Higher order and adaptive DG methods for compressible flows (2)

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Outline

Definition of consistency and adjoint consistency for nonlinear problems

Primal problem:

$$Nu = 0 \quad \text{in } \Omega, \qquad\qquad Bu = 0 \quad \text{on } \Gamma.$$

Target quantity:

$$J(u) = \int_\Omega j_\Omega(u)\,\mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma(Cu)\,\mathrm{d}s,$$

with Fréchet derivative

$$J'[u](w) = \int_\Omega j'_\Omega[u]\,w\,\mathrm{d}\mathbf{x} + \int_\Gamma j'_\Gamma[Cu]\;C'[u]w\,\mathrm{d}s.$$

Compatibility condition: $J(\cdot)$ is compatible to the primal problem if

$$(N'[u]w, z)_\Omega + (B'[u]w, (C'[u])^* z)_\Gamma = (w, (N'[u])^* z)_\Omega + (C'[u]w, (B'[u])^* z)_\Gamma.$$

Adjoint problem:

$$(N'[u])^* z = j'_\Omega[u] \quad \text{in } \Omega, \qquad\qquad (B'[u])^* z = j'_\Gamma[Cu] \quad \text{on } \Gamma.$$

Definition of consistency for nonlinear problems

Primal problem:

$$Nu = 0 \quad \text{in } \Omega, \qquad\qquad Bu = 0 \quad \text{on } \Gamma.$$

Discretization: Find $u_h \in V_h$ such that

$$N_h(u_h, v_h) = 0 \quad \forall v_h \in V_h.$$

Consistency: The exact solution u to the primal problem satisfies:

$$N_h(u, v) = 0 \quad \forall v \in V.$$

Consistency analysis

Rewrite the discrete problem: Find $u_h \in V_h$ such that

$$N_h(u_h, v_h) = 0 \quad \forall v_h \in V_h$$

in following element-based **primal residual form**: Find $u_h \in V_h$ such that

$$\int_\Omega R(u_h) v_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} r(u_h) v_h \,\mathrm{d}s + \int_\Gamma r_\Gamma(u_h) v_h \,\mathrm{d}s \;=\; 0 \quad \forall v_h \;\in\; V_h.$$

The discretization is **consistent**
if the exact solution u to the primal problem satisfies

$$\begin{aligned} R(u) &= 0 && \text{in } \kappa, \kappa \in \mathcal{T}_h, \\ r(u) &= 0 && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ r_\Gamma(u) &= 0 && \text{on } \Gamma. \end{aligned}$$

Definition of adjoint consistency for nonlinear problems

Discretization: Find $u_h \in V_h$ such that

$$N_h(u_h, v_h) = 0 \quad \forall v_h \in V_h,$$

Compatible target quantity: $J(u)$
consistent discretization $J_h(u_h)$ with $J_h(u) = J(u)$.

Discrete adjoint problem: find $z_h \in V_h$ such that

$$N_h'[u_h](w_h, z_h) = J_h'[u_h](w_h) \quad \forall w_h \in V_h.$$

Adjoint consistency: The exact solution z to the adjoint problem satisfies:

$$N_h'[u](w, z) = J_h'[u](w) \quad \forall w \in V.$$

Adjoint consistency analysis

Rewrite the **discrete adjoint problem**: find $z_h \in V_h$ such that

$$N'_h[u_h](w_h, z_h) = J'_h[u_h](w_h) \quad \forall w_h \in V_h,$$

in **adjoint residual form**: find $z_h \in V_h$ such that

$$\sum_{\kappa \in \mathcal{T}_h} \int_\kappa w_h \, R^*[u_h](z_h) \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} w_h \, r^*[u_h](z_h) \, \mathrm{d}s + \int_\Gamma w_h \, r^*_\Gamma[u_h](z_h) \, \mathrm{d}s = 0,$$

The discrete adjoint problem is a **consistent** discretization of the adjoint problem if the exact solution z to the adjoint problem satisfies

$$R^*[u](z) = 0 \ \text{ in } \kappa, \quad r^*[u](z) = 0 \ \text{ on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \quad r^*_\Gamma[u](z) = 0 \ \text{ on } \Gamma.$$

Then the discretization N_h in combination with J_h is **adjoint consistent**.

Outline

The compressible Euler equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\varrho\\ \varrho v_1\\ \varrho v_2\\ \varrho E\end{pmatrix}+\frac{\partial}{\partial x_1}\begin{pmatrix}\varrho v_1\\ \varrho v_1^2+p\\ \varrho v_1 v_2\\ v_1(\varrho E+p)\end{pmatrix}+\frac{\partial}{\partial x_2}\begin{pmatrix}\varrho v_2\\ \varrho v_1 v_2\\ \varrho v_2^2+p\\ v_2(\varrho E+p)\end{pmatrix}=0$$

The compressible Euler equations

$$\frac{\partial}{\partial t}\begin{pmatrix} \varrho \\ \varrho v_1 \\ \varrho v_2 \\ \varrho E \end{pmatrix} + \frac{\partial}{\partial x_1}\begin{pmatrix} \varrho v_1 \\ \varrho v_1^2 + p \\ \varrho v_1 v_2 \\ v_1(\varrho E + p) \end{pmatrix} + \frac{\partial}{\partial x_2}\begin{pmatrix} \varrho v_2 \\ \varrho v_1 v_2 \\ \varrho v_2^2 + p \\ v_2(\varrho E + p) \end{pmatrix} = 0$$

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x_1}\mathbf{f}_1^c(\mathbf{u}) + \frac{\partial}{\partial x_2}\mathbf{f}_2^c(\mathbf{u}) = 0$$

The compressible Euler equations

$$\frac{\partial}{\partial t}\begin{pmatrix} \varrho \\ \varrho v_1 \\ \varrho v_2 \\ \varrho E \end{pmatrix} + \frac{\partial}{\partial x_1}\begin{pmatrix} \varrho v_1 \\ \varrho v_1^2 + p \\ \varrho v_1 v_2 \\ v_1(\varrho E + p) \end{pmatrix} + \frac{\partial}{\partial x_2}\begin{pmatrix} \varrho v_2 \\ \varrho v_1 v_2 \\ \varrho v_2^2 + p \\ v_2(\varrho E + p) \end{pmatrix} = 0$$

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x_1}\mathbf{f}_1^c(\mathbf{u}) + \frac{\partial}{\partial x_2}\mathbf{f}_2^c(\mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla \cdot \mathcal{F}^c(\mathbf{u}) = 0$$

The compressible Euler equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\varrho\\ \varrho v_1\\ \varrho v_2\\ \varrho E\end{pmatrix}+\frac{\partial}{\partial x_1}\begin{pmatrix}\varrho v_1\\ \varrho v_1^2+p\\ \varrho v_1 v_2\\ v_1(\varrho E+p)\end{pmatrix}+\frac{\partial}{\partial x_2}\begin{pmatrix}\varrho v_2\\ \varrho v_1 v_2\\ \varrho v_2^2+p\\ v_2(\varrho E+p)\end{pmatrix}=0$$

$$\frac{\partial}{\partial t}\mathbf{u}+\frac{\partial}{\partial x_1}\mathbf{f}_1^c(\mathbf{u})+\frac{\partial}{\partial x_2}\mathbf{f}_2^c(\mathbf{u})=0$$

$$\frac{\partial}{\partial t}\mathbf{u}+\nabla\cdot\mathcal{F}^c(\mathbf{u})=0$$

Steady state compressible Euler equations:

$$\nabla\cdot\mathcal{F}^c(\mathbf{u})=0$$

Boundary conditions

- Supersonic inflow corresponds to Dirichlet boundary conditions where
$$\mathbf{u}_\Gamma(\mathbf{u}) = \mathbf{g}_D = \mathbf{u}_\infty.$$
- Supersonic outflow corresponds to Neumann boundary conditions where
$$\mathbf{u}_\Gamma(\mathbf{u}) = \mathbf{u}.$$
- The subsonic inflow boundary condition takes the pressure from the flow field and imposes all other variables based on freestream conditions $\mathbf{u}_\infty$, i.e.
$$\mathbf{u}_\Gamma(\mathbf{u}) = \left(\rho_\infty, \rho_\infty v_{1,\infty}, \rho_\infty v_{2,\infty}, \frac{p(\mathbf{u})}{\gamma-1} + \rho_\infty\left(v_{1,\infty}^2 + v_{2,\infty}^2\right)\right)^\top.$$
Here, $p \equiv p(\mathbf{u})$ denotes the pressure.
- The subsonic outflow boundary condition imposes an outflow pressure p_{out} and takes all other variables from the flow field, i.e.
$$\mathbf{u}_\Gamma(\mathbf{u}) = \left(u_1, u_2, u_3, \frac{p_{\text{out}}}{\gamma-1} + \frac{u_2^2+u_3^2}{2u_1}\right)^\top.$$

Slip wall boundary conditions

- For slip wall boundary conditions we set

$$\mathbf{u}_\Gamma(\mathbf{u}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-n_1^2 & -n_1 n_2 & 0 \\ 0 & -n_1 n_2 & 1-n_2^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{u} \quad \text{on} \quad \Gamma_W,$$

which originates from $\mathbf{u}$ by removing the normal velocity component of $\mathbf{u}$, i.e. $\mathbf{v} = (v_1, v_2)$ is replaced by $\mathbf{v}_\Gamma = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$.

This choice ensures a vanishing normal velocity,

$$B\mathbf{u}_\Gamma(\mathbf{u}) = n_1 u_{\Gamma,2} + n_2 u_{\Gamma,3} = \rho\, \mathbf{n} \cdot \mathbf{v}_\Gamma = 0,$$

for the boundary operator

$$B\mathbf{u} = n_1 u_2 + n_2 u_3 \quad \text{on } \Gamma_W.$$

The continuous adjoint equations

Given an inviscid compressible flow at an angle of attack α.
Then the aerodynamic force coefficients are given by

$$J(\mathbf{u}) = \int_\Gamma j(\mathbf{u})\,\mathrm{d}s = \int_{\Gamma_W} p\,\mathbf{n}\cdot\boldsymbol{\psi}\,\mathrm{d}s,$$

where $\boldsymbol{\psi} = \boldsymbol{\psi}_d = \frac{1}{C_\infty}(\cos(\alpha), \sin(\alpha))^\top$ for the drag coefficient
and $\boldsymbol{\psi} = \boldsymbol{\psi}_l = \frac{1}{C_\infty}(-\sin(\alpha), \cos(\alpha))^\top$ for the lift coefficient.

Primal problem with slip wall boundary conditions, $\mathbf{n}\cdot\mathbf{v} = n_1 v_1 + n_2 v_2 = 0$:

$$N\mathbf{u} = \nabla\cdot\mathcal{F}^c(\mathbf{u}) = 0 \quad \text{on } \Omega, \qquad\qquad B\mathbf{u} = n_1 u_2 + n_2 u_3 = 0 \quad \text{on } \Gamma_W.$$

Multiply left hand side by $\mathbf{z}$, integrate over Ω and integrate by parts:

$$(\nabla\cdot\mathcal{F}^c(\mathbf{u}), \mathbf{z})_\Omega = -(\mathcal{F}^c(\mathbf{u}), \nabla\mathbf{z})_\Omega + (\mathbf{n}\cdot\mathcal{F}^c(\mathbf{u}), \mathbf{z})_\Gamma.$$

Linearize about the exact solution $\mathbf{u}$

$$\begin{aligned}(\nabla\cdot(\mathcal{F}^c_{\mathbf{u}}[\mathbf{u}](\mathbf{w}))\,, \mathbf{z})_\Omega &= -\,(\mathcal{F}^c_{\mathbf{u}}[\mathbf{u}](\mathbf{w}), \nabla\mathbf{z})_\Omega + (\mathbf{n}\cdot\mathcal{F}^c_{\mathbf{u}}[\mathbf{u}](\mathbf{w}), \mathbf{z})_\Gamma \\ &= -\left(\mathbf{w}, (\mathcal{F}^c_{\mathbf{u}}[\mathbf{u}])^\top\,\nabla\mathbf{z}\right)_\Omega + \left(\mathbf{w}, (\mathbf{n}\cdot\mathcal{F}^c_{\mathbf{u}}[\mathbf{u}])^\top\,\mathbf{z}\right)_\Gamma\end{aligned}$$

The continuous adjoint equations

The **variational formulation** of the adjoint problem is given by: find $\mathbf{z}$ such that

$$-\left(\mathbf{w}, \left(\mathcal{F}^c_{\mathbf{u}}[\mathbf{u}]\right)^\top \nabla \mathbf{z}\right)_\Omega + \left(\mathbf{w}, \left(\mathbf{n} \cdot \mathcal{F}^c_{\mathbf{u}}[\mathbf{u}]\right)^\top \mathbf{z}\right)_\Gamma = J'[\mathbf{u}](\mathbf{w}) \quad \forall \mathbf{w} \in V,$$

with

$$J(\mathbf{u}) = \int_\Gamma j(\mathbf{u})\,\mathrm{d}s = \int_{\Gamma_W} p\,\mathbf{n} \cdot \boldsymbol{\psi}\,\mathrm{d}s,$$

$$J'[\mathbf{u}](\mathbf{w}) = \int_\Gamma j'[\mathbf{u}](\mathbf{w})\,\mathrm{d}s = \int_{\Gamma_W} p'[\mathbf{u}](\mathbf{w})\,\mathbf{n} \cdot \boldsymbol{\psi}\,\mathrm{d}s.$$

The continuous **adjoint problem** is

$$(N'[u])^* z = -\left(\mathcal{F}^c_{\mathbf{u}}[\mathbf{u}]\right)^\top \nabla \mathbf{z} = 0 \quad \text{in } \Omega, \qquad \left(\mathbf{n} \cdot \mathcal{F}^c_{\mathbf{u}}[\mathbf{u}]\right)^\top \mathbf{z} = j'[\mathbf{u}] \quad \text{on } \Gamma_W.$$

Using $\mathcal{F}^c(\mathbf{u}) \cdot \mathbf{n} = p(0, n_1, n_2, 0)^\top$ on Γ_W we obtain

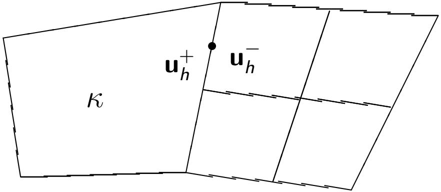
$$p'[\mathbf{u}](0, n_1, n_2, 0) \cdot \mathbf{z} = p'[\mathbf{u}]\mathbf{n} \cdot \boldsymbol{\psi} \quad \text{on } \Gamma_W,$$

which reduces to the boundary condition of the adjoint problem:

$$(B'[\mathbf{u}])^* \mathbf{z} = n_1 z_2 + n_2 z_3 = \mathbf{n} \cdot \boldsymbol{\psi} \quad \text{on } \Gamma_W.$$

Outline

The DG discretization of the compressible Euler equations



The problem:

$$\nabla \cdot \mathcal{F}^c(\mathbf{u}) = 0 \quad \text{in } \Omega \subset \mathbb{R}^2,$$

with $\mathbf{u} = (\varrho, \varrho v_1, \varrho v_2, \rho E)^T$.

The DG(p) discretization: Find $\mathbf{u}_h$ in $\mathbf{V}_h^p$ such that

$$N_h(\mathbf{u}_h, \mathbf{v}_h) \equiv \sum_{\kappa \in \mathcal{T}_h} \left\{ -\int_\kappa \mathcal{F}^c(\mathbf{u}_h) : \nabla \mathbf{v}_h \, \mathrm{d}\mathbf{x} + \int_{\partial\kappa \setminus \Gamma} \hat{\mathbf{h}}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}) \cdot \mathbf{v}_h^+ \, \mathrm{d}s \right\}$$
$$+ \int_\Gamma \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+, \mathbf{n}) \cdot \mathbf{v}_h^+ \, \mathrm{d}s = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h^p,$$

with

$$\mathbf{V}_h^p = \{\mathbf{v}_h \in \left[L^2(\Omega)\right]^m : \mathbf{v}_h|_\kappa \circ F_\kappa \in [Q_p(\hat{\kappa})]^m \text{ if } \hat{\kappa} \text{ is the unit square, and}$$
$$\mathbf{v}_h|_\kappa \circ F_\kappa \in [P_p(\hat{\kappa})]^m \text{ if } \hat{\kappa} \text{ is the unit triangle}, \kappa \in \mathcal{T}_h\}.$$

Numerical flux function $\hat{\mathbf{h}}$: (Local) Lax-Friedrichs, Vijayasundaram, Roe, ...

Consistency

The discretization: find $\mathbf{u}_h$ in $\mathbf{V}_h^p$ such that

$$N_h(\mathbf{u}_h, \mathbf{v}_h) \equiv -\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} \hat{\mathbf{h}}_h \cdot \mathbf{v}_h^+ \,\mathrm{d}s + \int_\Gamma \hat{\mathbf{h}}_{\Gamma,h} \cdot \mathbf{v}_h^+ \,\mathrm{d}s = 0$$

for all $\mathbf{v}_h \in \mathbf{V}_h^p$, with $\hat{\mathbf{h}}_h := \hat{\mathbf{h}}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n})$ on $\partial\kappa \setminus \Gamma$, and $\hat{\mathbf{h}}_{\Gamma,h} := \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+, \mathbf{n})$ on Γ,

is **consistent** if

- the numerical flux $\hat{\mathbf{h}}$ on interior edges $e \in \Gamma_\mathcal{I}$ is consistent, i.e.

$$\hat{\mathbf{h}}(\mathbf{v}, \mathbf{v}, \mathbf{n}) = \mathbf{n} \cdot \mathcal{F}^c(\mathbf{v}) \qquad \text{on } e \in \Gamma_\mathcal{I},$$

- and, the numerical flux $\hat{\mathbf{h}}_\Gamma$ on boundary edges is consistent, i.e., the exact solution $\mathbf{u}$ of the flow equations satisfies

$$\hat{\mathbf{h}}_\Gamma(\mathbf{u}, \mathbf{n}) = \mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}) \qquad \text{on } \Gamma.$$

Adjoint consistency

The discretization: find $\mathbf{u}_h$ in $\mathbf{V}_h^p$ such that

$$N_h(\mathbf{u}_h, \mathbf{v}_h) \equiv -\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla \mathbf{v}_h \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \hat{\mathbf{h}}_h \cdot \mathbf{v}_h^+ \, \mathrm{d}s + \int_\Gamma \hat{\mathbf{h}}_{\Gamma,h} \cdot \mathbf{v}_h^+ \, \mathrm{d}s = 0$$

for all $\mathbf{v}_h \in \mathbf{V}_h^p$, with $\hat{\mathbf{h}}_h := \hat{\mathbf{h}}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n})$ on $\partial\kappa \setminus \Gamma$, and $\hat{\mathbf{h}}_{\Gamma,h} := \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+, \mathbf{n})$ on Γ.

Adjoint consistency

The discretization: find $\mathbf{u}_h$ in $\mathbf{V}_h^p$ such that

$$N_h(\mathbf{u}_h, \mathbf{v}_h) \equiv -\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \hat{\mathbf{h}}_h \cdot \mathbf{v}_h^+ \,\mathrm{d}s + \int_\Gamma \hat{\mathbf{h}}_{\Gamma,h} \cdot \mathbf{v}_h^+ \,\mathrm{d}s = 0$$

for all $\mathbf{v}_h \in \mathbf{V}_h^p$, with $\hat{\mathbf{h}}_h := \hat{\mathbf{h}}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n})$ on $\partial\kappa \setminus \Gamma$, and $\hat{\mathbf{h}}_{\Gamma,h} := \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+, \mathbf{n})$ on Γ.
The (compatible) target quantity:

$$J(\mathbf{u}) = \int_{\Gamma_W} p \,\mathbf{n} \cdot \psi \,\mathrm{d}s,$$

Task: Find a discretization $J_h(\mathbf{u}_h)$ of $J(\mathbf{u})$ which is **consistent** and which (in combination with N_h) is **adjoint consistent**.

Adjoint consistency

The discretization: find $\mathbf{u}_h$ in $\mathbf{V}_h^p$ such that

$$N_h(\mathbf{u}_h, \mathbf{v}_h) \equiv -\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla \mathbf{v}_h \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \hat{\mathbf{h}}_h \cdot \mathbf{v}_h^+ \, \mathrm{d}s + \int_\Gamma \hat{\mathbf{h}}_{\Gamma,h} \cdot \mathbf{v}_h^+ \, \mathrm{d}s = 0$$

for all $\mathbf{v}_h \in \mathbf{V}_h^p$, with $\hat{\mathbf{h}}_h := \hat{\mathbf{h}}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n})$ on $\partial\kappa \setminus \Gamma$, and $\hat{\mathbf{h}}_{\Gamma,h} := \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+, \mathbf{n})$ on Γ.
The (compatible) target quantity:

$$J(\mathbf{u}) = \int_{\Gamma_W} p \, \mathbf{n} \cdot \boldsymbol{\psi} \, \mathrm{d}s,$$

Task: Find a discretization $J_h(\mathbf{u}_h)$ of $J(\mathbf{u})$ which is **consistent** and which (in combination with N_h) is **adjoint consistent**.

Consider following discretization of $J(\mathbf{u})$:

$$J_h(\mathbf{u}_h) = \int_{\Gamma_W} \hat{\mathbf{h}}_{\Gamma,h} \cdot \tilde{\boldsymbol{\psi}} \, \mathrm{d}s,$$

with $\tilde{\boldsymbol{\psi}} = (0, \psi_1, \psi_2, 0)^\top$ on Γ_W for $\boldsymbol{\psi} = (\psi_1, \psi_2)^\top$.

Adjoint consistency

Consider the target quantity and its discretization

$$J(\mathbf{u}) = \int_{\Gamma_W} p\,\mathbf{n}\cdot\boldsymbol{\psi}\,\mathrm{d}s, \qquad J_h(\mathbf{u}_h) = \int_{\Gamma_W} \hat{\mathbf{h}}_{\Gamma,h}\cdot\tilde{\boldsymbol{\psi}}\,\mathrm{d}s,$$

with $\tilde{\boldsymbol{\psi}} = (0,\psi_1,\psi_2,0)^\top$ for $\boldsymbol{\psi} = (\psi_1,\psi_2)^\top$, and $\hat{\mathbf{h}}_{\Gamma,h} := \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+,\mathbf{n})$ on Γ.

Adjoint consistency

Consider the target quantity and its discretization

$$J(\mathbf{u}) = \int_{\Gamma_W} p\,\mathbf{n}\cdot\boldsymbol{\psi}\,\mathrm{d}s, \qquad J_h(\mathbf{u}_h) = \int_{\Gamma_W} \hat{\mathbf{h}}_{\Gamma,h}\cdot\tilde{\boldsymbol{\psi}}\,\mathrm{d}s,$$

with $\tilde{\boldsymbol{\psi}} = (0,\psi_1,\psi_2,0)^\top$ for $\boldsymbol{\psi} = (\psi_1,\psi_2)^\top$, and $\hat{\mathbf{h}}_{\Gamma,h} := \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+,\mathbf{n})$ on Γ.

Assume $\hat{\mathbf{h}}_\Gamma$ is consistent. Then, $J_h(\mathbf{u}_h)$ is a **consistent** discretization of $J(\mathbf{u})$, as the exact solution $\mathbf{u}$ satisfies

$$\hat{\mathbf{h}}_\Gamma(\mathbf{u},\mathbf{n})\cdot\tilde{\boldsymbol{\psi}} = (\mathbf{n}\cdot\mathcal{F}^c(\mathbf{u}))\cdot\tilde{\boldsymbol{\psi}} = p(\mathbf{u})(0,n_1,n_2,0)^\top\cdot\tilde{\boldsymbol{\psi}} = p(\mathbf{u})\mathbf{n}\cdot\boldsymbol{\psi},$$

and thereby $J_h(\mathbf{u}) = J(\mathbf{u})$.

Adjoint consistency

Consider the target quantity and its discretization

$$J(\mathbf{u}) = \int_{\Gamma_W} p\,\mathbf{n}\cdot\boldsymbol{\psi}\,\mathrm{d}s, \qquad J_h(\mathbf{u}_h) = \int_{\Gamma_W} \hat{\mathbf{h}}_{\Gamma,h}\cdot\tilde{\boldsymbol{\psi}}\,\mathrm{d}s,$$

with $\tilde{\boldsymbol{\psi}} = (0,\psi_1,\psi_2,0)^\top$ for $\boldsymbol{\psi} = (\psi_1,\psi_2)^\top$, and $\hat{\mathbf{h}}_{\Gamma,h} := \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+,\mathbf{n})$ on Γ.

Assume $\hat{\mathbf{h}}_\Gamma$ is consistent. Then, $J_h(\mathbf{u}_h)$ is a **consistent** discretization of $J(\mathbf{u})$, as the exact solution $\mathbf{u}$ satisfies

$$\hat{\mathbf{h}}_\Gamma(\mathbf{u},\mathbf{n})\cdot\tilde{\boldsymbol{\psi}} = (\mathbf{n}\cdot\mathcal{F}^c(\mathbf{u}))\cdot\tilde{\boldsymbol{\psi}} = p(\mathbf{u})(0,n_1,n_2,0)^\top\cdot\tilde{\boldsymbol{\psi}} = p(\mathbf{u})\mathbf{n}\cdot\boldsymbol{\psi},$$

and thereby $J_h(\mathbf{u}) = J(\mathbf{u})$.

Furthermore, one can show (cf. Theorem 5.13) that N_h in combination with J_h is **adjoint consistent**.

With a numerical flux function at the boundary ...

1. ... based on the *normal boundary flux*

$$\hat{\mathbf{h}}_{\Gamma,h} = \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+, \mathbf{n}) = \mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)),$$

the discretization is given by

$$-\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla_h \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \hat{\mathbf{h}}_h \cdot \mathbf{v}_h^+ \,\mathrm{d}s + \int_\Gamma \mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)) \cdot \mathbf{v}_h^+ \,\mathrm{d}s = 0.$$

(a) This discretization is **adjoint consistent** in combination with

$$J_h(\mathbf{u}_h) = \int_{\Gamma_W} \hat{\mathbf{h}}_{\Gamma,h} \cdot \tilde{\psi} \,\mathrm{d}s = \int_{\Gamma_W} \left(\mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)\right) \cdot \tilde{\psi} \,\mathrm{d}s$$
$$= \int_{\Gamma_W} p\left(\mathbf{u}_\Gamma(\mathbf{u}_h^+)\right) \mathbf{n} \cdot \psi \,\mathrm{d}s = J(\mathbf{u}_\Gamma(\mathbf{u}_h^+)).$$

(b) It is **adjoint inconsistent** in combination with following direct discretization

$$J(\mathbf{u}_h) = \int_{\Gamma_W} p(\mathbf{u}_h)\, \mathbf{n} \cdot \psi \,\mathrm{d}s.$$

With a numerical flux function at the boundary ...

2. ... based on the *interior numerical flux*

$$\hat{\mathbf{h}}_{\Gamma,h} = \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+, \mathbf{n}) = \hat{\mathbf{h}}(\mathbf{u}_h^+, \mathbf{u}_\Gamma^-(\mathbf{u}_h^+), \mathbf{n}),$$

where the *boundary exterior state* $\mathbf{u}_\Gamma^-(\mathbf{u}_h^+)$ is obtained by

$$\frac{1}{2}\left(\mathbf{u}_h^+ + \mathbf{u}_\Gamma^-(\mathbf{u}_h^+)\right) = \mathbf{u}_\Gamma(\mathbf{u}_h^+), \qquad \text{i.e., } \mathbf{u}_\Gamma^-(\mathbf{u}_h^+) = 2\mathbf{u}_\Gamma(\mathbf{u}_h^+) - \mathbf{u}_h^+,$$

$$\mathbf{u}_\Gamma(\mathbf{u}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-n_1^2 & -n_1 n_2 & 0 \\ 0 & -n_1 n_2 & 1-n_2^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{u}, \quad \mathbf{u}_\Gamma^-(\mathbf{u}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2n_1^2 & -2n_1 n_2 & 0 \\ 0 & -2n_1 n_2 & 1-2n_2^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{u}.$$

Then, the discretization is given by

$$-\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla_h \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} \hat{\mathbf{h}}_h \cdot \mathbf{v}_h^+ \,\mathrm{d}s + \int_\Gamma \hat{\mathbf{h}}(\mathbf{u}_h^+, \mathbf{u}_\Gamma^-(\mathbf{u}_h^+), \mathbf{n}) \cdot \mathbf{v}_h^+ \,\mathrm{d}s = 0.$$

With a numerical flux function at the boundary ...

2. ... based on the *interior numerical flux*, the discretization,

$$-\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla_h \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} \hat{\mathbf{h}}_h \cdot \mathbf{v}_h^+ \,\mathrm{d}s + \int_\Gamma \hat{\mathbf{h}}(\mathbf{u}_h^+, \mathbf{u}_\Gamma^-(\mathbf{u}_h^+), \mathbf{n}) \cdot \mathbf{v}_h^+ \,\mathrm{d}s = 0,$$

(a) ... is **adjoint consistent** in combination with following discretization of $J(\cdot)$,

$$J_h(\mathbf{u}_h) = \int_{\Gamma_W} \hat{\mathbf{h}}_{\Gamma,h} \cdot \tilde{\psi} \,\mathrm{d}s = \int_{\Gamma_W} \hat{\mathbf{h}}(\mathbf{u}_h^+, \mathbf{u}_\Gamma^-(\mathbf{u}_h^+), \mathbf{n}) \cdot \tilde{\psi} \,\mathrm{d}s,$$

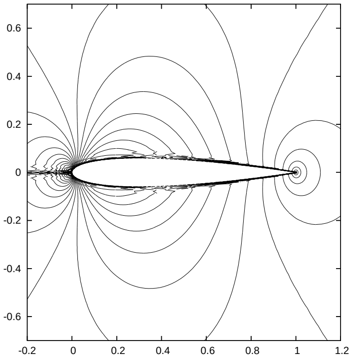
(b) ... is **adjoint inconsistent** in combination with the direct discretization

$$J(\mathbf{u}_h) = \int_{\Gamma_W} p(\mathbf{u}_h) \,\mathbf{n} \cdot \psi \,\mathrm{d}s,$$

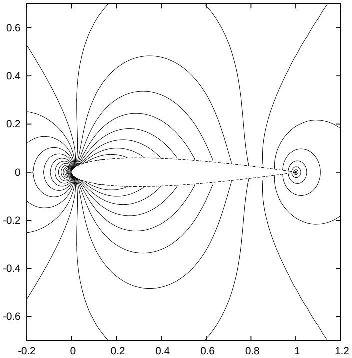
(c) ... is **adjoint inconsistent** in combination with

$$J(\mathbf{u}_\Gamma(\mathbf{u}_h^+)).$$

Example: Inviscid flow around NACA0012 airfoil at $M = 0.5$, $\alpha = 0^\circ$



z_1 isolines of the discrete adjoint solution $\mathbf{z}_h$ for C_{dp} for $J(\mathbf{u}_h) = \int_{\Gamma_W} p(\mathbf{u}_h)\,\mathbf{n}\cdot\psi\,\mathrm{d}s$ (**adjoint inconsistent**)



z_1 isolines of the discrete adjoint solution $\mathbf{z}_h$ for C_{dp} for $J_h(\mathbf{u}_h) = \int_{\Gamma_W} \hat{\mathbf{h}}_{\Gamma,h}\cdot\tilde{\psi}\,\mathrm{d}s$ (**adjoint consistent**).

Outline

The compressible Navier-Stokes equations

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x_1}\mathbf{f}_1^c(\mathbf{u}) + \frac{\partial}{\partial x_2}\mathbf{f}_2^c(\mathbf{u}) - \frac{\partial}{\partial x_1}\mathbf{f}_1^v(\mathbf{u}, \nabla\mathbf{u}) - \frac{\partial}{\partial x_2}\mathbf{f}_2^v(\mathbf{u}, \nabla\mathbf{u}) = 0$$

The compressible Navier-Stokes equations

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x_1}\mathbf{f}_1^c(\mathbf{u}) + \frac{\partial}{\partial x_2}\mathbf{f}_2^c(\mathbf{u}) - \frac{\partial}{\partial x_1}\mathbf{f}_1^v(\mathbf{u},\nabla\mathbf{u}) - \frac{\partial}{\partial x_2}\mathbf{f}_2^v(\mathbf{u},\nabla\mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla\cdot\mathcal{F}^c(\mathbf{u}) - \nabla\cdot\mathcal{F}^v(\mathbf{u},\nabla\mathbf{u}) = 0$$

The compressible Navier-Stokes equations

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x_1}\mathbf{f}_1^c(\mathbf{u}) + \frac{\partial}{\partial x_2}\mathbf{f}_2^c(\mathbf{u}) - \frac{\partial}{\partial x_1}\mathbf{f}_1^v(\mathbf{u}, \nabla\mathbf{u}) - \frac{\partial}{\partial x_2}\mathbf{f}_2^v(\mathbf{u}, \nabla\mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla\cdot\mathcal{F}^c(\mathbf{u}) - \nabla\cdot\mathcal{F}^v(\mathbf{u}, \nabla\mathbf{u}) = 0$$

We consider the steady state equations

$$\nabla\cdot\mathcal{F}^c(\mathbf{u}) - \nabla\cdot\mathcal{F}^v(\mathbf{u}, \nabla\mathbf{u}) = 0,$$

with the no-slip wall boundary decomposed in isothermal and adiabiatic boundaries $\Gamma_W = \Gamma_{\text{iso}} \cup \Gamma_{\text{adia}}$ and following boundary conditions imposed

$$\mathbf{v} = 0 \text{ on } \Gamma_W, \qquad T = T_{\text{wall}} \text{ on } \Gamma_{\text{iso}}, \qquad \mathbf{n}\cdot\nabla T = 0 \text{ on } \Gamma_{\text{adia}}.$$

The adjoint equations

Primal problem:

$$\nabla \cdot \mathcal{F}^c(\mathbf{u}) - \nabla \cdot \mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u}) = 0 \quad \text{on } \Omega,$$

with adiabatic or isothermal wall boundary conditions.
Target quantity: Total drag or lift coefficient:

$$J(\mathbf{u}) = \int_\Gamma j(\mathbf{u})\,\mathrm{d}s = \int_{\Gamma_W} (p\,\mathbf{n} - \underline{\tau}\,\mathbf{n}) \cdot \psi\,\mathrm{d}s$$

Adjoint problem:

$$-\left(\mathcal{F}^c_{\mathbf{u}} - \mathcal{F}^v_{\mathbf{u}}\right)^\top \nabla \mathbf{z} - \nabla \cdot \left(\left(\mathcal{F}^v_{\nabla \mathbf{u}}\right)^\top \nabla \mathbf{z}\right) = 0,$$

subject to boundary conditions

$$z_2 = \psi_1,\ z_3 = \psi_2 \ \text{ on } \Gamma_W, \qquad z_4 = 0 \ \text{ on } \Gamma_{\text{iso}}, \qquad \mathbf{n} \cdot \nabla z_4 = 0 \ \text{ on } \Gamma_{\text{adia}}.$$

Outline

1. Consistency and adjoint consistency
   - Derivation of the adjoint problem
2. DG discretization of the compressible Euler equations
   - The compressible Euler and its adjoint equations
   - The DG discretization
3. DG discretization of the compressible Navier-Stokes equations
   - The compressible Navier-Stokes and its adjoint equations
   - The DG discretization
4. Adjoint-based error estimation and adaptive mesh refinement
   - Error estimation and adaptive mesh refinement
   - Residual-based mesh refinement
5. Numerical results

DG discretization of the viscous part of the Navier-Stokes equations

$$-\nabla \cdot \mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u}) = -\nabla \cdot \left(G(\mathbf{u})\nabla \mathbf{u}\right) = 0 \quad \text{in } \Omega,$$

System of first order equations

$$\underline{\sigma} = G(\mathbf{u})\nabla \mathbf{u}, \qquad\qquad -\nabla \cdot \underline{\sigma} = 0 \quad \text{in } \Omega.$$

Similar to for Poisson's equation we obtain: find $\mathbf{u}_h \in \mathbf{V}_h^p$ such that

$$\int_\Omega G(\mathbf{u}_h)\nabla_h \mathbf{u}_h : \nabla_h \mathbf{v}_h \,\mathrm{d}\mathbf{x} - \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \underline{\hat{\sigma}}_h : \mathbf{v}_h \otimes \mathbf{n} \,\mathrm{d}s$$
$$+ \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} (\hat{\mathbf{u}}_h - \mathbf{u}_h) \otimes \mathbf{n} : \left(G^\top(\mathbf{u}_h)\nabla \mathbf{v}_h\right) \,\mathrm{d}s = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h^p,$$

with numerical flux functions

$$\hat{\mathbf{u}}_h = \hat{\mathbf{u}}(\mathbf{u}_h) = \hat{\mathbf{u}}(\mathbf{u}_h^+, \mathbf{u}_h^-), \qquad \hat{\mathbf{u}}_h|_\Gamma = \hat{\mathbf{u}}_{\Gamma,h} = \hat{\mathbf{u}}_\Gamma(\mathbf{u}_h^+),$$
$$\underline{\hat{\sigma}}_h = \underline{\hat{\sigma}}(\mathbf{u}_h, \nabla \mathbf{u}_h) = \underline{\hat{\sigma}}(\mathbf{u}_h^+, \mathbf{u}_h^-, \nabla \mathbf{u}_h^+, \nabla \mathbf{u}_h^-), \qquad \underline{\hat{\sigma}}_h|_\Gamma = \underline{\hat{\sigma}}_{\Gamma,h} = \underline{\hat{\sigma}}_\Gamma(\mathbf{u}_h^+, \nabla \mathbf{u}_h^+).$$

DG discretization of the compressible Navier-Stokes equations

Combine with the discretization of the compressible Euler equations to get

$$N_h(\mathbf{u}_h, \mathbf{v}_h) \equiv \int_\Omega \left(-\mathcal{F}^c(\mathbf{u}_h) + \mathcal{F}^v(\mathbf{u}_h, \nabla_h \mathbf{u}_h)\right) : \nabla_h \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \left(\hat{\mathbf{h}}_h - \underline{\hat{\sigma}}_h \mathbf{n}\right) \cdot \mathbf{v}_h \,\mathrm{d}s$$
$$+ \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} (\hat{\mathbf{u}}_h - \mathbf{u}_h) \otimes \mathbf{n} : \left(G^\top(\mathbf{u}_h)\nabla \mathbf{v}_h\right) \,\mathrm{d}s = 0,$$

with $\hat{\mathbf{h}}|_\Gamma = \hat{\mathbf{h}}_{\Gamma,h} = \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+, \mathbf{n})$ and $\underline{\hat{\sigma}}_h|_\Gamma = \underline{\hat{\sigma}}_{\Gamma,h} = \underline{\hat{\sigma}}_\Gamma(\mathbf{u}_h^+, \nabla \mathbf{u}_h^+)$.

DG discretization of the compressible Navier-Stokes equations

Combine with the discretization of the compressible Euler equations to get

$$N_h(\mathbf{u}_h, \mathbf{v}_h) \equiv \int_\Omega (-\mathcal{F}^c(\mathbf{u}_h) + \mathcal{F}^v(\mathbf{u}_h, \nabla_h \mathbf{u}_h)) : \nabla_h \mathbf{v}_h \, \mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \left(\hat{\mathbf{h}}_h - \hat{\underline{\sigma}}_h \mathbf{n}\right) \cdot \mathbf{v}_h \, \mathrm{d}s$$

$$+ \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} (\hat{\mathbf{u}}_h - \mathbf{u}_h) \otimes \mathbf{n} : \left(G^\top(\mathbf{u}_h)\nabla \mathbf{v}_h\right) \, \mathrm{d}s = 0,$$

with $\hat{\mathbf{h}}|_\Gamma = \hat{\mathbf{h}}_{\Gamma,h} = \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+, \mathbf{n})$ and $\hat{\underline{\sigma}}_h|_\Gamma = \hat{\underline{\sigma}}_{\Gamma,h} = \hat{\underline{\sigma}}_\Gamma(\mathbf{u}_h^+, \nabla \mathbf{u}_h^+)$.
The (compatible) target quantity

$$J(\mathbf{u}) = \int_{\Gamma_W} (p\,\mathbf{n} - \underline{\tau}\,\mathbf{n}) \cdot \boldsymbol{\psi} \, \mathrm{d}s$$

Task: Find a discretization $J_h(\mathbf{u}_h)$ of $J(\mathbf{u})$ which is **consistent** and which (in combination with N_h) is **adjoint consistent**.

DG discretization of the compressible Navier-Stokes equations

Combine with the discretization of the compressible Euler equations to get

$$N_h(\mathbf{u}_h, \mathbf{v}_h) \equiv \int_\Omega \left(-\mathcal{F}^c(\mathbf{u}_h) + \mathcal{F}^v(\mathbf{u}_h, \nabla_h \mathbf{u}_h)\right) : \nabla_h \mathbf{v}_h \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \left(\hat{\mathbf{h}}_h - \hat{\underline{\sigma}}_h \mathbf{n}\right) \cdot \mathbf{v}_h \, \mathrm{d}s$$

$$+ \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} (\hat{\mathbf{u}}_h - \mathbf{u}_h) \otimes \mathbf{n} : \left(G^\top(\mathbf{u}_h) \nabla \mathbf{v}_h\right) \, \mathrm{d}s = 0,$$

with $\hat{\mathbf{h}}|_\Gamma = \hat{\mathbf{h}}_{\Gamma,h} = \hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+, \mathbf{n})$ and $\hat{\underline{\sigma}}_h|_\Gamma = \hat{\underline{\sigma}}_{\Gamma,h} = \hat{\underline{\sigma}}_\Gamma(\mathbf{u}_h^+, \nabla \mathbf{u}_h^+)$.
The (compatible) target quantity

$$J(\mathbf{u}) = \int_{\Gamma_W} (p\,\mathbf{n} - \underline{\tau}\,\mathbf{n}) \cdot \boldsymbol{\psi} \, \mathrm{d}s$$

Task: Find a discretization $J_h(\mathbf{u}_h)$ of $J(\mathbf{u})$ which is **consistent** and which (in combination with N_h) is **adjoint consistent**.

$$J_h(\mathbf{u}_h) = \int_{\Gamma_W} \left(\hat{\mathbf{h}}_{\Gamma,h} - \hat{\underline{\sigma}}_{\Gamma,h} \mathbf{n}\right) \cdot \tilde{\boldsymbol{\psi}} \, \mathrm{d}s,$$

with $\tilde{\boldsymbol{\psi}} = (0, \psi_1, \psi_2, 0)^\top$ for $\boldsymbol{\psi} = (\psi_1, \psi_2)^\top$.

DG discretization of the compressible Navier-Stokes equations

Consider the target quantity and its discretization

$$J(\mathbf{u}) = \int_{\Gamma_W} (p\,\mathbf{n} - \underline{\tau}\,\mathbf{n}) \cdot \boldsymbol{\psi}\,\mathrm{d}s, \qquad J_h(\mathbf{u}_h) = \int_{\Gamma_W} \left(\hat{\mathbf{h}}_{\Gamma,h} - \underline{\hat{\sigma}}_{\Gamma,h}\mathbf{n}\right) \cdot \tilde{\boldsymbol{\psi}}\,\mathrm{d}s,$$

with $\tilde{\boldsymbol{\psi}} = (0, \psi_1, \psi_2, 0)^\top$ for $\boldsymbol{\psi} = (\psi_1, \psi_2)^\top$.
Assume $\hat{\mathbf{h}}_\Gamma$ and $\underline{\hat{\sigma}}_\Gamma$ are consistent. Then, $J_h(\mathbf{u}_h)$ is a **consistent** discretization of $J(\mathbf{u})$, as the exact solution $\mathbf{u}$ satisfies

$$\left(\hat{\mathbf{h}}_\Gamma(\mathbf{u},\mathbf{n}) - (\underline{\hat{\sigma}}_\Gamma(\mathbf{u},\nabla\mathbf{u})\mathbf{n})\right) \cdot \tilde{\boldsymbol{\psi}} = (\mathbf{n}\cdot\mathcal{F}^c(\mathbf{u}) - \mathbf{n}\cdot\mathcal{F}^v(\mathbf{u},\nabla\mathbf{u}))\cdot\tilde{\boldsymbol{\psi}} = (p\mathbf{n} - \tau\mathbf{n})\cdot\boldsymbol{\psi},$$

due to $\mathbf{n}\cdot\mathcal{F}^c(\mathbf{u}) = (0, pn_1, pn_2, 0)^\top$ and
$\mathbf{n}\cdot\mathcal{F}^v(\mathbf{u},\nabla\mathbf{u}) = (0, (\tau\mathbf{n})_1, (\tau\mathbf{n})_2, \mathcal{K}\mathbf{n}\cdot\nabla T)^\top$ on Γ_W. Thus $J_h(\mathbf{u}) = J(\mathbf{u})$.

DG discretization of the compressible Navier-Stokes equations

Consider the target quantity and its discretization

$$J(\mathbf{u}) = \int_{\Gamma_W} (p\,\mathbf{n} - \underline{\tau}\,\mathbf{n}) \cdot \psi \,\mathrm{d}s, \qquad J_h(\mathbf{u}_h) = \int_{\Gamma_W} \left(\hat{\mathbf{h}}_{\Gamma,h} - \underline{\hat{\sigma}}_{\Gamma,h}\mathbf{n}\right) \cdot \tilde{\psi} \,\mathrm{d}s,$$

with $\tilde{\boldsymbol{\psi}} = (0, \psi_1, \psi_2, 0)^\top$ for $\boldsymbol{\psi} = (\psi_1, \psi_2)^\top$.
Assume $\hat{\mathbf{h}}_\Gamma$ and $\underline{\hat{\sigma}}_\Gamma$ are consistent. Then, $J_h(\mathbf{u}_h)$ is a **consistent** discretization of $J(\mathbf{u})$, as the exact solution $\mathbf{u}$ satisfies

$$\left(\hat{\mathbf{h}}_\Gamma(\mathbf{u}, \mathbf{n}) - (\underline{\hat{\sigma}}_\Gamma(\mathbf{u}, \nabla\mathbf{u})\mathbf{n})\right) \cdot \tilde{\psi} = (\mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}) - \mathbf{n} \cdot \mathcal{F}^v(\mathbf{u}, \nabla\mathbf{u})) \cdot \tilde{\psi} = (p\mathbf{n} - \tau\mathbf{n}) \cdot \boldsymbol{\psi},$$

due to $\mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}) = (0, pn_1, pn_2, 0)^\top$ and
$\mathbf{n} \cdot \mathcal{F}^v(\mathbf{u}, \nabla\mathbf{u}) = (0, (\tau\mathbf{n})_1, (\tau\mathbf{n})_2, \mathcal{K}\mathbf{n} \cdot \nabla T)^\top$ on Γ_W. Thus $J_h(\mathbf{u}) = J(\mathbf{u})$.

Furthermore, one can show (cf. Theorem 6.9) that N_h in combination with J_h is **adjoint consistent**.

Numerical flux functions

For SIPG and BR2 the fluxes are given by

$$\hat{\mathbf{u}}_h = \{\!\{\mathbf{u}_h\}\!\}, \qquad \hat{\underline{\sigma}}_h = \{\!\{G(\mathbf{u}_h)\nabla_h \mathbf{u}_h\}\!\} - \underline{\delta}(\mathbf{u}_h) \qquad \text{on } \Gamma_{\mathcal{I}},$$

with

$$\underline{\delta}(\mathbf{u}_h) = C_{\mathrm{IP}} \frac{p^2}{h_e} \mu \underline{[\![\mathbf{u}_h]\!]} \qquad \text{for IP (Hartmann \& Houston, 2006a),}$$

$$\underline{\delta}(\mathbf{u}_h) = C_{\mathrm{IP}} \frac{p^2}{h_e} \{\!\{G(\mathbf{u}_h)\}\!\} \underline{[\![\mathbf{u}_h]\!]} \qquad \text{for IP (Hartmann \& Houston, 2008),}$$

$$\underline{\delta}(\mathbf{u}_h) = C_{\mathrm{BR2}} \{\!\{G(\mathbf{u}_h) \underline{L}_0^e(\mathbf{u}_h)\}\!\} \qquad \text{for BR2 (Bassi et al. 2005),}$$

$$\underline{\delta}(\mathbf{u}_h) = C_{\mathrm{BR2}} \{\!\{\underline{\tilde{L}}_0^e(\mathbf{u}_h)\}\!\} \qquad \text{for BR2 (Bassi \& Rebay, 2000a, 2002).}$$

Numerical flux functions

For SIPG and BR2 the fluxes are given by

$$\hat{\mathbf{u}}_h = \{\!\{\mathbf{u}_h\}\!\}, \qquad \underline{\hat{\sigma}}_h = \{\!\{G(\mathbf{u}_h)\nabla_h\mathbf{u}_h\}\!\} - \underline{\delta}(\mathbf{u}_h) \qquad \text{on } \Gamma_{\mathcal{I}},$$

with

$$\underline{\delta}(\mathbf{u}_h) = C_{\text{IP}}\frac{p^2}{h_e}\mu\underline{[\![\mathbf{u}_h]\!]} \qquad \text{for IP (Hartmann \& Houston, 2006a),}$$

$$\underline{\delta}(\mathbf{u}_h) = C_{\text{IP}}\frac{p^2}{h_e}\{\!\{G(\mathbf{u}_h)\}\!\}\underline{[\![\mathbf{u}_h]\!]} \qquad \text{for IP (Hartmann \& Houston, 2008),}$$

$$\underline{\delta}(\mathbf{u}_h) = C_{\text{BR2}}\{\!\{G(\mathbf{u}_h)\underline{L}_0^e(\mathbf{u}_h)\}\!\} \qquad \text{for BR2 (Bassi et al. 2005),}$$

$$\underline{\delta}(\mathbf{u}_h) = C_{\text{BR2}}\{\!\{\underline{\tilde{L}}_0^e(\mathbf{u}_h)\}\!\} \qquad \text{for BR2 (Bassi \& Rebay, 2000a, 2002).}$$

Then the DG discretization is given by: find $\mathbf{u}_h \in \mathbf{V}_h^p$ such that

$$\begin{aligned} N_h(\mathbf{u}_h, \mathbf{v}_h) = {}& \int_\Omega \left(-\mathcal{F}^c(\mathbf{u}_h) + \mathcal{F}^v(\mathbf{u}_h, \nabla_h\mathbf{u}_h)\right) : \nabla_h\mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma} \mathbf{h}_h\cdot\mathbf{v}_h \,\mathrm{d}s \\ & - \int_{\Gamma_{\mathcal{I}}} \underline{[\![\mathbf{u}_h]\!]} : \{\!\{G^\top(\mathbf{u}_h)\nabla\mathbf{v}_h\}\!\}\,\mathrm{d}s - \int_{\Gamma_{\mathcal{I}}} \{\!\{G(\mathbf{u}_h)\nabla_h\mathbf{u}_h\}\!\} : \underline{[\![\mathbf{v}_h]\!]} \\ & + \int_{\Gamma_{\mathcal{I}}} \underline{\delta}(\mathbf{u}_h) : \underline{[\![\mathbf{v}_h]\!]}\,\mathrm{d}s + N_{\Gamma,h}(\mathbf{u}_h,\mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h^p. \end{aligned}$$

With a numerical flux function at the boundary . . .

1. . . . based on the *normal boundary flux*

$$\hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+,\mathbf{n}) = \mathbf{n}\cdot\mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)),\quad \hat{\mathbf{u}}_{\Gamma,h} = \mathbf{u}_\Gamma(\mathbf{u}_h^+),\quad \hat{\underline{\sigma}}_{\Gamma,h} = \tilde{\mathcal{F}}^v(\mathbf{u}_\Gamma(\mathbf{u}_h^+),\nabla\mathbf{u}_h^+) - \underline{\delta}_\Gamma(\mathbf{u}_h^+),$$

the discretization at the boundary is given by

$$\begin{aligned} N_{\Gamma_W,h}(\mathbf{u}_h,\mathbf{v}_h) = &\int_{\Gamma_W} \mathbf{n}\cdot\left(\mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)) - \tilde{\mathcal{F}}^v(\mathbf{u}_\Gamma(\mathbf{u}_h^+),\nabla\mathbf{u}_h^+) + \underline{\delta}_\Gamma(\mathbf{u}_h^+)\right)\cdot\mathbf{v}_h^+\,\mathrm{d}s \\ &- \int_{\Gamma_W}\left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\right)\otimes\mathbf{n} : \left(G^\top(\mathbf{u}_h^+)\nabla\mathbf{v}_h^+\right)\,\mathrm{d}s \end{aligned}$$

(a) This discretization is **adjoint consistent** in combination with

$$J_h(\mathbf{u}_h) = \int_{\Gamma_W}\left(\hat{\mathbf{h}}_{\Gamma,h} - \hat{\underline{\sigma}}_{\Gamma,h}\mathbf{n}\right)\cdot\tilde{\psi}\,\mathrm{d}s = J(\mathbf{u}_\Gamma(\mathbf{u}_h^+)) + \int_{\Gamma_W}\left(\mathbf{n}\cdot\underline{\delta}_\Gamma(\mathbf{u}_h^+)\right)\cdot\tilde{\psi}\,\mathrm{d}s,$$

with $J(\mathbf{u}) = \int_{\Gamma_W}(p\,\mathbf{n} - \underline{\tau}\,\mathbf{n})\cdot\psi\,\mathrm{d}s.$

(b) It is **adjoint inconsistent** in combination with any other discretization, like

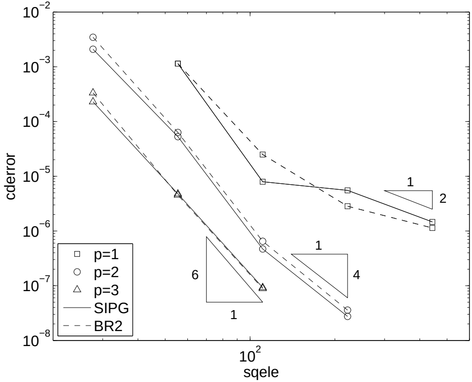
$$J(\mathbf{u}_h),\qquad \text{or}\qquad J(\mathbf{u}_\Gamma(\mathbf{u}_h^+)).$$

Example: $M = 0.5$, $\alpha = 0°$, $Re = 5000$ viscous flow, NACA0012 airfoil

Target quantity: Total drag coefficient: $J(\mathbf{u}) = \int_{\Gamma_W} (p\,\mathbf{n} - \underline{\tau}\,\mathbf{n}) \cdot \boldsymbol{\psi}\,\mathrm{d}s$

Adjoint consistent discretization:

$$J_h(\mathbf{u}_h) = \int_{\Gamma_W} \left(\hat{\mathbf{h}}_{\Gamma,h} - \underline{\hat{\sigma}}_{\Gamma,h}\mathbf{n}\right) \cdot \tilde{\boldsymbol{\psi}}\,\mathrm{d}s = J(\mathbf{u}_\Gamma(\mathbf{u}_h^+)) + \int_{\Gamma_W} \left(\mathbf{n} \cdot \underline{\delta}_\Gamma(\mathbf{u}_h^+)\right) \cdot \tilde{\boldsymbol{\psi}}\,\mathrm{d}s$$



The error $|J(\mathbf{u}) - J_h(\mathbf{u}_h)|$ of SIPG(p) and BR2(p) is of $\mathcal{O}(h^{2p})$

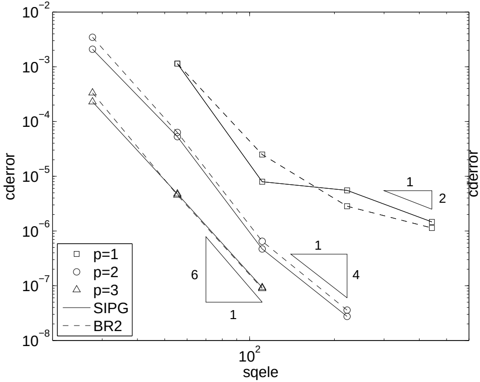
adjoint consistent

Example: $M = 0.5$, $\alpha = 0°$, $Re = 5000$ viscous flow, NACA0012 airfoil

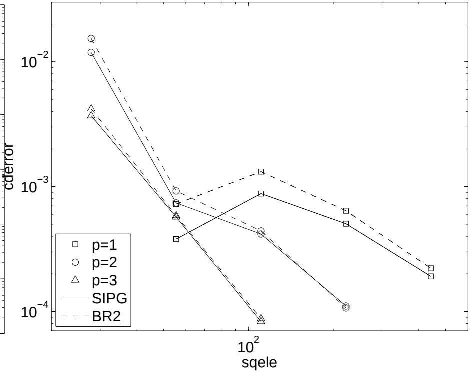
Target quantity: Total drag coefficient: $J(\mathbf{u}) = \int_{\Gamma_W} (p\,\mathbf{n} - \underline{\tau}\,\mathbf{n}) \cdot \boldsymbol{\psi}\, ds$

Adjoint consistent discretization:

$$J_h(\mathbf{u}_h) = \int_{\Gamma_W} \left(\hat{\mathbf{h}}_{\Gamma,h} - \underline{\hat{\sigma}}_{\Gamma,h}\mathbf{n} \right) \cdot \tilde{\boldsymbol{\psi}}\, ds = J(\mathbf{u}_\Gamma(\mathbf{u}_h^+)) + \int_{\Gamma_W} \left(\mathbf{n} \cdot \underline{\delta}_\Gamma(\mathbf{u}_h^+) \right) \cdot \tilde{\boldsymbol{\psi}}\, ds$$



The error $|J(\mathbf{u}) - J_h(\mathbf{u}_h)|$ of SIPG(p) and BR2(p) is of $\mathcal{O}(h^{2p})$
adjoint consistent



The error $|J(\mathbf{u}) - J(\mathbf{u}_\Gamma(\mathbf{u}_h))|$ of SIPG(p) and BR2(p) is of reduced order
adjoint inconsistent

With a numerical flux function at the boundary ...

1. ... based on the *interior numerical fluxes*

$$\hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+,\mathbf{n}) = \hat{\mathbf{h}}(\mathbf{u}_h^+,\mathbf{u}_\Gamma^-(\mathbf{u}_h^+),\mathbf{n}), \quad \hat{\mathbf{u}}_{\Gamma,h} = \mathbf{u}_\Gamma(\mathbf{u}_h^+), \quad \underline{\hat{\sigma}}_{\Gamma,h} = \{\!\{\tilde{\mathcal{F}}^v(\mathbf{u}_h,\nabla\mathbf{u}_h)\}\!\}_\Gamma - \underline{\tilde{\delta}}_\Gamma(\mathbf{u}_h^+),$$

where $\{\!\{\cdot\}\!\}_\Gamma$ denotes the mean value of a function evaluated at the interior state $\mathbf{u}_h^+$ and the (mirrored) *boundary exterior state* $\mathbf{u}_\Gamma^-(\mathbf{u}_h^+)$ given by

$$\frac{1}{2}\left(\mathbf{u}_h^+ + \mathbf{u}_\Gamma^-(\mathbf{u}_h^+)\right) = \mathbf{u}_\Gamma(\mathbf{u}_h^+), \qquad \text{i.e., } \mathbf{u}_\Gamma^-(\mathbf{u}_h^+) = 2\mathbf{u}_\Gamma(\mathbf{u}_h^+) - \mathbf{u}_h^+,$$

the discretization at the boundary is given by

$$N_{\Gamma_W,h}(\mathbf{u}_h,\mathbf{v}_h) = \int_{\Gamma_W}\left(\hat{\mathbf{h}}(\mathbf{u}_h^+,\mathbf{u}_\Gamma^-(\mathbf{u}_h^+),\mathbf{n}) - \{\!\{\tilde{\mathcal{F}}^v(\mathbf{u}_h,\nabla\mathbf{u}_h)\}\!\}_\Gamma + \underline{\tilde{\delta}}_\Gamma(\mathbf{u}_h^+)\right)\cdot\mathbf{v}_h^+\,\mathrm{d}s$$
$$- \int_{\Gamma_W}\left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\right)\otimes\mathbf{n} : \left(G^\top(\mathbf{u}_h^+)\nabla\mathbf{v}_h^+\right)\,\mathrm{d}s$$

(a) This discretization is **adjoint consistent** in combination with

$$J_h(\mathbf{u}_h) = \int_{\Gamma_W}\left(\hat{\mathbf{h}}_{\Gamma,h} - \underline{\hat{\sigma}}_{\Gamma,h}\mathbf{n}\right)\cdot\tilde{\psi}\,\mathrm{d}s$$

(b) It is **adjoint inconsistent** in combination with any other $J_h(\mathbf{u}_h)$.

Outline

Error estimation for nonlinear problems

Discretization: find $\mathbf{u}_h \in \mathbf{V}_h^p$ such that

$$N_h(\mathbf{u}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h^p.$$

Error representation:

$$J(\mathbf{u}) - J_h(\mathbf{u}_h) = R_h(\mathbf{u}_h, \mathbf{z}),$$

where z is the exact (but unknown) solution to the adjoint equations. Replace $\mathbf{z}$ by the solution to following discrete adjoint problem: Find $\bar{\mathbf{z}}_h \in \bar{\mathbf{V}}_h^p$ such that

$$N_h'[\mathbf{u}_h](\mathbf{w}_h, \bar{\mathbf{z}}_h) = J'[\mathbf{u}_h](\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \bar{\mathbf{V}}_h^p.$$

We obtain the error estimate (approximate error representation):

$$J(\mathbf{u}) - J(\mathbf{u}_h) \approx R_h(\mathbf{u}_h, \bar{\mathbf{z}}_h) = \sum_{\kappa \in \mathcal{T}_h} \bar{\eta}_\kappa.$$

Error estimation for nonlinear problems

Discretization: find $\mathbf{u}_h \in \mathbf{V}_h^p$ such that

$$N_h(\mathbf{u}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h^p.$$

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We obtain the error estimate (approximate error representation):

$$J(\mathbf{u}) - J(\mathbf{u}_h) \approx R_h(\mathbf{u}_h, \bar{\mathbf{z}}_h) = \sum_{\kappa \in \mathcal{T}_h} \bar{\eta}_\kappa.$$

Note, that $R_h(\mathbf{u}_h, \mathbf{z}_h) = -N_h(\mathbf{u}_h, \mathbf{z}_h) = 0$ for any $\mathbf{z}_h \in \mathbf{V}_h^p$. Thereby,

$$R_h(\mathbf{u}_h, \bar{\mathbf{z}}_h) = R_h(\mathbf{u}_h, \bar{\mathbf{z}}_h - \mathbf{z}_h) = \begin{cases} 0 & \text{for } \bar{\mathbf{z}}_h \in \mathbf{V}_h^p, \\ E_h \neq 0 & \text{for } \bar{\mathbf{z}}_h \in \bar{\mathbf{V}}_h^p \not\subset \mathbf{V}_h^p. \end{cases}$$

Take, for example, $\bar{\mathbf{V}}_h^p = \mathbf{V}_h^{\bar{p}}$, with $\bar{p} = p + 1$, on the same mesh $\mathcal{T}_h$.

Single-target adaptive algorithm

.. for the accurate and efficient approximation of a single target quantity $J(\mathbf{u})$.
The error estimate:

$$J(\mathbf{u}) - J(\mathbf{u}_h) \approx R_h(\mathbf{u}_h, \bar{\mathbf{z}}_h) = \sum_{\kappa \in \mathcal{T}_h} \bar{\eta}_\kappa$$

includes the so-called *adjoint-based* indicators $\bar{\eta}_\kappa$.

Algorithm:

1. Construct an initial mesh $\mathcal{T}_h$.
2. Compute $\mathbf{u}_h \in \mathbf{V}_h^p$ on the current mesh $\mathcal{T}_h$.
3. Compute $\bar{\mathbf{z}}_h \in \bar{\mathbf{V}}_h^p = \mathbf{V}_h^{\bar{p}}$ on the same mesh employed for $\mathbf{u}_h$, with $\bar{p} = p + 1$.
4. Evaluate the approximate error representation $R_h(\mathbf{u}_h, \bar{\mathbf{z}}_h) = \sum_{\kappa \in \mathcal{T}_h} \bar{\eta}_\kappa$.
5. If $|\sum_{\kappa \in \mathcal{T}_h} \bar{\eta}_\kappa| \leq \texttt{TOL}$, where `TOL` is a given tolerance, then `STOP`.
6. Otherwise, refine and coarsen a fixed fraction of the total number of elements according to the size of $|\bar{\eta}_\kappa|$ and generate a new mesh $\mathcal{T}_h$; `GOTO` 2.

Outline

Derivation of residual-based indicators

The error representation:

$$J(\mathbf{u}) - J(\mathbf{u}_h) = R_h(\mathbf{u}_h, \mathbf{z}) = R_h(\mathbf{u}_h, \mathbf{z} - \mathbf{z}_h).$$

Choose $\mathbf{z}_h = \Pi_h \mathbf{z}$ and write R_h in primal residual form:

$$\begin{aligned} J(\mathbf{u}) - J(\mathbf{u}_h) &= \int_\Omega \mathbf{R}(\mathbf{u}_h) \cdot (\mathbf{z} - \Pi_h \mathbf{z}) \, \mathrm{d}\mathbf{x} \\ &\quad + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \mathbf{r}(\mathbf{u}_h) \cdot (\mathbf{z} - \Pi_h \mathbf{z})^+ + \underline{\rho}(\mathbf{u}_h) : \nabla (\mathbf{z} - \Pi_h \mathbf{z})^+ \, \mathrm{d}s \\ &\quad + \int_\Gamma \mathbf{r}_\Gamma(\mathbf{u}_h) \cdot (\mathbf{z} - \Pi_h \mathbf{z})^+ + \underline{\rho}_\Gamma(\mathbf{u}_h) : \nabla (\mathbf{z} - \Pi_h \mathbf{z})^+ \, \mathrm{d}s, \end{aligned}$$

Assume some smoothness properties of the adjoint solution, apply approximation estimates and obtain

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \le C \left(\sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^{(\mathrm{res})} \right)^2 \right)^{1/2}, \qquad \text{with}$$

$$\eta_\kappa^{\mathrm{res}} = h_\kappa \|\mathbf{R}(\mathbf{u}_h)\|_{L^2(\kappa)} + h_\kappa^{1/2} \|\mathbf{r}_{\partial\kappa}(\mathbf{u}_h)\|_{L^2(\partial\kappa)} + h_\kappa^{-1/2} \|\underline{\rho}_{\partial\kappa}(\mathbf{u}_h)\|_{L^2(\partial\kappa)}.$$

Numerical examples

Compare

- adjoint-based mesh refinement (using η_κ) against
- residual-based mesh refinement (using $\eta_\kappa^{(res)}$).

Investigate the accuracy of the error estimation

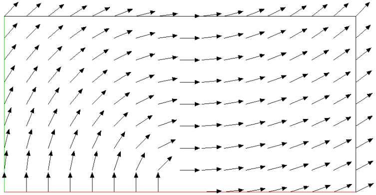
$$J(\mathbf{u}) - J(\mathbf{u}_h) = R_h(\mathbf{u}_h, \mathbf{z}) \approx R_h(\mathbf{u}_h, \bar{\mathbf{z}}_h).$$

Use the error estimate for improving/enhancing the computed target quantity $J_h(\mathbf{u}_h)$ as follows

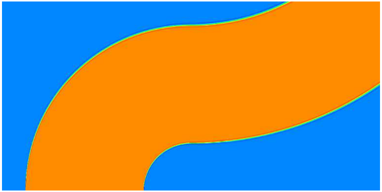
$$\widetilde{J}_h(\mathbf{u}_h) = J_h(\mathbf{u}_h) + R_h(\mathbf{u}_h, \bar{\mathbf{z}}_h).$$

Numerical example: Linear advection equation

$$
\begin{aligned}
Lu := \nabla \cdot (\mathbf{b}u) = 0 &\quad \text{in } \Omega = [0,2] \times [0,1] \in \mathbb{R}^2, \\
u = 1 &\quad \text{on } \left[\frac{1}{8}, \frac{3}{4}\right] \times \{0\} \\
u = 0 &\quad \text{elsewhere on } \Gamma_-.
\end{aligned}
$$



vector field $\mathbf{b}$



primal solution

Numerical example: Linear advection equation

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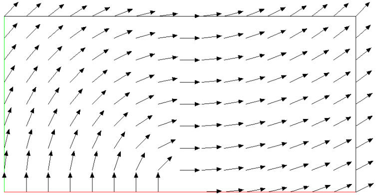
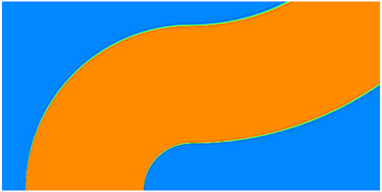


vector field **b**



primal solution

Interest in the solution on right boundary part: $\mathbf{x} \in \{2\} \times (\frac{1}{4}, 1)$.

Numerical example: Linear advection equation

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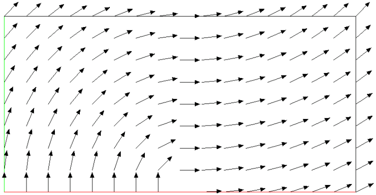
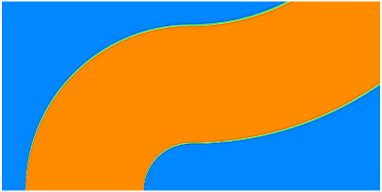
$$u = 0 \quad \text{elsewhere on } \Gamma_-.$$



vector field **b**



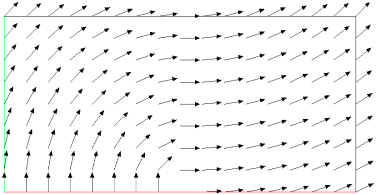
primal solution

Interest in the solution on right boundary part: $\mathbf{x} \in \{2\} \times (\frac{1}{4}, 1)$.

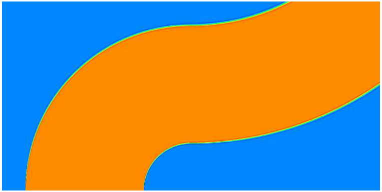
Define target quantity $J(u) = \int_{\Gamma_+} j_\Gamma \, u \, ds$

Numerical example: Linear advection equation

$$\begin{aligned} Lu := \nabla \cdot (\mathbf{b}u) &= 0 \quad \text{in } \Omega = [0,2] \times [0,1] \in \mathbb{R}^2, \\ u &= 1 \quad \text{on } \left[\frac{1}{8}, \frac{3}{4}\right] \times \{0\} \\ u &= 0 \quad \text{elsewhere on } \Gamma_-. \end{aligned}$$



vector field $\mathbf{b}$



primal solution

Interest in the solution on right boundary part: $\mathbf{x} \in \{2\} \times (\frac{1}{4}, 1)$.

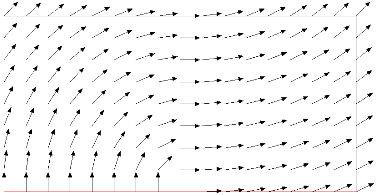
Define target quantity $J(u) = \int_{\Gamma_+} j_\Gamma \, u \, ds$, with

$j_\Gamma(2,y) = \exp\left(\left(\frac{3}{8}\right)^{-2} - \left((y - \frac{5}{8})^2 - \frac{3}{8}\right)^{-2} \right)$ for $\frac{1}{4} < y < 1$ and 0 elsewhere.

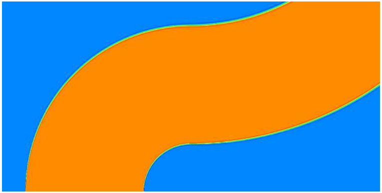
Numerical example: Linear advection equation

Target quantity: $J(u) = \int_{\Gamma_+} j_\Gamma \, u \, ds$
with $j_\Gamma \neq 0$ and smooth on
right outflow boundary

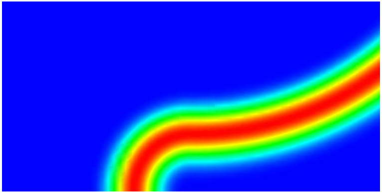
$$\begin{aligned} -\mathbf{b} \cdot \nabla z &= 0 \quad \text{in } \Omega, \\ \mathbf{b} \cdot \mathbf{n} \, z &= j_\Gamma \quad \text{on } \Gamma_+. \end{aligned}$$



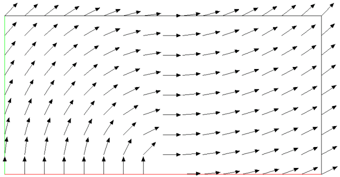
The primal and adjoint solutions:



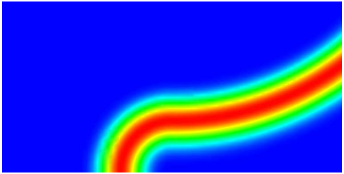
primal solution



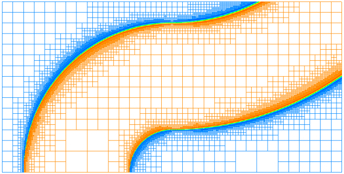
adjoint solution



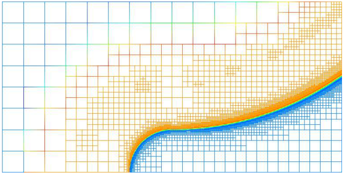
vector field **b**



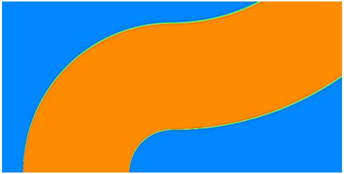
adjoint solution



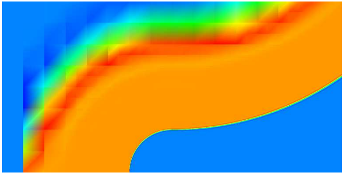
residual-based refined mesh



adjoint-based refined mesh



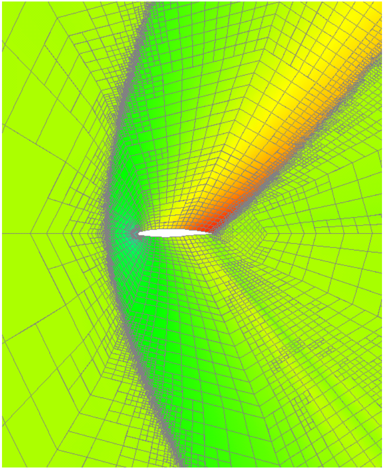
solution on residual-based refined mesh



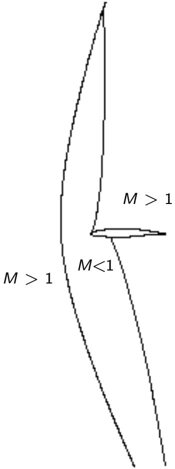
solution on adjoint-based refined mesh

Supersonic flow past a BAC3-11 airfoil

Inviscid flow at $M = 1.2$ and an angle $\alpha = 5^\circ$ past the BAC3-11 airfoil



Mach number on residual-based refined mesh



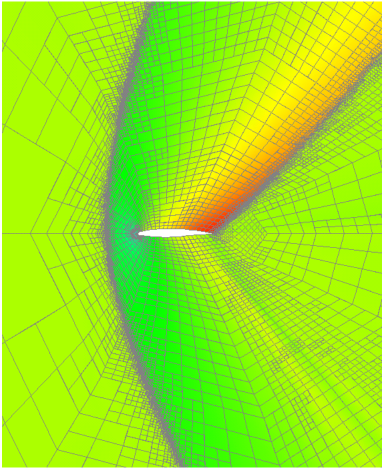
sonic lines ($M = 1$ lines)

Supersonic flow past a BAC3-11 airfoil

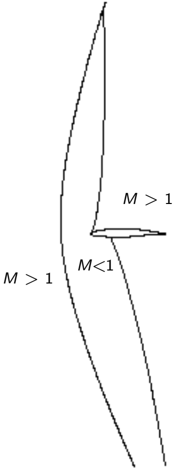
Inviscid flow at $M = 1.2$
and an angle $\alpha = 5^\circ$
past the BAC3-11 airfoil

Target quantity:
$J(u) = p(\mathbf{x}_0)$
(pressure at leading edge)

Problem: find pressure at leading edge to best accuracy.



Mach number on residual-based refined mesh



sonic lines
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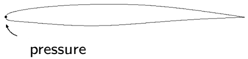

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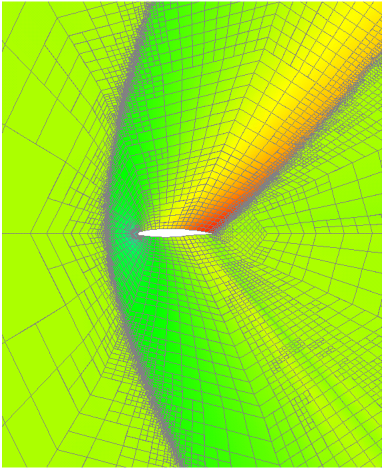
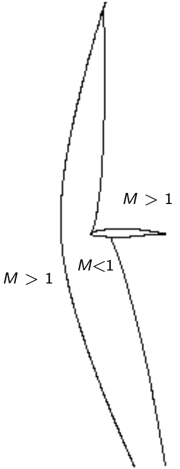


Mach number on
residual-based refined mesh

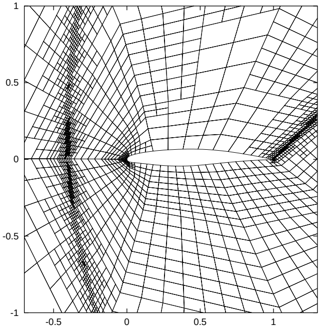


sonic lines
($M = 1$ lines)

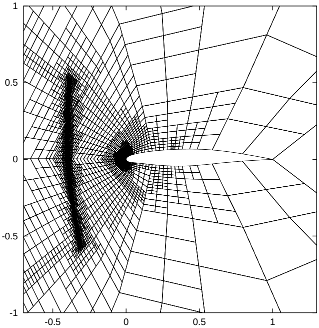
How to create an efficient mesh for this?

Supersonic flow past a BAC3-11 airfoil

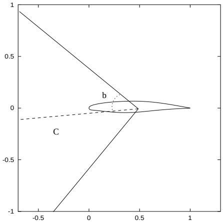
Inviscid flow at $M = 1.2$ and an angle $\alpha = 5^\circ$ past the BAC3-11 airfoil
Target quantity (pressure at leading edge): $J(\mathbf{u}) = p(\mathbf{x}_0)$
Reference value (fine mesh computation): $J(\mathbf{u}) = 2.393$



residual-based refined
13719 elements
$J(\mathbf{u}) - J(\mathbf{u}_h) = 3.5 \cdot 10^{-2}$

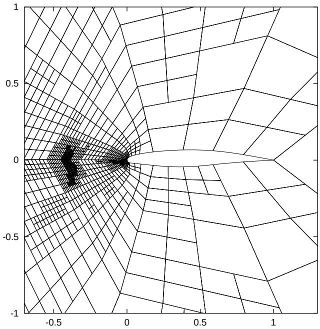


modified residual-based
9516 elements
$J(\mathbf{u}) - J(\mathbf{u}_h) = 7.9 \cdot 10^{-3}$

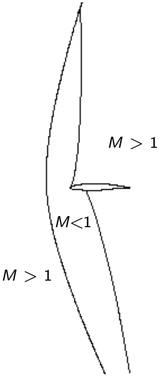


Supersonic flow past a BAC3-11 airfoil

Inviscid flow at $M = 1.2$ and an angle $\alpha = 5^\circ$ past the BAC3-11 airfoil
Target quantity (pressure at leading edge): $J(\mathbf{u}) = p(\mathbf{x}_0)$



adjoint-based refined
1803 elements
$J(\mathbf{u}) - J(\mathbf{u}_h) = 3.0 \cdot 10^{-3}$

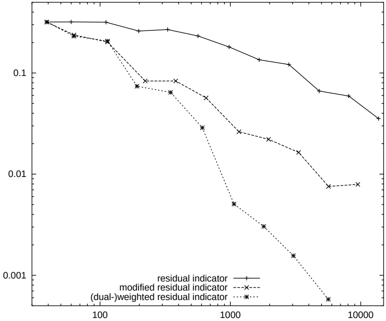


sonic lines
($M = 1$ isolines)

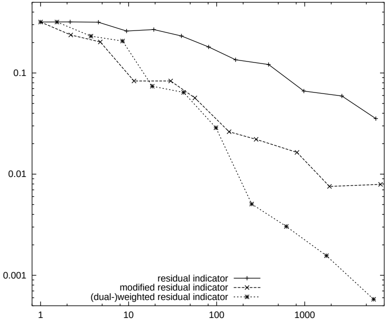
adjoint solution z_1

Supersonic flow past a BAC3-11 airfoil

Inviscid flow at $M = 1.2$ and an angle $\alpha = 5^\circ$ past the BAC3-11 airfoil
Target quantity (pressure at leading edge): $J(\mathbf{u}) = p(\mathbf{x}_0)$



$|J(\mathbf{u}) - J(\mathbf{u}_h)|$
over number of cells

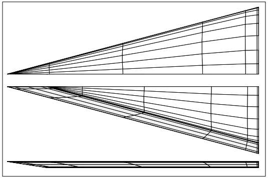


$|J(\mathbf{u}) - J(\mathbf{u}_h)|$
over number of time units

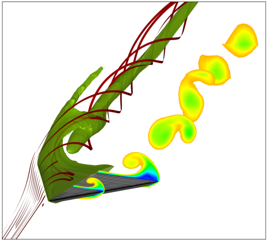
ADIGMA BTC3 test case

Laminar flow at $M = 0.3$, $Re = 4000$ and $\alpha = 12.5^\circ$ around a delta wing

Reference values by fine grid computations:
$C_\mathrm{l}^\mathrm{ref} = 0.34865$, $C_\mathrm{d}^\mathrm{ref} = 0.16608$, and $C_\mathrm{m}^\mathrm{ref} = -0.03065$



ADIGMA industrial accuracy requirements:
$\mathtt{TOL}_{C_\mathrm{l}} = 10^{-2}$, $\mathtt{TOL}_{C_\mathrm{d}} = \mathtt{TOL}_{C_\mathrm{m}} = 10^{-3}$

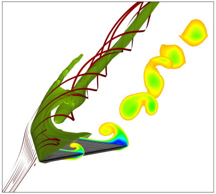


Performance of

- residual-based refinement
- adjoint-based refinement (single-target and multi-target)
- error estimation (single-target and multi-target)

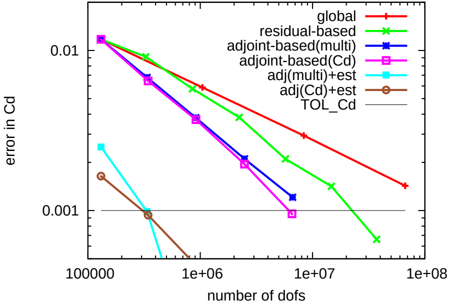
ADIGMA BTC3 test case

Laminar flow at $M = 0.3$, $Re = 4000$, $\alpha = 12.5^\circ$ around a delta wing

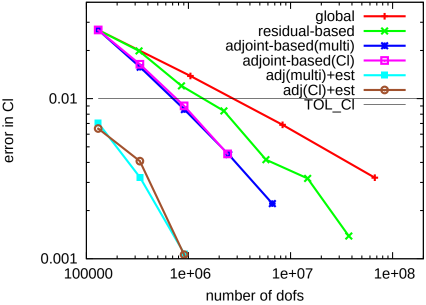


Multi-target adjoint-based mesh refinement for the sum of relative errors of C_l, C_d and C_m

Error in C_d:

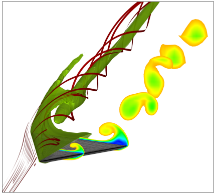


Error in C_l:



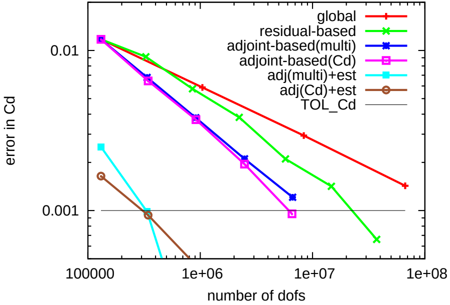
ADIGMA BTC3 test case

Laminar flow at $M = 0.3$, $Re = 4000$, $\alpha = 12.5^\circ$ around a delta wing

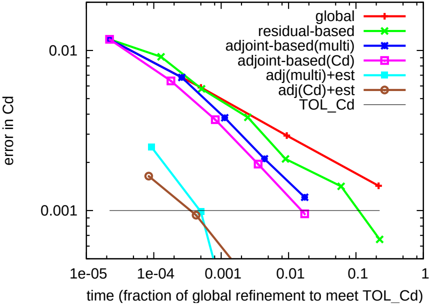


Multi-target adjoint-based mesh refinement for the sum of relative errors of C_l, C_d and C_m

Error in C_d:



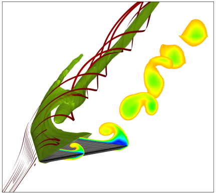
Error in C_d:

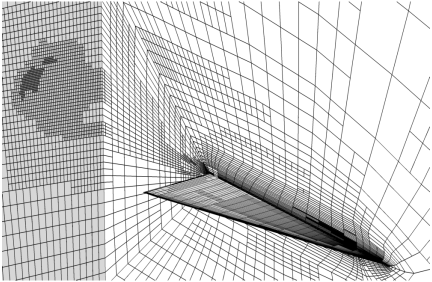


ADIGMA BTC3 test case

Laminar flow at $M = 0.3$, $Re = 4000$, $\alpha = 12.5^\circ$ around a delta wing

Multi-target adjoint-based mesh refinement for the sum of relative errors of C_l, C_d and C_m

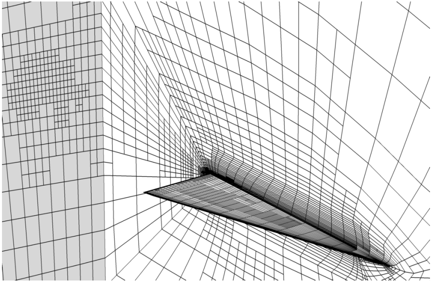




After 5 residual-based ref. steps: 14.7 mio. DoFs

Sum of relative errors: 5%

Rel. computing time: 0.06 (no error est.)

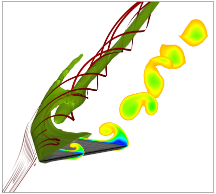


After 4 adjoint-based ref. steps: 6.6 mio. DoFs

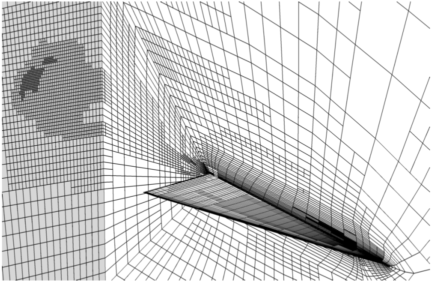
Sum of relative errors: 1.6%

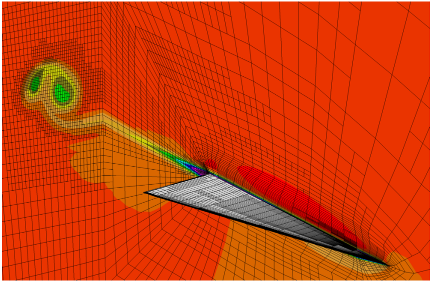
Rel. computing time: 0.017 (incl. error est.)

ADIGMA BTC3 test case



Laminar flow at $M = 0.3$, $Re = 4000$, $\alpha = 12.5^\circ$ around a delta wing

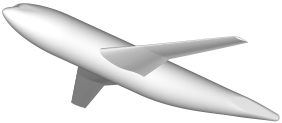




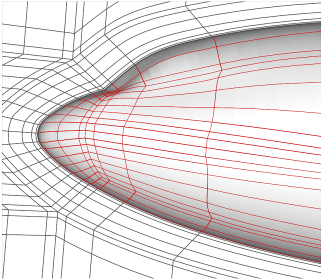
After 5 residual-based mesh refinement steps: 14.7 mio. DoFs

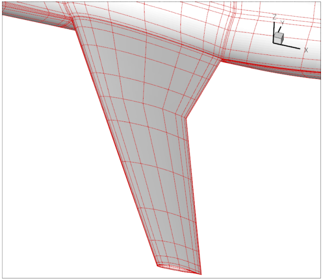
The DLR-F6 wing-body configuration without fairing

- The original mesh of 3.24×10^6 elements has been agglomerated twice.
- The elements of the coarse mesh of 50618 elements are curved based on additional points taken from the original mesh



geometry





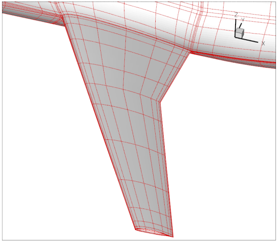
curved mesh with lines given by polynomials of degree 4

Subsonic turbulent flow around the DLR-F6 wing-body

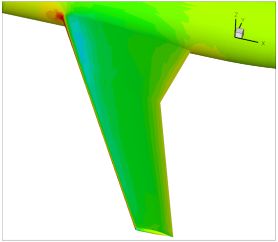
Modification of the DPW III test case:

- $M = 0.5$ (instead of $M = 0.75$)
- $\alpha = -0.141$ (instead of target lift $C_l = 0.5$)
- $Re = 5 \times 10^6$

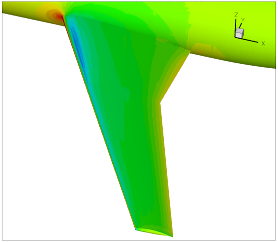
DG solutions on coarse mesh of 50618 curved elements.



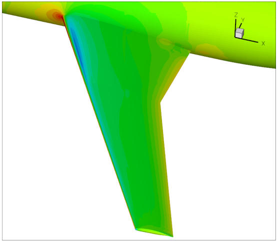
coarse mesh



2^{nd} order solution



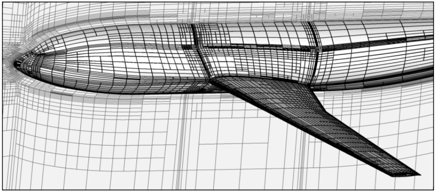
3^{rd} order solution



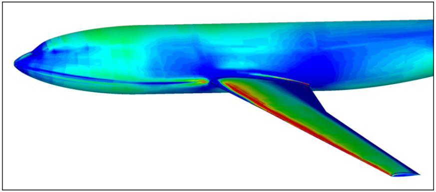
4^{th} order solution

Subsonic turbulent flow around the DLR-F6 wing-body

Adjoint-based refinement for C_{d}:



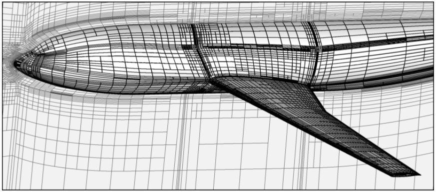
Mesh after 2 adjoint-based refinement steps



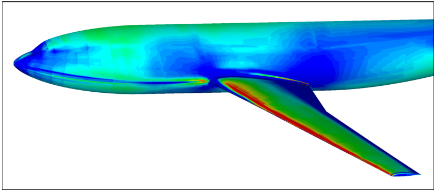
Density adjoint

Subsonic turbulent flow around the DLR-F6 wing-body

Adjoint-based refinement for C_{d}:

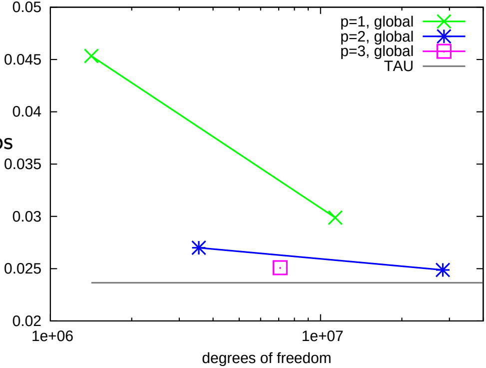


Mesh after 2 adjoint-based refinement steps



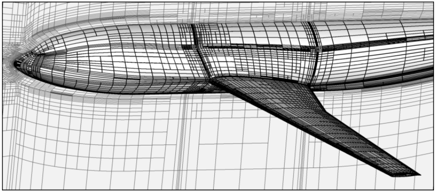
Density adjoint

Convergence of C_{d}
(global mesh refinement):

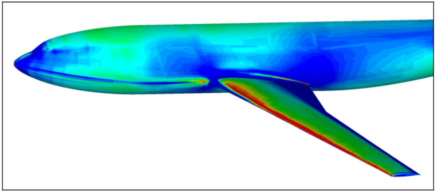


Subsonic turbulent flow around the DLR-F6 wing-body

Adjoint-based refinement for C_d:

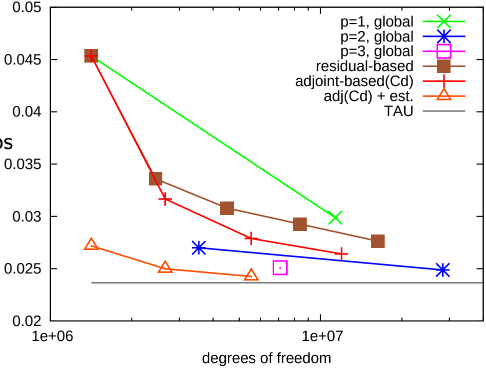


Mesh after 2 adjoint-based refinement steps



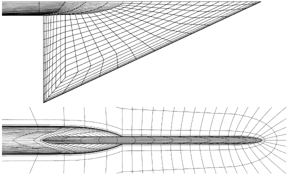
Density adjoint

Convergence of C_d
(global & anisotropic h-refinement):

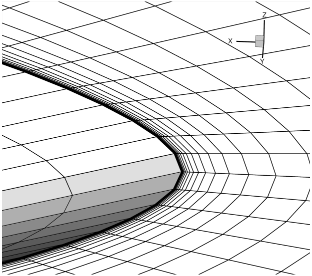


The VFE-2 delta wing with medium rounded leading edge

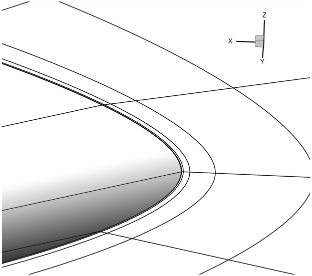
- The original mesh of 884 224 elements has been agglomerated twice.
- The elements of the coarse mesh of 13 816 elements are curved based on additional points taken from the original mesh



geometry



original mesh
with straight lines



curved coarse mesh with lines
given by polynomials of degree 4

Fully turbulent flow around the VFE-2 delta wing configuration

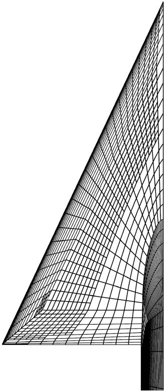
Underlying flow case **U.1** in the EU-project **IDIHOM**

The VFE-2 delta wing with medium rounded leading edge
at two different flow conditions:

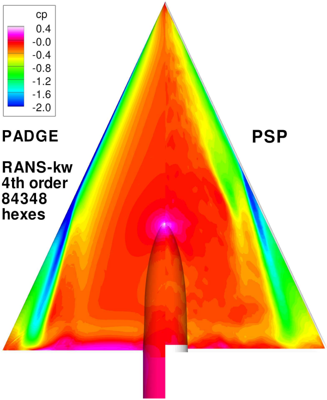
- **U.1b**: RANS-$k\omega$, **subsonic** flow at $M = 0.4$, $\alpha = 13.3^\circ$ and $Re = 3 \times 10^6$
- **U.1c**: RANS-$k\omega$, **transonic** flow at $M = 0.8$, $\alpha = 20.5^\circ$ and $Re = 2 \times 10^6$

Subsonic flow around the VFE-2 delta wing

U.1b: Fully turbulent flow at $M = 0.4$, $\alpha = 13.3^\circ$ and $Re = 3 \times 10^6$



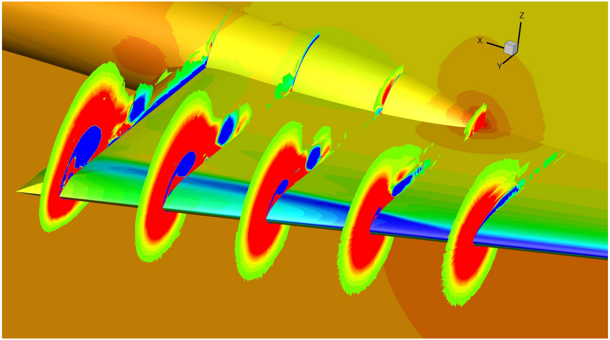
residual-based refined mesh with 84 348 curved elements



c_p distribution
4th order solution vs. experiment (PSP)

Subsonic flow around the VFE-2 delta wing

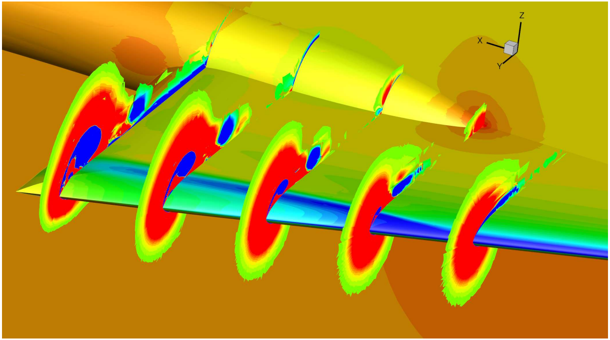
U.1b: Fully turbulent flow at $M = 0.4$, $\alpha = 13.3^\circ$ and $Re = 3 \times 10^6$



4^{th}-order solution on residual-based refined mesh with 84 348 curved elements

Subsonic flow around the VFE-2 delta wing

U.1b: Fully turbulent flow at $M = 0.4$, $\alpha = 13.3^\circ$ and $Re = 3 \times 10^6$



4^{th}-order solution on residual-based refined mesh with 84 348 curved elements

2^{nd}-order solution on residual-based refined mesh with 562 892 curved elements

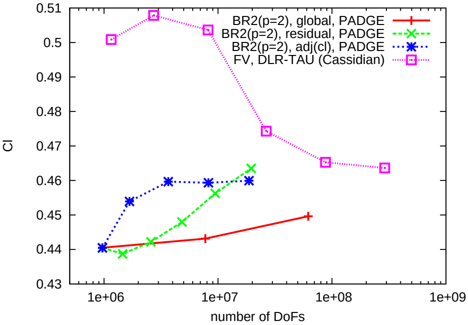
Mesh convergence study (EU-project IDIHOM)

U.1b: Fully turbulent flow at $M = 0.4$, $\alpha = 13.3^\circ$ and $Re = 3 \times 10^6$

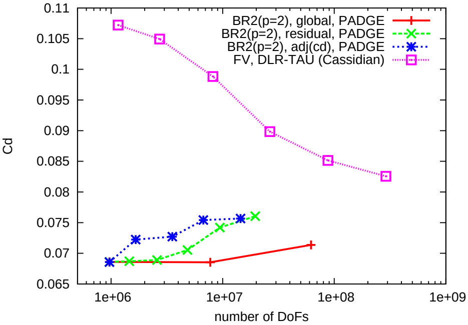
	DLR-TAU Code	DLR-PADGE Code
numerical scheme	finite volume	discontinuous Galerkin
design order	2	3
grids	hybrid unstructured q1 (linear) elements grid sequence	hexahedral q4 elements refinement of starting grid
# elements	$0.6 - 146 \cdot 10^6$	$14 - 884 \cdot 10^3$ (global ref.) $14 - 280 \cdot 10^3$ (local ref.)
degrees of freedom	7 per node	70 per element
$\sum$ degrees of freedom	$1.2 - 290 \cdot 10^6$	$1.6 - 62 \cdot 10^6$ (global ref.) $1.6 - 20 \cdot 10^6$ (local ref.)

Mesh convergence study (EU-project IDIHOM)

U.1b: Fully turbulent flow at $M = 0.4$, $\alpha = 13.3^\circ$ and $Re = 3 \times 10^6$



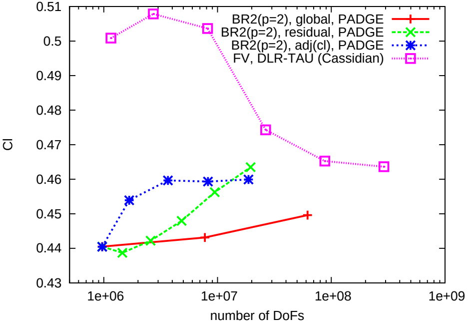
C_l vs. # DoFs



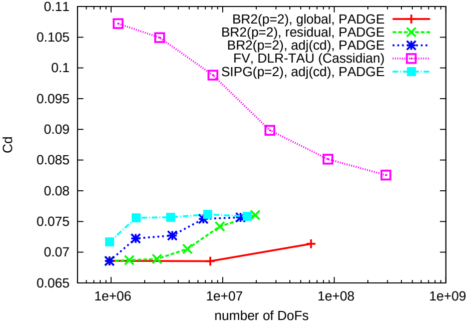
C_d vs. # DoFs

Mesh convergence study (EU-project IDIHOM)

U.1b: Fully turbulent flow at $M = 0.4$, $\alpha = 13.3^\circ$ and $Re = 3 \times 10^6$



C_l vs. # DoFs



C_d vs. # DoFs

Fully turbulent flow around the VFE-2 delta wing configuration

Underlying flow case **U.1** in the EU-project **IDIHOM**

The VFE-2 delta wing with medium rounded leading edge
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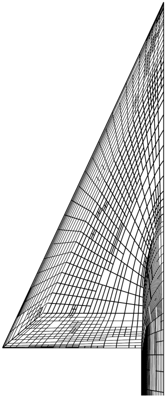
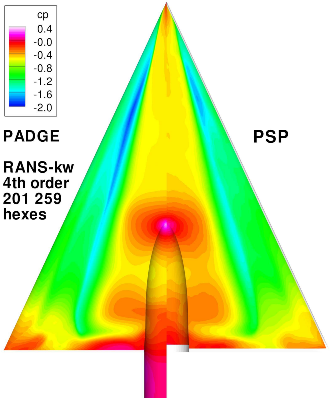
Transonic flow around the VFE-2 delta wing

U.1c: Fully turbulent flow at $M = 0.8$, $\alpha = 20.5^\circ$ and $Re = 2 \times 10^6$



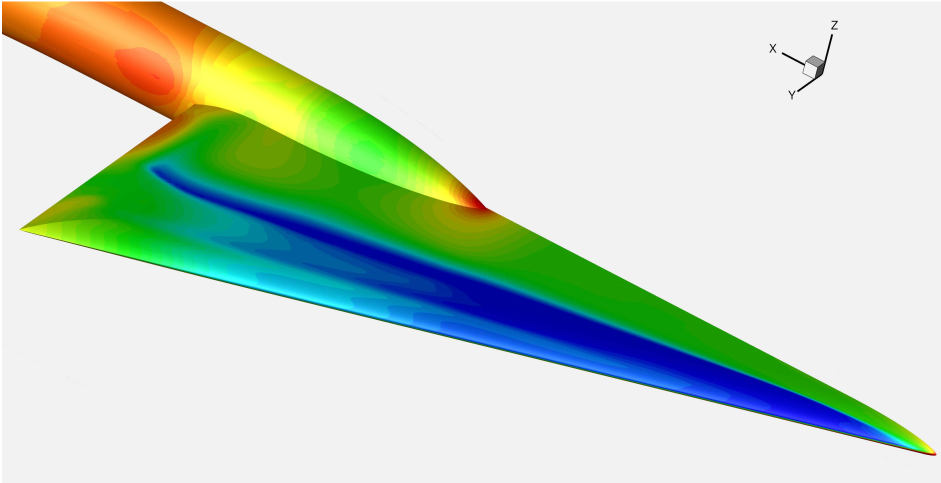
refined mesh with
201 259 curved elements



c_p distribution
4^{th} order solution vs. experiment (PSP)

Transonic flow around the VFE-2 delta wing

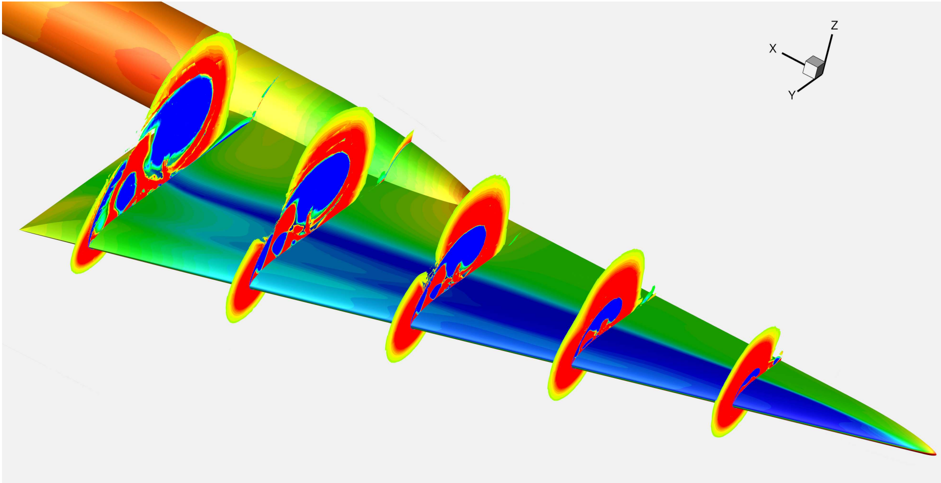
U.1c: Fully turbulent flow at $M = 0.8$, $\alpha = 20.5^\circ$ and $Re = 2 \times 10^6$



4^{th}-order solution on residual-based refined mesh with 201 259 curved elements

Transonic flow around the VFE-2 delta wing

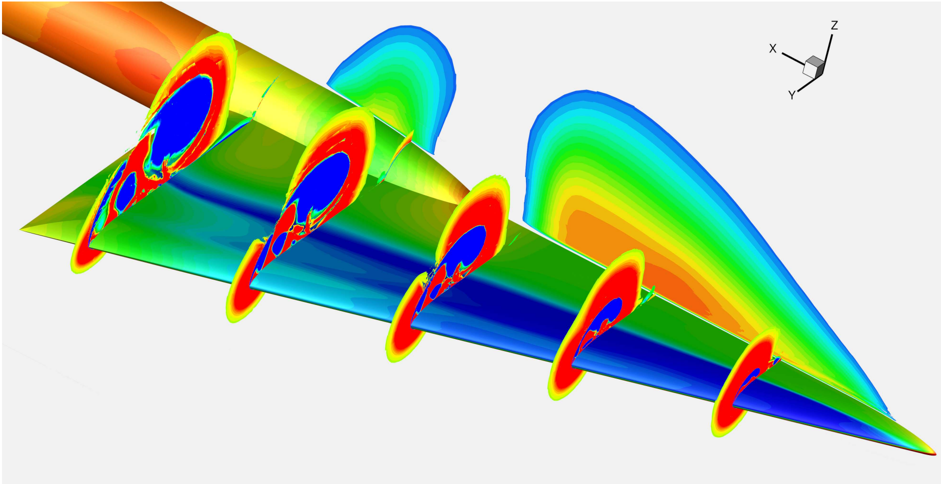
U.1c: Fully turbulent flow at $M = 0.8$, $\alpha = 20.5^\circ$ and $Re = 2 \times 10^6$



4th-order solution on residual-based refined mesh with 201 259 curved elements

Transonic flow around the VFE-2 delta wing

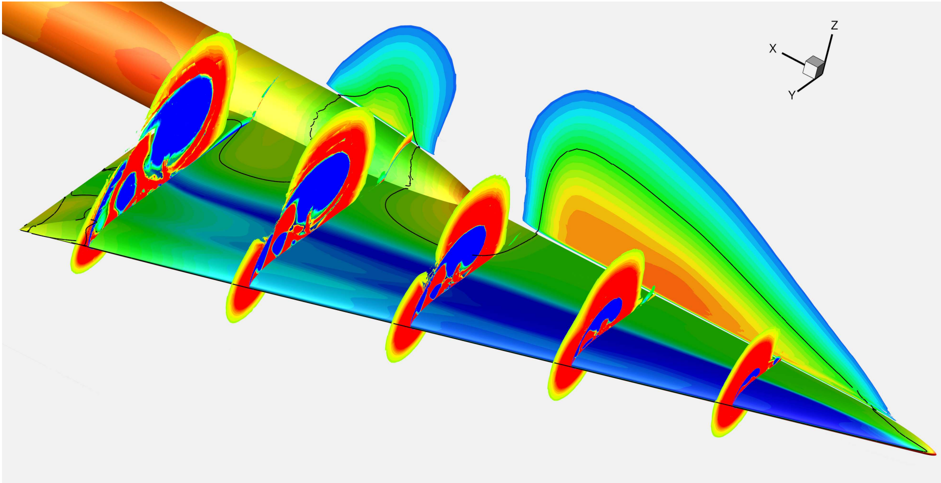
U.1c: Fully turbulent flow at $M = 0.8$, $\alpha = 20.5^\circ$ and $Re = 2 \times 10^6$



4^{th}-order solution on residual-based refined mesh with 201 259 curved elements

Transonic flow around the VFE-2 delta wing

U.1c: Fully turbulent flow at $M = 0.8$, $\alpha = 20.5^\circ$ and $Re = 2 \times 10^6$



4^{th}-order solution on residual-based refined mesh with 201 259 curved elements

Summary

Adjoint consistency

- . . . is available for **compatible** target quantities $J(\cdot)$ only:
  - pressure-induced drag, lift and moment coefficients for compr. Euler
  - total drag, lift and moment coefficients for compr. Navier-Stokes
- There are many consistent discretizations of $J(\cdot)$ but the discretization N_h in combination with only *one* discretization $J_h(\cdot)$ is **adjoint consistent**.
- Given a consistent DG discretization with adjoint consistent (interior) faces terms (like SIPG, BR2). For any discretization of boundary terms it **is** possible to a provide a discretization of the target quantity which results in an **adjoint consistent** discretization (force coefficients are evaluated based on the numerical boundary fluxes employed in the discretization N_h).

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Adjoint-based error estimation and adaptive mesh refinement

- Single-target (and multi-target) error estimation and adaptivity
- Residual-based mesh refinement

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Adjoint-based error estimation and adaptive mesh refinement

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Thank you. Questions?