j = \frac{\mu_1}{a} + \frac{\mu_2}{R_2} + 2\sqrt{a\,(1-e^2)}\cos(i) \tag{9}$$

To determine at which distance R_2, the two terms become of the same order of magnitude (or the R_2 part becoming even dominant), the following equation must be fulfilled:

$$R_2 \leq \overset{!}{=} \frac{m_2}{m_1} a, \tag{10}$$

where m denotes the masses of the Sun (1) and the planet (2).

This distance is within the planet's Sphere of Influence [1], at which's border the heliocentric dominance switches over to planetcentric. Since heliocentric trajectories are to be considered, the limit is assumed to be the Sphere of Influence for this purpose of this comparison.

To determine the possible error magnitude for diversion from the R_2 is a large quantity assumption, sample calculations have been made for all planets. To compare the respective values of Equ. (9) and (1) the position R_2 was set to the radius of the respective planet's Sphere of Influence and the eccentricity was varied (0-1 in steps of 0.1 size) as was the semi-major axes (up to 5 times the solar distance of the planet).

The largest difference between the two equations' results (1.5%) occurred for Jupiter and an eccentricity of 0.99 at a semi-major axes of 1.5 times of Jupiter's solar distance. For all other planets, due to their significantly smaller masses, the errors were below 1%. Consequently the violation of this presumption does not result in a very noticeable error.

III. DERIVATION OF A CORRECTION TERM

To incorporate the Tisserand Graphs' methodology into optimization of gravity-assist sequences for low-thrust missions, despite violating the gravity-only premise, it is also possible to include a correction term addressing the necessary changes to incorporate the energy change caused by thrust into the energy quantity that Tisserand's Criterion represents.

Starting of similarly as Equations (2a) to (2c) just with an added term for the thrust acceleration T for each direction of the coordinate system, the same steps of modifying the equations is used to arrive at a formulation like Equ. (5):

$$\ddot{x} - 2\,n\,\cdot\dot{y} = \frac{\partial U}{\partial x} + T_x, \tag{11a}$$

$$\ddot{y} + 2\,n\,\cdot\dot{x} = \frac{\partial U}{\partial y} + T_y, \tag{11b}$$

$$\ddot{z} = \frac{\partial U}{\partial z} + T_z. \tag{11c}$$

The modified Jacobi Integral, C_j^*, is similar to the original one, only containing an integral over time of the velocity multiplied with the thrust acceleration:

$$C_j^* = C_j + 2\int \vec{V}\cdot\vec{T}\,dt. \tag{12}$$

Finishing the reformulation finally leads to Equ. (13):

$$\begin{aligned}&\frac{1}{2\,a_2} + \sqrt{a_2\,(1-e_2{}^2)}\cos i_2 \\ &= \frac{1}{2\,a_1} + \sqrt{a_1\,(1-e_1{}^2)}\cos i_1 \\ &+ 2\int_{t_1}^{t_2}\vec{V}\cdot\vec{T}\,dt\end{aligned} \tag{13}$$

The integral allows the orbital energy introduced by the thrust of the spacecraft to be accounted for, but is otherwise similar to Equ. (5), as desired. The condition (2) has to be equal to the condition (1) plus the energy gained by thrusting between the two points in time.

Depending on certain assumptions to be made, e.g. a circular velocity and tangential thrust, the term can be further simplified (leading to a formulation only containing r as the integration variable).

It should be noted that the thrust energy part is not a state quantity as is e.g. the gravity potential, so it is not sufficient to know the points in time t_1 and t_2 to determine its magnitude, but the whole path between these two points in time has to be known, to calculate the integral correctly. This also means that the corrected Tisserand Parameter value changes constantly during the course of the trajectory in case of low-thrust propulsion.

IV. DISCUSSION

It has been shown that a correction term exists that includes the energy effects of thrusting and therefore eliminates the problem of violating the Tisserand's Criterion's gravity-only premise. However this also means that the graphs would get a different look, depending hugely on the actual integral part of the velocity and thrust acceleration. The graphs would not be universally usable, but depend on the trajectory the spacecraft takes, i.e. be individual for each mission design.

Considering errors due to violations of other premises of the Tisserand Criterion, the most striking is the error occurring due to the non-circular orbits of the planets in Solar System. The largest value found due to this diversion has been 25%, however no regularity is visible in Figure 3. The error is not visible to be a function of the eccentricity directly, so not anticipation about its maximum limit can be made. Considering that random sampling already produced an error of 25% maximum, it is possible that other samples might produce even larger ones. In general, the diversion of circular orbits creates a significant error in the Tisserand Parameter.

From a forces point of view, the thrust is not a significant deviation from the Tisserand Criterion premises, i.e. the non-constant spacecraft mass is neither. While it violates the gravity-only assumptions for its derivation, the gravity force dominates over the thrust by a factor of minimum 100 for all investigated cases. For a realistic thrust decline over the increasing solar distance, the factor was even larger by several orders of magnitude, showing that thrust acceleration is not relevant to consider regarding violations of the restricted three-body system.

However the Tisserand Criterion is an energy quantity and therefore the impact of the thrust regarding the orbital energy of the spacecraft should be considered. Based on the assumption that for a given moment the spacecraft is on a trajectory that represents a Keplerian orbit (with altering properties for each time step, caused by the changes due to thrust), its orbital energy can be summarized as:

$$\xi(t) = -\frac{\mu}{2\,a(t)}, \tag{13}$$

which is the orbital energy of an elliptical orbit [1], but with an added dependency on time.

Assuming, for instance, a mission going from Earth's solar distance to Jupiter's, i.e. from 1 AU to ca. 5 AU, this would mean a change of the specific energy of:

$$|\xi_2 - \xi_1| = |-\frac{\mu}{2 \cdot 5AU} + \frac{\mu}{2 \cdot 1AU}| \tag{14}$$

$$|\xi_2 - \xi_1| = \left|\frac{1}{5}\xi_1 - \xi_1\right| = |-\frac{4}{5}\xi_1|$$

This change has to be achieved by the thrust and of course depends on the exact mission. In this example 80% of the specific energy needs to be created by the thrusting of the spacecraft to complete the mission. This exceeds the errors previously explored and is not negligible.

While missions with less demanding energy changes are possible and thinkable, energy changes of only 10% to 20%, which would be within the error magnitude of the non-circular planetary orbits, are certainly not large enough to warrant low-thrust propulsion in the first place. The reason to use low-thrust propulsion was to enable highly challenging missions (and combine them with gravity assists). Therefore these demanding missions should be selected as benchmark.

As the low-thrust's contribution to orbital energy is significant it cannot be argued that the Tisserand Criterion can be used without modification for gravity-assist sequencing of low-thrust missions. Therefore it is proposed to apply the correction term as described in Section III for this kind of optimization. It should also be noted that the correction term has been derived without the assumption that the thrust is small, therefore it can be used for any thrust regime, when necessary.

V. CONCLUSION

Comparing the violations of the presumptions belonging to the three-body system and those used to derive the Tisserand Criterion, it becomes obvious that the majority does not create deviations of the Tisserand Parameter.

A noteworthy exception is the difference caused by eccentric orbits vs. circular orbits. Random sampling in a realistic solar system model found deviations of up to 25% in comparison to the circular three-body system. Since a direct function of the error in dependence of the eccentricity has not been found, it cannot be determined what the maximum error is.

The major deviation is however caused energywise by the thrust, as – depending on the mission – the majority of the specific orbital energy change is introduced by the thrust of the spacecraft.

To address the errors caused by the usage of (low) continuous thrust for a given mission, a correction term has been derived and is recommended to be used.

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