Simple estimations of thermodynamic properties of Yukawa systems

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Motivation

- Equation of state for complex (dusty) plasmas
  - Thermodynamics
  - Hydrodynamic description of the particle component
  - Waves and instabilities

- Specifics of complex plasmas
  - Open systems
  - Particle charge depends on particle density (charge cannibalism)
  - Plasma composition can vary

- Strategy
  - Develop simple analytical approximations for the “basic” case
  - Study relative importance of various specific phenomena
Model

- Two–component system consisting of
  - Point-like particles of charge $Q$ and density $n_0$
  - Neutralizing background (uniform – OCP; linear response - Yukawa)

- Main parameters
  - Wigner-Seitz radius $a = (3/4\pi n_0)^{1/3}$ and the screening length $\lambda$
  - Coupling parameter, $\Gamma = Q^2/aT$
  - Screening parameter, $\kappa = a/\lambda$

- Main quantities of interest (in reduced units)
  - Internal energy, $u = U/NT$
  - Helmholtz free energy, $f = F/NT$
  - Pressure, $p = P V/NT$
Debye-Hückel + Hole (DHH) Approximation

- Conventional Debye-Hückel approach results in unphysical negative density

- Main idea behind DHH is to introduce a cut off (hole radius) $h$, below which the particle density is zero

- The hole radius, $h$, has to be found self-consistently via electrostatic consideration

DHH has been explicitly introduced for the OCP by Nordholm (1984)
Similar relations have been known earlier, e.g. Gryaznov&Iosilevskiy (1973)
DHH for Yukawa systems: Procedure

- Solve Poisson equation

\[ \Delta \phi = -4\pi (Qn - en_m) \]

- Two solutions, inside and outside the hole

\[ n = \begin{cases} 
0, & r \leq h \\
 n_0(1 - Q\phi/T), & r > h
\end{cases} \]

- Match the two solutions at the hole boundary to determine \( h \) and one unknown parameter in the expression for \( \phi \) inside the hole

- Determine excess energy via the conventional expression

\[ u_{ex} = \frac{1}{2} \frac{Q}{T} \left[ \phi(r) - \frac{Q}{r} \right]_{r \to 0} \]
Results: Excess Energy

Numerical results from Hamaguchi et al. (1996, 1997)
Results: Helmholtz Free Energy at Weak Coupling

Excess free energy:

\[ f_{\text{ex}} = \int_0^\Gamma d\Gamma' \frac{u_{\text{ex}}(\kappa, \Gamma')}{\Gamma'} \]

Debye-Hückel (DH) approximation:

\[ u_{\text{ex}}(\kappa, \Gamma) = -\frac{1}{2} \Gamma \kappa \sqrt{1 + 3\Gamma/\kappa^2} \]
\[ f_{\text{ex}}(\kappa, \Gamma) = -\frac{\kappa^3}{9} \left[ \left(1 + \frac{3\Gamma}{\kappa^2}\right)^{3/2} - 1 \right] \]

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MD results from Hamaguchi et al. (1997)
Application: Dust Acoustic Waves (DAW) at strong coupling

- Simplest hydrodynamic approach (particles)

\[
\frac{\partial N_d}{\partial t} + \nabla (N_d \mathbf{V}_d) = 0,
\]

\[
\frac{\partial \mathbf{V}_d}{\partial t} + (\mathbf{V}_d \cdot \nabla) \mathbf{V}_d = \frac{Q \mathbf{E}}{M_d} - \frac{\nabla (P_d \gamma)}{M_d N_d}
\]

- Boltzmann response of the neutralizing medium + Poisson equation

- Resulting dispersion relation

\[
\frac{\omega^2}{\omega_p^2} = \frac{q^2}{q^2 + \kappa^2} + \frac{q^2}{3 \Gamma} \gamma \mu_p
\]

where \( q = k \alpha \), \( \gamma \approx 1 \), and \( \mu_p = 1 + p_{cx} + \frac{\Gamma}{3} \frac{\partial p_{cx}}{\partial \Gamma} - \frac{\kappa}{3} \frac{\partial p_{cx}}{\partial \kappa} \)
Dust Acoustic Waves at Strong Coupling

Numerical results: Ohta & Hamaguchi (2000);
Solid curves: Hydrodynamics with DHH, Dashed curves: sum rule analysis OCP
Recent Developments: Ion Sphere Model (ISM)

- Fixed hole radius = Wigner-Seitz radius
- Pure electrostatics to estimate the static excess energy, $u_{ex}$
- Simple approximation to estimate the thermal contribution to $u_{ex}$
- ISM is simple and more accurate than DHH at strong coupling

Numerical results from Hamaguchi et al. (1996, 1997)
Conclusion

- The ultimate goal of these studies is to produce reliable equation(s) of state for complex (dusty) plasmas

- The first element of the project is to develop simple analytic approximations for the “basic” case

- Two such approximations have been proposed

- DHH approximation is an extension of the DH approach and is suitable in the weak/moderate coupling regime

- ISM approximation is more appropriate in the moderate/strong coupling regime
  - paper in preparation
Thank you for your attention!