A Direct 2D Position Solution for an APNT-System

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ABSTRACT

A future APNT-System will be composed of ground based ranging sources and other sensors (e.g. barometric altimeter, IMU). The vertical Dilution of Precision for a system of ground stations can be significantly higher than for GNSS, therefore the integration of altitude measurements is especially important. We propose two different algorithms to integrate an altitude measurement: an iterative position solution with an additional equation for the altitude and a direct position solution which introduces a quadratic constraint for the altitude measurement to a least squares problem using Lagrange multipliers.

INTRODUCTION

Global navigation satellite systems (GNSS) have been selected to be the primary means of navigation in aeronautics. They open up a lot of new possibilities e.g. increase of capacity and cost efficiency. However, especially the inherent low signal strength creates the need for a backup system with sufficient accuracy, availability, continuity and integrity to compensate for major outages or degradation of GNSS. This need shall be covered by a future Alternative Positioning, Navigation and Timing (APNT) System that will use (pseudo-)ranging with ground stations amongst other sensors. This setup rises new challenges concerning ranging as well as positioning.

In an APNT Setup the geometrical constellation of the ranging sources generally leads to a high vertical Dilution of Precision (VDOP) (see [5]) because of their position on the ground. Because of the high VDOP the vertical component of the position needs to be obtained by other means.

Since air data systems are installed in most aircraft types the barometric altimeter is the obvious source of information that can and should be used to compensate in case of difficult geometric constellations. Similar to 2D GNSS PVT solutions, it is possible to use barometric altitude measurements in different ways. If only three pseudorange measurements are available additional information e.g. an altitude measurement is necessary to obtain a position solution. Then the position can be obtained as an intersection of the surface at the measured altitude above the Earth's ellipsoid and two confocal hyperboloids of two sheets. These hyperboloids consist of the positions in space that have the same range difference to two ranging sources at a time (see e.g. [2]). In the very unlikely situation of all pseudoranges being equal this algorithm will become singular and thus needs special treatment, which also has been addressed in [2].

When four or more range measurements are available a three dimensional position solution is generally possible by either using a direct method (e.g. [1] or [4]) or an iterative method, like the Newton-Raphson algorithm, or a combination of both algorithms (see [5]). But they do not make use of additional altitude information and hence lead to suboptimal results in geometries with high VDOP.

In the following we will describe an iterative and a direct positioning algorithm for four or more pseudorange measurements and an altitude measurement. Then we will compare the two algorithms with each other in simulations.

THE ALGORITHMS

Iterative Method: Gauss-Newton

For comparison we will introduce an iterative method based on the Gauss-Newton algorithm, to integrate altitude measurements. To introduce an equation for the altitude measurement h we use

$$\mathbf{A}(h) := \operatorname{diag}\left(\frac{1}{(\alpha+h)^2}, \frac{1}{(\alpha+h)^2}, \frac{1}{(\beta+h)^2}\right)$$

with α and β for the semi-major and semi-minor axis of the WGS84 ellipsoid and h the altitude above the ellipsoid. So all points $\mathbf{x}$ satisfying

$$\mathbf{x}^T\mathbf{A}(h)\mathbf{x} - 1 = 0 \tag{1}$$

are located on an ellipsoid with semi-major and semi-minor axis extended by the amount of h. It is known that the surface described by (1) does not represent the surface consisting of the local normal vectors of length h of the WGS84 ellipsoid - so a the surface is not fully "parallel". However (1) is close to this surface on centimeter level for altitudes h up to 10km (see [6]). Which is sufficiently close for this application. In addition to the altitude measurement we use k pseudorange measurements. For the ith ground station the pseudorange equation reads as follows:

$$\rho_i = \|\mathbf{s}_i - \mathbf{x}\| + b$$

Pseudoranges and altitude measurements together define a function F on which we want to apply the Gauss-Newton algorithm to compute the zeros:

$$\mathrm{F}(\mathbf{x}, b) := \begin{pmatrix} \rho_1 - (\| \mathbf{s}_1 - \mathbf{x} \| + b) \\ \vdots \\ \rho_k - (\| \mathbf{s}_k - \mathbf{x} \| + b) \\ \mathbf{x}^T\mathbf{A}(h)\mathbf{x} - 1 \end{pmatrix}$$

So we need to compute Jacobian matrix of F:

$$\mathbf{G}(\mathbf{x}) := \left(\begin{array}{c|c} \frac{1}{\|\mathbf{s}_1-\mathbf{x}\|}(\mathbf{s}_1 - \mathbf{x})^T & -1 \\ \vdots & \vdots \\ \frac{1}{\|\mathbf{s}_k-\mathbf{x}\|}(\mathbf{s}_k - \mathbf{x})^T & -1 \\ \hline 2\mathbf{x}^T\mathbf{A}(h) & 0 \end{array}\right).$$

Using a starting vector $\begin{pmatrix} \mathbf{x}_0 \\ b_0 \end{pmatrix}$ which is sufficiently close to the user position, we get the position solution iteratively by:

$$\begin{pmatrix} \mathbf{x}_{n+1} \\ b_{n+1} \end{pmatrix} := \begin{pmatrix} \mathbf{x}_n \\ b_n \end{pmatrix} - (\mathbf{G}(\mathbf{x}_n)^T\mathbf{W}\mathbf{G}(\mathbf{x}_n))^{-1}\mathbf{G}(\mathbf{x}_n)^T\mathbf{W}\mathrm{F}(\mathbf{x}_n, b_n)$$

To get good results the weighting matrix $\mathbf{W}$ has to be chosen appropriately to give enough weight to the last row of G because of the way the altitude h is used here. This algorithm works with three or more stations. The following scatter plots show the performance in different geometrical constellations in simulations. For the ranges a standard deviation of $\sigma_{range} = 20m$ and for the altitude $\sigma_{altitude} = 10m$ were used. The resulting standard deviation in the position domain is less than $\sigma_{position} = 44$m (see Fig. 1). The arrows in the plots indicate the directions from the user position to the ground stations. The first scenario has a well distributed set of stations at the following coordinates:

station 1	station 2	station 3
-30528m	84780m	9802m
-17062m	79002m	90012m
-2842m	-3929m	-3585m

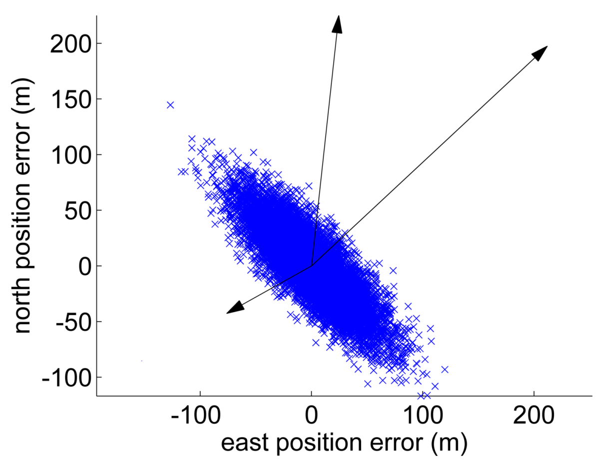

Fig. 1 Horizontal position error distribution for three well distributed ground stations

But in the second scenario the stations are all in a similar direction from the user, as it might occur adjacent to a

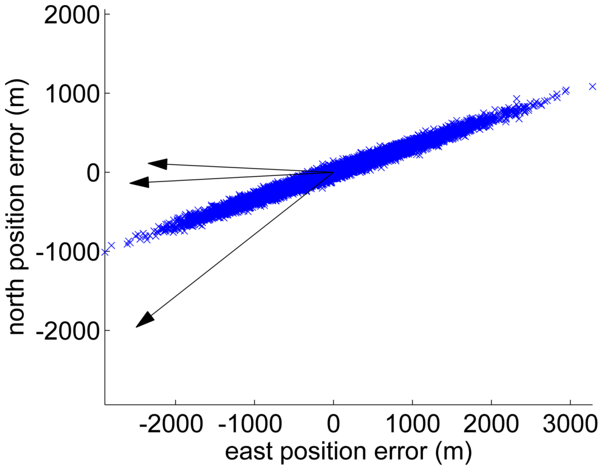

Fig. 2 Horizontal position error distribution for three ground stations in one direction

cost line or to mountains, shadowing stations from the other side. This scenario is geometrically very extreme and has been chosen to analyze the robustness of the method. The resulting standard deviation in the position domain is less than $\sigma_{position} = 865$m (see Fig. 2).

station 1	station 2	station 3
-77443m	-74985m	-70468m
-4134m	-58806m	3405m
-3372m	-3607m	-3259m

Direct Method

We will start with the pseudorange equation and derive a system of equations for spherical positioning from there following an approach from [7]. Let ρ_i be the pseudorange to the i-th ranging source located at $\mathbf{s}_i$. $\mathbf{x}$ and b denote the user position and clock error respectively.

$$\rho_i = \|\mathbf{s}_i - \mathbf{x}\| + b$$

For $i \neq j$ we take the difference of the squared pseudorange equations and get a hyperplane in space time:

$$\Rightarrow \rho_i^2 - \rho_j^2 + 2(\rho_j - \rho_i)b = \|\mathbf{s}_i\|^2 - \|\mathbf{s}_j\|^2 + 2(\mathbf{s}_j - \mathbf{s}_i)^T\mathbf{x}$$

These equations can be summarized in a linear system of equations

$$\underbrace{\left(\begin{array}{c|c} (\mathbf{s}_2 - \mathbf{s}_1)^T & \rho_1 - \rho_2 \\ \vdots & \vdots \\ (\mathbf{s}_k - \mathbf{s}_1)^T & \rho_1 - \rho_k \end{array}\right)}_{=:\mathbf{H}} \begin{pmatrix} \mathbf{x} \\ b \end{pmatrix} = \underbrace{\frac{1}{2}\begin{pmatrix} \rho_1^2 - \rho_2^2 + \|\mathbf{s}_2\|^2 - \|\mathbf{s}_1\|^2 \\ \vdots \\ \rho_1^2 - \rho_k^2 + \|\mathbf{s}_k\|^2 - \|\mathbf{s}_1\|^2 \end{pmatrix}}_{=:\mathbf{y}} \quad (2)$$

which can be solved for the position $\mathbf{x}$ and the user clock bias b.

The next step will be to shape the constraint for the altitude. We write α and β for the semi-major and semi-minor axis of the WGS84 reference ellipsoid. With this we define:

$$\mathbf{A}(h) := \begin{pmatrix} \frac{1}{(\alpha+h)^2} & 0 & 0 \\ 0 & \frac{1}{(\alpha+h)^2} & 0 \\ 0 & 0 & \frac{1}{(\beta+h)^2} \end{pmatrix}$$

for the altitude h above the reference ellipsoid. So for a user position $\mathbf{x}$ at altitude h the following holds in a global approximation:

$$\mathbf{x}^T\mathbf{A}(h)\mathbf{x} = 1.$$

This leads to a quadratic constraint of our least squares problem. For a general description of least squares problems with quadratic constraints and their solutions see [3], we will present a solution for given problem in the following. So we can introduce the Lagrange multiplier ℓ to define the minimization criterium for the position solution:

$$\underset{(\mathbf{x}^T,b,\ell)\in\mathbb{R}^5}{\operatorname{argmin}} \underbrace{(\mathbf{y} - \mathbf{H}\begin{pmatrix} \mathbf{x} \\ b \end{pmatrix})^T(\mathbf{y} - \mathbf{H}\begin{pmatrix} \mathbf{x} \\ b \end{pmatrix}) + \ell(\mathbf{x}^T\mathbf{A}\mathbf{x} - 1)}_{=:J(\mathbf{x},b,\ell)} \quad (3)$$

Optionally a weighting matrix $\mathbf{W}$ can be introduced, but for simplicity of the derivation we will not consider this here. To solve the minimization problem we need to derive J for $\mathbf{x}$, b and ℓ. To do so we first rewrite J such that we separate $\mathbf{x}$ and b

$$\begin{aligned} J(\mathbf{x}, b, \ell) = {} & \mathbf{y}^T\mathbf{y} - 2(\mathbf{y}^T\mathbf{H}_{\mathbf{x}}\mathbf{x} + \mathbf{y}^T\mathbf{H}_b b) + \mathbf{x}^T\mathbf{K}_{\mathbf{x}}\mathbf{x} \\ & + 2b\mathbf{K}_b^T\mathbf{x} + b^2 k_{44} + \ell(\mathbf{x}^T\mathbf{A}\mathbf{x} - 1) \end{aligned}$$

with

$$\mathbf{H} =: \left(\begin{array}{c|c} \mathbf{H}_{\mathbf{x}} & \mathbf{H}_b \end{array} \right)$$

$$\mathbf{H}^T\mathbf{H} =: \mathbf{K} = \left(\begin{array}{c|c} \mathbf{K}_{\mathbf{x}} & \mathbf{K}_b \\ \hline \mathbf{K}_b^T & k_{44} \end{array}\right)$$

Now the derivatives are es follows:

$$\begin{aligned} \frac{dJ}{d\mathbf{x}} &= 2(-\mathbf{y}^T\mathbf{H}_{\mathbf{x}} + \mathbf{x}^T\mathbf{K}_{\mathbf{x}} + b\mathbf{K}_b^T + \ell\mathbf{x}^T\mathbf{A}) &= 0 \quad (4) \\ \frac{dJ}{db} &= 2(-\mathbf{y}^T\mathbf{H}_b + \mathbf{K}_b^T\mathbf{x} + bk_{44}) &= 0 \quad (5) \\ \frac{dJ}{d\ell} &= \mathbf{x}^T\mathbf{A}\mathbf{x} - 1 &= 0 \quad (6) \end{aligned}$$

Note that (6) is just the quadratic constraint. Now we solve (5) for b:

$$b = \frac{1}{k_{44}}\left(\mathbf{y}^T\mathbf{H}_b - \mathbf{K}_b^T\mathbf{x}\right)$$

where k_{44} may not equal zero, which is only the case if all pseudoranges have equal length (in this case we have a singularity) and substitute this into (4)

$$-\mathbf{y}^T\mathbf{H}_\mathbf{x}+\mathbf{x}^T\mathbf{K}_\mathbf{x}+\frac{1}{k_{44}}(\mathbf{y}^T\mathbf{H}_b-\mathbf{K}_b^T\mathbf{x})\mathbf{K}_b^T+\ell\mathbf{x}^T\mathbf{A}=0$$
$$\Leftrightarrow$$
$$\mathbf{x}^T=\mathbf{y}^T\underbrace{(\mathbf{H}_\mathbf{x}-\frac{1}{k_{44}}\mathbf{H}_b\mathbf{K}_b^T)}_{=:\mathbf{V}}(\mathbf{K}_\mathbf{x}-\frac{1}{k_{44}}\mathbf{K}_b\mathbf{K}_b^T+\ell\mathbf{A})^{-1}.$$

So we computed an expression for $\mathbf{x}$ which we can substitute into (6). Furthermore we want to use the principal axis transformation instead of $(\mathbf{K}_\mathbf{x}-\frac{1}{k_{44}}\mathbf{K}_b\mathbf{K}_b^T)\mathbf{A}^{-1}=:\mathbf{U}\Lambda\mathbf{U}^{-1}$, which is existent due to the symmetry of the matrix.

$$\begin{aligned}&\mathbf{y}^T\mathbf{V}^T(\mathbf{U}\Lambda\mathbf{U}^{-1}\mathbf{A}-\ell\mathbf{A})^{-1}\mathbf{A}\\ &\qquad(\mathbf{U}\Lambda\mathbf{U}^{-1}\mathbf{A}+\ell\mathbf{A})^{-1}\mathbf{V}\mathbf{y}-1 &&=0\\ \Leftrightarrow\;&\mathbf{y}^T\mathbf{V}^T\mathbf{A}^{-1}(\mathbf{U}\Lambda\mathbf{U}^{-1}+\ell\mathbf{U}\mathbf{U}^{-1})^{-2}\mathbf{V}\mathbf{y}-1 &&=0\\ \Leftrightarrow\;&\underbrace{\mathbf{y}^T\mathbf{V}^T\mathbf{A}^{-1}\mathbf{U}}_{=:\mathbf{p}^T}(\Lambda-\ell\mathbf{I})^{-2}\underbrace{\mathbf{U}^{-1}\mathbf{V}\mathbf{y}}_{=:\mathbf{q}}-1 &&=0\end{aligned}$$

Which is equivalent to

$$\sum_{i=1}^{3}\frac{p_iq_i}{(\lambda_i+\ell)^2}-1=0 \qquad (7)$$

where p_i and q_i are the i-th entry of $\mathbf{p}$ and $\mathbf{q}$ respectively and λ_i is the i-th diagonal element of Λ. Then we solve (7). The different solutions of (7) lead to different options for ζ that could be extremal. To check for a minimal solution we can simply substitute each solution into the minimization criterium (3).

COMPARISON

The following plot shows simulation results of both the direct and the iterative method with 10000 Monte Carlo runs. For the ranges a standard deviation $\sigma_{range}=20$m was used and for the altitude $\sigma_{altitude}=10$m. In the first scenario (Figures 3-6) we use 6 stations in the directions indicated by the arrows in the scatter plot. The coordinates of the stations in a local ENU frame, with respect to the user position are the following:

station 1	station 2	station 3	station 4	station 5	station 6
-30528m	-64709m	84780m	7389m	9802m	22736m
-17062m	-6247m	79002m	88505m	90012m	65749m
-2842m	-3086m	-3929m	-3554m	-3585m	-3259m

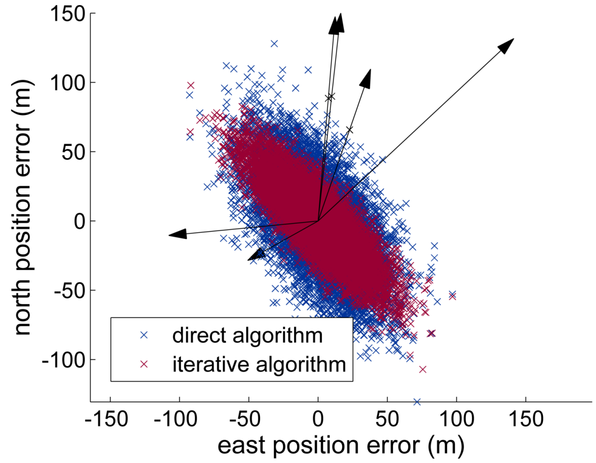

Fig. 3 Scatter plot of horizontal position error for 6 stations with HDOP 1.9

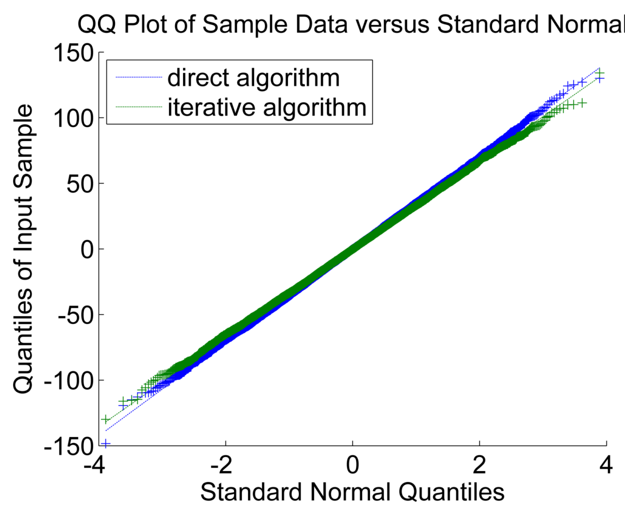

Fig. 4 Quantile-Quantile plot of horizontal position error in dominant direction for 6 stations with HDOP 1.9

Without the altitude measurement these stations lead to the following covariance matrix in ENU coordinates:

$$\mathrm{Cov}(\begin{pmatrix}\delta\mathbf{x}\\ \delta b\end{pmatrix})=$$
$$\sigma^2_{range}\begin{pmatrix}1.41 & -1.16 & -4.52 & -0.51\\ -1.16 & 2.34 & -38.37 & 3.14\\ -4.52 & -38.37 & 1999.1 & -115.09\\ -0.51 & 3.14 & -115.09 & 7.34\end{pmatrix}$$

and the horizontal Dilution of Precision is 1.9. The errors in the altitude component are very similar and according to the introduced error in the altitude measurements. For the direct method this is due to the constraint and for the iterative solution the altitude measurement was given a high weight compared to the ranges. We can overbound the horizontal position error, in the dominant direction of the error, by 38m for the iterative solution and 41m for the direct solution.

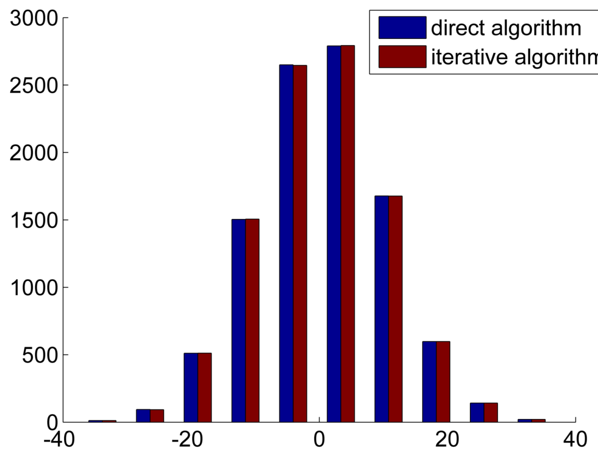

Fig. 5 Histogram of vertical position error in meters for 6 stations with HDOP 1.9

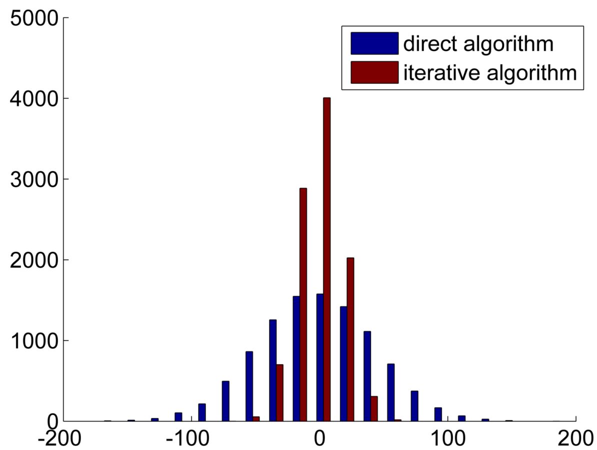

Fig. 6 Histogram of timing error in meters for 6 stations with HDOP 1.9

The second scenario (Figures 7-10) uses different stations with a higher horizontal Dilution of Precision (5.1) but otherwise the same parameters. The stations are again in the directions indicated by the arrows plus one additional station approximately below the user position. The coordinates of the stations in a local ENU frame, with respect to the user position are the following:

station 1	station 2	station 3	station 4	station 5
81544m	-60547m	56885m	37256m	0
73767m	-10039m	29796m	33149m	0
-3706m	-2954m	-2932m	-2866m	-3000m

The coordinates above lead to the following covariance matrix:

$$\mathrm{Cov}(\begin{pmatrix}\delta\mathbf{x}\\ \delta b\end{pmatrix}) =$$

$$\sigma^2_{range}\begin{pmatrix} 4.20 & -9.04 & 2.34 & -2.34 \\ -9.04 & 21.61 & -5.97 & 5.97 \\ 2.34 & -5.97 & 3.08 & -2.03 \\ -2.34 & 5.97 & -2.03 & 1.98 \end{pmatrix}$$

In this scenario the horizontal error is greater due to the

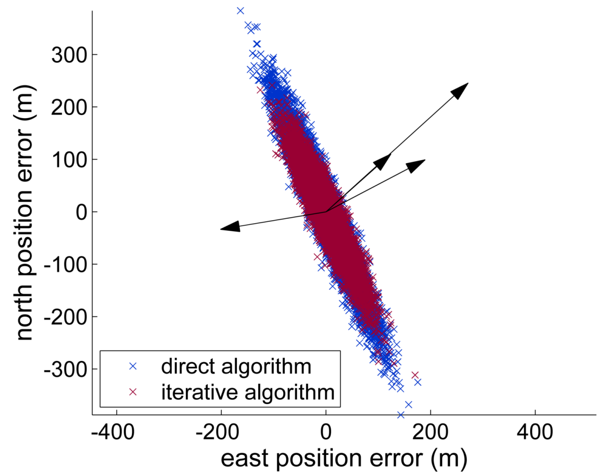

Fig. 7 Scatter plot of horizontal position error for 5 stations and HDOP 5.1

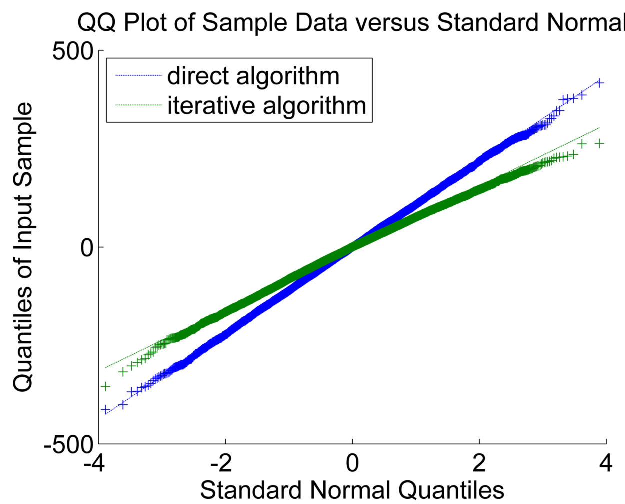

Fig. 8 Quantile-Quantile plot of horizontal position error in dominant direction for 5 stations and HDOP 5.1

increased horizontal dilution of precision and can be overbounded by 96m (iterative) and 128m (direct).

CONCLUSIONS

In this paper we proposed two different kinds of algorithms, one iterative and one direct, to introduce an altitude measurement to a position solution. The iterative algorithm shows a better performance and has the advantage of requiring only three pseudorange measurements. The iterative solution performs a linearization, which is only valid locally, hence a sufficiently good approximation has to be available for a starting point. It is necessary to ensure convergence, but it is not trivial to determine how close sufficiently close is. It can vary due to the geometrical constellation and the distance to the stations (a similar analysis as

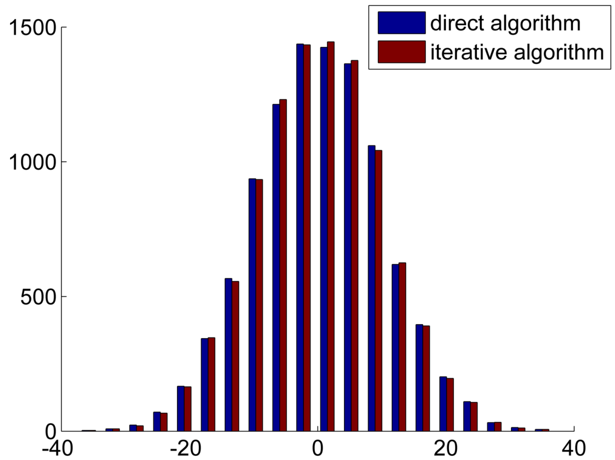

Fig. 9 Histogram of vertical position error in meters for 5 stations and HDOP 5.1

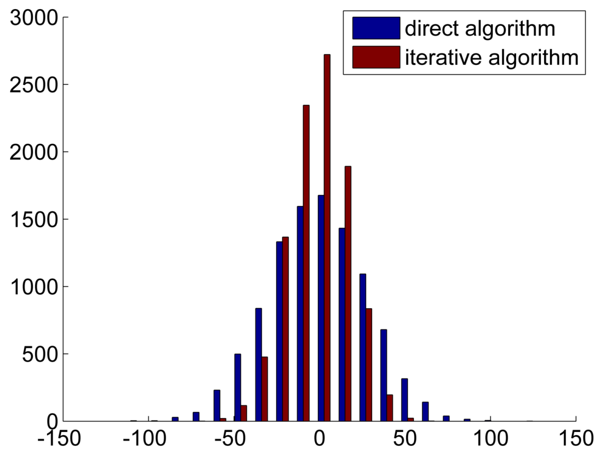

Fig. 10 Histogram of timing error in meters for 5 stations and HDOP 5.1

in [5] would have to be made for methods including altitude measurements). We also see potential for improvement of this method by bringing the altitude and the pseudorange equations to a common frame.

The direct algorithm does not need an approximation of the position, but leads to multiple possible position solutions. One of these solutions has to be identified as the right solution of the user position. Furthermore the direct method suffers from a loss of information due to the spherical positioning approach (see (2)). We are currently investigating how to improve the direct algorithm to prevent this loss of information respectively its impact. Also a combination with the iterative method is to be considered.

In our view these algorithms show promising results but further analysis and improvements of the methods are ongoing to provide a solution that can be used in an real world APNT system.

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