Higher order and adaptive DG methods for compressible flows (1)

Ralf Hartmann and Tobias Leicht

Institute of Aerodynamic and Flow Technology
DLR (German Aerospace Center)

11. Dec. 2013

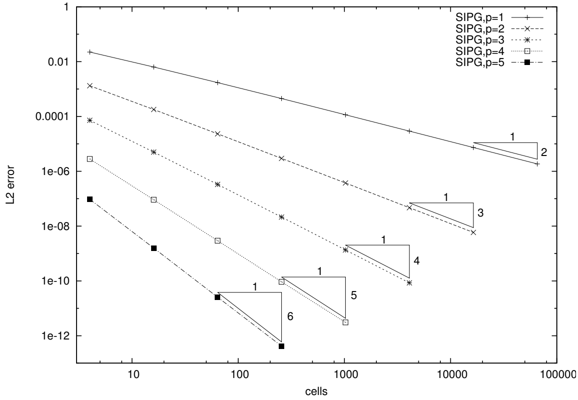
Outline

Higher order discretization methods

A discretization method is of order n if the discretization error behaves like $\mathcal{O}(h^n)$. This means:

- Reducing the mesh size from h to $h/2$ (one global mesh refinement step), the discretization error is reduced by a factor of 2^n.

Example:



DG discretization of Poisson's equation:

The $L^2(\Omega)$-error of the DG(p), $p = 1, \ldots, 5$, discretization behaves like $\mathcal{O}(h^{p+1})$

Discontinuous Galerkin Discretization

Basic properties:

- finite element method with discontinuous trial and test functions
- uses numerical flux functions
- has a local and global conservation property
- DG of 1st order is comparable to a basic finite volume method
- higher order simply by increasing the polynomial degree p
- higher order on unstructured and locally refined meshes
- different polynomial degree in different parts of the domain
- allows error estimation, *hp*-refinement

Outline

Topics in the numerical analysis of Discontinuous Galerkin methods

... which **will be** covered in this lecture:

- Consistency
- Coercivity and stability
- Adjoint consistency
- Order of convergence in the L^2-norm
- Order of convergence in specific target quantities $J(\cdot)$
- *A priori* error estimation
- *A posteriori* error estimation
- Derivation of indicators for local (isotropic) mesh refinement (h-refinement)

Topics in the numerical analysis of Discontinuous Galerkin methods

... which **will be** covered in this lecture:

- Consistency
- Coercivity and stability
- Adjoint consistency
- Order of convergence in the L^2-norm
- Order of convergence in specific target quantities $J(\cdot)$
- *A priori* error estimation
- *A posteriori* error estimation
- Derivation of indicators for local (isotropic) mesh refinement (h-refinement)

... which **will not be** covered in this lecture

- Derivation of indicators for local anisotropic mesh refinement
- Derivation of indicators for hp-refinement

The problem and its discretization

Primal problem: Consider a linear PDE of the form

$$Lu = f \quad \text{in } \Omega, \qquad\qquad Bu = g \quad \text{on } \Gamma,$$

with $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$, where L denotes a linear differential operator on Ω, and B denotes a linear differential (boundary) operator on the boundary Γ.

The problem and its discretization

Primal problem: Consider a linear PDE of the form

$$Lu = f \quad \text{in } \Omega, \qquad\qquad Bu = g \quad \text{on } \Gamma,$$

with $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$, where L denotes a linear differential operator on Ω, and B denotes a linear differential (boundary) operator on the boundary Γ.

Consider the finite element **discretization**: find $u_h \in V_h$ such that

$$L_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h.$$

V_h is a discrete function space and $L_h : V \times V \to \mathbb{R}$ is a bilinear form.
Here V is a function space such that $V_h \subset V$ and $u \in V$, where u is the exact, i.e. analytical, solution to the primal problem.

Consistency and Galerkin orthogonality

The discretization: find $u_h \in V_h \subset V$ such that

$$L_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h,$$

is **consistent** if the exact solution $u \in V$ to the primal problem satisfies

$$L_h(u, v) = F_h(v) \quad \forall v \in V.$$

This answers the question: Do we solve the right equations?

Subtracting both equations for $v_h \in V_h \subset V$ we obtain
the **Galerkin orthogonality**:

$$L_h(u - u_h, v_h) = 0 \quad \forall v_h \in V_h.$$

Coercivity & Stability

Coercivity of L_h: Is there a constant $\gamma > 0$, such that

$$L_h(v_h, v_h) \geq \gamma |||v_h|||^2 \quad \forall v_h \in V_h,$$

where $|||v|||$ is a norm (or seminorm) on V.

Continuity of F_h: Is there a constant $C_F > 0$ such that

$$F_h(v_h) \leq C_F |||v_h||| \quad \forall v_h \in V_h.$$

Then, for the solution $u_h \in V_h$ to the discrete problem

$$L_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h,$$

we obtain

$$\gamma |||u_h|||^2 \leq L_h(u_h, u_h) = F_h(u_h) \leq C_F |||u_h|||,$$

and thus **stability**: $\quad |||u_h||| \leq \frac{C_F}{\gamma}.$

If $|||\cdot|||$ is a norm (and not only a semi-norm) on V then the discretization is **stable**.

Convergence and order of convergence

- Does the discrete solution u_h converge to the exact solution u?

Convergence and order of convergence

- Does the discrete solution u_h converge to the exact solution u?
- What is the order of convergence, i.e., given a solution u with $\|u\|_{**} < \infty$, what is (the maximum) r such that

$$\|u - u_h\|_* \leq ch^r \|u\|_{**}.$$

  - Here, $\|\cdot\|_*$ is an appropriate (global) norm to measure the error in, e.g. $\|\cdot\|_* = \|\cdot\|_{L^2}$,
  - and $\|\cdot\|_{**}$ is a norm on (possibly a subset of) V.

Convergence in specific target quantities $J(\cdot)$

The target quantity $J(u)$ may represent a physically relevant quantity

- weighted mean value of the solution
- weighted boundary integral of the solution or its normal derivative
- aerodynamic force coefficients: drag, lift and moment coefficients

Given a solution u with $\|u\|_{**} < \infty$, what is (the maximum) s such that

$$|J(u) - J(u_h)| \leq ch^s \|u\|_{**}.$$

A priori and a posteriori error estimates

A priori error estimates: e.g.

$$\|u - u_h\|_* \leq ch^r \|u\|_{**},$$
$$|J(u) - J(u_h)| \leq ch^s \|u\|_{**}$$

A priori and a posteriori error estimates

A priori error estimates: e.g.

$$\|u - u_h\|_* \leq ch^r \|u\|_{**},$$
$$|J(u) - J(u_h)| \leq ch^s \|u\|_{**}$$

A posteriori error estimates: e.g.

$$|J(u) - J(u_h)| \leq E(u_h),$$
$$|J(u) - J(u_h)| \approx E(u_h, z_h)$$

Adjoint-based error estimates and adjoint consistency

Error estimates in the L^2-norm or in target quantities $J()$ require the use of duality arguments:

- Define an appropriate adjoint problem connected to the primal problem and the L^2-norm or the target quantity.
- Some analysis reveals that the discretization is of optimal order only if the discretization is adjoint consistent.

In addition to **consistency** require **adjoint consistency** for optimality

Outline

Definition of consistency and adjoint consistency for linear problems

Primal problem: $Lu = f \quad \text{in } \Omega, \qquad Bu = g \quad \text{on } \Gamma,$

Target quantity:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s = (j_\Omega, u)_\Omega + (j_\Gamma, Cu)_\Gamma$$

Definition of consistency and adjoint consistency for linear problems

Primal problem: $Lu = f \quad \text{in } \Omega, \qquad Bu = g \quad \text{on } \Gamma,$

Target quantity:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s = (j_\Omega, u)_\Omega + (j_\Gamma, Cu)_\Gamma$$

Compatibility condition: $J(\cdot)$ is compatible to the primal problem if

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma.$$

Definition of consistency and adjoint consistency for linear problems

Primal problem: $Lu = f \quad \text{in } \Omega, \qquad Bu = g \quad \text{on } \Gamma,$

Target quantity:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s = (j_\Omega, u)_\Omega + (j_\Gamma, Cu)_\Gamma$$

Compatibility condition: $J(\cdot)$ is compatible to the primal problem if

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma.$$

Adjoint problem: $L^* z = j_\Omega \quad \text{in } \Omega, \qquad B^* z = j_\Gamma \quad \text{on } \Gamma.$

Definition of consistency and adjoint consistency for linear problems

Primal problem: $Lu = f \quad \text{in } \Omega, \qquad Bu = g \quad \text{on } \Gamma,$

Target quantity:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s = (j_\Omega, u)_\Omega + (j_\Gamma, Cu)_\Gamma$$

Compatibility condition: $J(\cdot)$ is compatible to the primal problem if

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma.$$

Adjoint problem: $L^* z = j_\Omega \quad \text{in } \Omega, \qquad B^* z = j_\Gamma \quad \text{on } \Gamma.$

Let the primal problem be discretized: Find $u_h \in V_h$ such that

$$L_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h,$$

and evaluate the discrete target quantity, $J_h(u_h)$.

Consistency: The exact solution u to the primal problem satisfies:

$$L_h(u, v) = F_h(v) \quad \forall v \in V, \qquad J_h(u) = J(u).$$

Adjoint consistency: The exact solution z to the adjoint problem satisfies:

$$L_h(w, z) = J_h(w) \quad \forall w \in V.$$

Outline

Derivation of the adjoint problem

Given the primal problem

$$Lu = f \quad \text{in } \Omega, \qquad\qquad Bu = g \quad \text{on } \Gamma,$$

and the target quantity

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s = (j_\Omega, u)_\Omega + (j_\Gamma, Cu)_\Gamma.$$

Find the operator C and the adjoint operators L^*, B^* and C^* via the compatibility condition

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma.$$

Then the adjoint problem is given by

$$L^* z = j_\Omega \quad \text{in } \Omega, \qquad\qquad B^* z = j_\Gamma \quad \text{on } \Gamma.$$

Consistency analysis of the discrete primal problem

Rewrite the discrete problem: Find $u_h \in V_h$ such that

$$L_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h$$

in following element-based **primal residual form**: Find $u_h \in V_h$ such that

$$\int_\Omega R(u_h) v_h \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} r(u_h) v_h \, \mathrm{d}s + \int_\Gamma r_\Gamma(u_h) v_h \, \mathrm{d}s = 0 \quad \forall v_h \in V_h.$$

The discretization is **consistent**
if the exact solution u to the primal problem satisfies

$$\begin{aligned} R(u) &= 0 && \text{in } \kappa, \kappa \in \mathcal{T}_h, \\ r(u) &= 0 && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ r_\Gamma(u) &= 0 && \text{on } \Gamma. \end{aligned}$$

Adjoint consistency of element, interior face and boundary terms

Rewrite the discrete adjoint problem: find $z_h \in V_h$ such that

$$L_h(w_h, z_h) = J_h(w_h) \quad \forall w_h \in V_h,$$

in following element-based **adjoint residual form**: find $z_h \in V_h$ such that

$$\int_\Omega w_h\, R^*(z_h)\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} w_h\, r^*(z_h)\,\mathrm{d}s + \int_\Gamma w_h\, r^*_\Gamma(z_h)\,\mathrm{d}s = 0 \quad \forall w_h \in V_h.$$

The discrete adjoint problem is a **consistent** discretization of the adjoint problem if the exact solution z to the adjoint problem satisfies

$$\begin{aligned} R^*(z) &= 0 && \text{in } \kappa, \kappa \in \mathcal{T}_h, \\ r^*(z) &= 0 && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ r^*_\Gamma(z) &= 0 && \text{on } \Gamma. \end{aligned}$$

Then, the discretization L_h in combination with J_h is **adjoint consistent**.

Outline

The linear advection equation and its adjoint equation

Consider the linear advection equation

$$Lu := \nabla \cdot (\mathbf{b}u) + cu = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_- = \{\mathbf{x} \in \Gamma, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}.$$

The linear advection equation and its adjoint equation

Consider the linear advection equation

$$Lu := \nabla \cdot (\mathbf{b}u) + cu = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_- = \{\mathbf{x} \in \Gamma, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}.$$

Multiply by z, integrate over Ω and integrate by parts

$$\int_\Omega (\nabla \cdot (\mathbf{b}u) + cu)\, z \,\mathrm{d}\mathbf{x} = -\int_\Omega (\mathbf{b}u) \cdot \nabla z \,\mathrm{d}\mathbf{x} + \int_\Omega cuz \,\mathrm{d}\mathbf{x} + \int_\Gamma \mathbf{b} \cdot \mathbf{n}\, uz \,\mathrm{d}s.$$

The linear advection equation and its adjoint equation

Consider the linear advection equation

$$Lu := \nabla \cdot (\mathbf{b}u) + cu = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_- = \{\mathbf{x} \in \Gamma, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}.$$

Multiply by z, integrate over Ω and integrate by parts

$$\int_\Omega (\nabla \cdot (\mathbf{b}u) + cu)\, z \,\mathrm{d}\mathbf{x} = -\int_\Omega (\mathbf{b}u) \cdot \nabla z \,\mathrm{d}\mathbf{x} + \int_\Omega cuz \,\mathrm{d}\mathbf{x} + \int_\Gamma \mathbf{b} \cdot \mathbf{n}\, uz \,\mathrm{d}s.$$

After splitting the boundary $\Gamma = \Gamma_- \cup \Gamma_+$ we obtain:

$$(\nabla \cdot (\mathbf{b}u) + cu, z)_\Omega + (u, -\mathbf{b} \cdot \mathbf{n}\, z)_{\Gamma_-} = (u, -\mathbf{b} \cdot \nabla z + cz)_\Omega + (u, \mathbf{b} \cdot \mathbf{n}\, z)_{\Gamma_+}.$$

The linear advection equation and its adjoint equation

Consider the linear advection equation

$$Lu := \nabla\cdot(\mathbf{b}u) + cu = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_- = \{\mathbf{x}\in\Gamma, \mathbf{b}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) < 0\}.$$

Multiply by z, integrate over Ω and integrate by parts

$$\int_\Omega (\nabla\cdot(\mathbf{b}u) + cu)\, z\,\mathrm{d}\mathbf{x} = -\int_\Omega (\mathbf{b}u)\cdot\nabla z\,\mathrm{d}\mathbf{x} + \int_\Omega cuz\,\mathrm{d}\mathbf{x} + \int_\Gamma \mathbf{b}\cdot\mathbf{n}\, uz\,\mathrm{d}s.$$

After splitting the boundary $\Gamma = \Gamma_-\cup\Gamma_+$ we obtain:

$$(\nabla\cdot(\mathbf{b}u) + cu, z)_\Omega + (u, -\mathbf{b}\cdot\mathbf{n}\, z)_{\Gamma_-} = (u, -\mathbf{b}\cdot\nabla z + cz)_\Omega + (u, \mathbf{b}\cdot\mathbf{n}\, z)_{\Gamma_+}.$$

Comparing with the compatibility condition

$$(Lu, z)_\Omega + (Bu, C^*z)_\Gamma = (u, L^*z)_\Omega + (Cu, B^*z)_\Gamma,$$

we see that for $Lu = \nabla\cdot(\mathbf{b}u) + cu$ in Ω and

$$\begin{aligned} Bu &= u, & Cu &= 0 & &\text{on } \Gamma_-,\\ Bu &= 0, & Cu &= u & &\text{on } \Gamma_+, \end{aligned}$$

the adjoint operators are given by $L^*z = -\mathbf{b}\cdot\nabla z + cz$ in Ω and

$$\begin{aligned} B^*z &= 0, & C^*z &= -\mathbf{b}\cdot\mathbf{n}\, z & &\text{on } \Gamma_-,\\ B^*z &= \mathbf{b}\cdot\mathbf{n}\, z, & C^*z &= 0 & &\text{on } \Gamma_+. \end{aligned}$$

The linear advection equation and its adjoint equation

Primal problem:

$$Lu := \nabla\cdot(\mathbf{b}u) + cu = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma_-.$$

For the operators $Lu = \nabla\cdot(\mathbf{b}u) + cu$ in Ω and

$$\begin{aligned} Bu &= u, & Cu &= 0 & &\text{on } \Gamma_-,\\ Bu &= 0, & Cu &= u & &\text{on } \Gamma_+, \end{aligned}$$

the adjoint operators are given by $L^*z = -\mathbf{b}\cdot\nabla z + cz$ in Ω and

$$\begin{aligned} B^*z &= 0, & C^*z &= -\mathbf{b}\cdot\mathbf{n}\, z & &\text{on } \Gamma_-,\\ B^*z &= \mathbf{b}\cdot\mathbf{n}\, z, & C^*z &= 0 & &\text{on } \Gamma_+. \end{aligned}$$

In particular,

$$J(u) = \int_\Omega j_\Omega\, u \,\mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma\, Cu \,\mathrm{d}s = \int_\Omega j_\Omega\, u \,\mathrm{d}\mathbf{x} + \int_{\Gamma_+} j_\Gamma\, u \,\mathrm{d}s,$$

is **compatible** and the continuous adjoint problem is given by

$$-\mathbf{b}\cdot\nabla z + cz = j_\Omega \quad \text{in } \Omega, \qquad \mathbf{b}\cdot\mathbf{n}\, z = j_\Gamma \quad \text{on } \Gamma_+.$$

Outline

Derivation of the DG discretization

Consider the linear advection equation:

$$Lu := \nabla\cdot(\mathbf{b}u) + cu = f \quad \text{in } \Omega, \qquad\qquad u = g \quad \text{on } \Gamma_-.$$

Multiply by a test function v, integrate over κ

$$\int_\kappa (\nabla\cdot(\mathbf{b}u) + cu)\, v \,\mathrm{d}\mathbf{x} = \int_\kappa fv \,\mathrm{d}\mathbf{x},$$

and integrate by parts

$$-\int_\kappa (\mathbf{b}u)\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_\kappa cuv\,\mathrm{d}\mathbf{x} + \int_{\partial\kappa} \mathbf{b}\cdot\mathbf{n}\, uv\,\mathrm{d}s = \int_\kappa fv\,\mathrm{d}\mathbf{x}.$$

Sum over all $\kappa\in\mathcal{T}_h$ and replace u by g on Γ_- by g:

$$-\int_\Omega (\mathbf{b}u)\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_\Omega cuv\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma} \mathbf{b}\cdot\mathbf{n}\, uv\,\mathrm{d}s + \int_{\Gamma_+} \mathbf{b}\cdot\mathbf{n}\, uv\,\mathrm{d}s$$
$$= \int_\Omega fv\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, gv\,\mathrm{d}s.$$

Derivation of the DG discretization

$$- \int_\Omega (\mathbf{b}u) \cdot \nabla v \, \mathrm{d}\mathbf{x} + \int_\Omega c u v \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \mathbf{b} \cdot \mathbf{n} \, u v \, \mathrm{d}s + \int_{\Gamma_+} \mathbf{b} \cdot \mathbf{n} \, u v \, \mathrm{d}s = \int_\Omega f v \, \mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b} \cdot \mathbf{n} \, g v \, \mathrm{d}s.$$

Derivation of the DG discretization

$$-\int_\Omega (\mathbf{b}u)\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_\Omega cuv\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma}\mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s + \int_{\Gamma_+}\mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s = \int_\Omega fv\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,gv\,\mathrm{d}s.$$

Replace u and v by $u_h \in V_h^p$ and $v_h \in V_h^p$ where

$$V_h^p = \{v_h \in L^2(\Omega) : v_h|_\kappa \circ F_\kappa \in Q_p(\hat\kappa) \text{ if } \hat\kappa \text{ is the unit square, and } v_h|_\kappa \circ F_\kappa \in P_p(\hat\kappa) \text{ if } \hat\kappa \text{ is the unit triangle}, \kappa\in\mathcal{T}_h\},$$

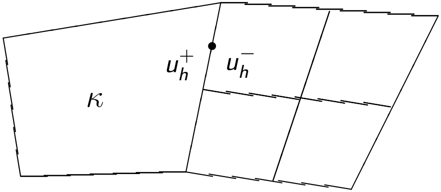
Derivation of the DG discretization

$$-\int_\Omega (\mathbf{b}u)\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_\Omega cuv\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma} \mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s + \int_{\Gamma_+}\mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s = \int_\Omega fv\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,gv\,\mathrm{d}s.$$

Replace u and v by $u_h \in V_h^p$ and $v_h \in V_h^p$ where

$$V_h^p = \{v_h \in L^2(\Omega) : v_h|_\kappa \circ F_\kappa \in Q_p(\hat\kappa) \text{ if } \hat\kappa \text{ is the unit square, and } v_h|_\kappa \circ F_\kappa \in P_p(\hat\kappa) \text{ if } \hat\kappa \text{ is the unit triangle}, \kappa \in \mathcal{T}_h\},$$

and replace $\mathbf{b}\cdot\mathbf{n}\,u$ on $\partial\kappa$ by a numerical flux function $\hat h(u_h^+, u_h^-, \mathbf{n})$ where u_h^+ and u_h^- are the interior and exterior traces of u_h on $\partial\kappa$.



Derivation of the DG discretization

Then, the DG discretization is given by: find $u_h \in V_h^p$ such that

$$L_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h^p,$$

with

$$\begin{aligned} L_h(u_h, v_h) = &- \int_\Omega (\mathbf{b} u_h) \cdot \nabla_h v_h \,\mathrm{d}\mathbf{x} + \int_\Omega c u_h v_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \hat{h}(u_h^+, u_h^-, \mathbf{n}) v_h \,\mathrm{d}s \\ &+ \int_{\Gamma_+} \mathbf{b} \cdot \mathbf{n}\, u_h^+ \, v_h \,\mathrm{d}s, \\ F_h(v_h) = &\int_\Omega f v_h \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b} \cdot \mathbf{n}\, g v_h \,\mathrm{d}s. \end{aligned}$$

The numerical flux function $\hat{h}(u_h^+, u_h^-, \mathbf{n})$ will be specified later.

Consistency

Integrating

$$-\int_\Omega (\mathbf{b}u_h)\cdot\nabla_h v_h\,\mathrm{d}\mathbf{x} + \int_\Omega c u_h v_h\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma}\hat{h}(u_h^+,u_h^-,\mathbf{n})v_h\,\mathrm{d}s + \int_{\Gamma_+}\mathbf{b}\cdot\mathbf{n}\,u_h^+\,v_h\,\mathrm{d}s = \int_\Omega f v_h\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,g\,v_h\,\mathrm{d}s$$

back by parts gives

$$\int_\Omega \nabla_h\cdot(\mathbf{b}u_h)\,v_h\,\mathrm{d}\mathbf{x} + \int_\Omega c u_h v_h\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma}\left(\hat{h}(u_h^+,u_h^-,\mathbf{n}) - \mathbf{b}\cdot\mathbf{n}\,u_h^+\right)v_h\,\mathrm{d}s - \int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,u_h^+\,v_h\,\mathrm{d}s = \int_\Omega f v_h\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,g\,v_h\,\mathrm{d}s.$$

Consistency

$$\int_\Omega \nabla_h \cdot (\mathbf{b} u_h)\, v_h \,\mathrm{d}\mathbf{x} + \int_\Omega c u_h v_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} \left(\hat{h}(u_h^+, u_h^-, \mathbf{n}) - \mathbf{b}\cdot\mathbf{n}\, u_h^+\right) v_h \,\mathrm{d}s$$
$$- \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, u_h^+\, v_h \,\mathrm{d}s = \int_\Omega f v_h \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, g v_h \,\mathrm{d}s.$$

Thus, we obtain the primal residual form: find $u_h \in V_h^p$ such that

$$\sum_{\kappa\in\mathcal{T}_h} \int_\kappa R(u_h) v_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} r(u_h) v_h \,\mathrm{d}s + \int_\Gamma r_\Gamma(u_h) v_h \,\mathrm{d}s = 0 \quad \forall v_h \in V_h^p,$$

with

$$\begin{aligned}
R(u_h) &= f - \nabla_h \cdot (\mathbf{b} u_h) - c u_h && \text{in } \kappa, \kappa \in \mathcal{T}_h,\\
r(u_h) &= \mathbf{b}\cdot\mathbf{n}\, u_h^+ - \hat{h}(u_h^+, u_h^-, \mathbf{n}) && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h,\\
r_\Gamma(u_h) &= \mathbf{b}\cdot\mathbf{n}\,(u_h^+ - g) && \text{on } \Gamma_-,\\
r_\Gamma(u_h) &\equiv 0 && \text{on } \Gamma_+.
\end{aligned}$$

Consistency

$$
\begin{aligned}
R(u_h) &= f - \nabla_h \cdot (\mathbf{b} u_h) - c u_h && \text{in } \kappa, \kappa \in \mathcal{T}_h,\\
r(u_h) &= \mathbf{b}\cdot\mathbf{n}\, u_h^+ - \hat{h}(u_h^+, u_h^-, \mathbf{n}) && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h,\\
r_\Gamma(u_h) &= \mathbf{b}\cdot\mathbf{n}\,(u_h^+ - g) && \text{on } \Gamma_-,\\
r_\Gamma(u_h) &\equiv 0 && \text{on } \Gamma_+.
\end{aligned}
$$

$R(u) = 0$ and $r_\Gamma(u) = 0$ for the exact solution to

$$\nabla \cdot (\mathbf{b} u) + c u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma_-.$$

Furthermore, $r(u) = 0$ if and only if $\hat{h}(u, u, \mathbf{n}) = \mathbf{b}\cdot\mathbf{n}\, u$.

Definition: A numerical flux function $\hat{h}$ is said to be *consistent* if

$$\hat{h}(v, v, \mathbf{n}) = \mathbf{b}\cdot\mathbf{n}\, v.$$

Global conservation property

Setting $c = 0$ and $v \equiv 1$ in the variational formulation we obtain

$$\sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} \hat{h}(u_h^+, u_h^-, \mathbf{n})\,\mathrm{d}s + \int_{\Gamma_+} \mathbf{b}\cdot\mathbf{n}\, u_h^+\,\mathrm{d}s = \int_\Omega f\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, g\,\mathrm{d}s.$$

Rewriting in terms of interior edges $e \in \Gamma_{\mathcal{I}}$ we obtain

$$\sum_{e \in \Gamma_{\mathcal{I}}} \int_e \hat{h}(u_h^+, u_h^-, \mathbf{n}) + \hat{h}(u_h^-, u_h^+, -\mathbf{n})\,\mathrm{d}s + \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, g\,\mathrm{d}s + \int_{\Gamma_+} \mathbf{b}\cdot\mathbf{n}\, u_h^+\,\mathrm{d}s = \int_\Omega f\,\mathrm{d}\mathbf{x}.$$

Hence, the discretization is **conservative**, i.e.

$$\int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, g\,\mathrm{d}s + \int_{\Gamma_+} \mathbf{b}\cdot\mathbf{n}\, u_h^+\,\mathrm{d}s = \int_\Omega f\,\mathrm{d}\mathbf{x},$$

if and only if the numerical flux function $\hat{h}$ is **conservative**, i.e.

$$\hat{h}(u_h^+, u_h^-, \mathbf{n}) = -\hat{h}(u_h^-, u_h^+, -\mathbf{n}).$$

Numerical flux functions for the linear advection equation

The **mean value flux** (or central flux):

$$\hat{h}_{\text{mv}}(u_h^+, u_h^-, \mathbf{n}) = \mathbf{b}\cdot\mathbf{n}\,\{u_h\}, \qquad \text{where } \{u_h\} = \frac{1}{2}\left(u_h^+ + u_h^-\right).$$

The **upwind flux:**

$$\hat{h}_{\text{uw}}(u_h^+, u_h^-, \mathbf{n}) = \begin{cases} \mathbf{b}\cdot\mathbf{n}\,u_h^-, & \text{for } (\mathbf{b}\cdot\mathbf{n})(\mathbf{x}) < 0, \text{ i.e. } \mathbf{x}\in\partial\kappa_-, \\ \mathbf{b}\cdot\mathbf{n}\,u_h^+, & \text{for } (\mathbf{b}\cdot\mathbf{n})(\mathbf{x}) \geq 0, \text{ i.e. } \mathbf{x}\in\partial\kappa_+, \end{cases}$$

where $\partial\kappa_-$ and $\partial\kappa_+$ are the inflow and outflow boundaries of element κ:

$$\begin{aligned} \partial\kappa_- &= \{\mathbf{x}\in\partial\kappa, \mathbf{b}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) < 0\}, \\ \partial\kappa_+ &= \{\mathbf{x}\in\partial\kappa, \mathbf{b}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) \geq 0\} = \partial\kappa\setminus\partial\kappa_-. \end{aligned}$$

The **generic flux:**

$$\hat{h}_{b_0}(u_h^+, u_h^-, \mathbf{n}) = \mathbf{b}\cdot\mathbf{n}\,\{u_h\} + b_0\,[u_h], \qquad \text{where } [u_h] = u_h^+ - u_h^-.$$

- represents the mean value flux for $b_0 = 0$
- represents the upwind flux for $b_0 = \frac{1}{2}|\mathbf{b}\cdot\mathbf{n}|$

Coercivity

Let $L_h(\cdot,\cdot)$ be given by

$$L_h(u_h, v_h) = -\int_\Omega (\mathbf{b}u_h)\cdot\nabla_h v_h\,\mathrm{d}\mathbf{x} + \int_\Omega c u_h v_h\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma} \hat{h}_{b_0}(u_h^+, u_h^-, \mathbf{n}) v_h\,\mathrm{d}s + \int_{\Gamma_+} \mathbf{b}\cdot\mathbf{n}\, u_h^+ v_h\,\mathrm{d}s,$$

where $\hat{h}_{b_0}$ represents

- the mean value flux for $b_0 = 0$
- the upwind flux for $b_0 = \frac{1}{2}|\mathbf{b}\cdot\mathbf{n}|$

Then for all $v_h \in V_h^p$ we have

$$L_h(v_h, v_h) = \|c_0 v_h\|^2 + \sum_{e\in\Gamma_\mathcal{I}}\int_e b_0\,[v_h]^2\,\mathrm{d}s + \frac{1}{2}\int_\Gamma |\mathbf{b}\cdot\mathbf{n}|\, v_h^2\,\mathrm{d}s =: |||v_h|||_{b_0}^2,$$

where we assume that $c(\mathbf{x}) + \frac{1}{2}\nabla\cdot\mathbf{b}(\mathbf{x}) > 0$ and set $c_0^2(\mathbf{x}) = c(\mathbf{x}) + \frac{1}{2}\nabla\cdot\mathbf{b}(\mathbf{x})$.

Stability

We have coercivity of $L_h(\cdot,\cdot)$

$$L_h(v_h, v_h) = \|c_0 v_h\|^2 + \sum_{e\in\Gamma_{\mathcal{I}}} \int_e b_0 \, [v_h]^2 \, \mathrm{d}s + \frac{1}{2}\int_\Gamma |\mathbf{b}\cdot\mathbf{n}| \, v_h^2 \, \mathrm{d}s =: |||v_h|||_{b_0}^2.$$

and continuity of $F(\cdot)$

$$F(v_h) \le C_F |||v_h|||_{b_0}$$

Thereby,

$$|||v_h|||_{b_0}^2 = L_h(v_h, v_h) = F(v_h) \le C_F |||v_h|||_{b_0}$$
$$|||v_h|||_{b_0} \le C_F$$

and we have control over all terms in

$$|||v_h|||_{b_0}^2 = \|c_0 v_h\|^2 + \sum_{e\in\Gamma_{\mathcal{I}}} \int_e b_0 \, [v_h]^2 \, \mathrm{d}s + \frac{1}{2}\int_\Gamma |\mathbf{b}\cdot\mathbf{n}| \, v_h^2 \, \mathrm{d}s \le C_F^2,$$

with $b_0 = 0$ for the mean value flux and $b_0 = \frac{1}{2}|\mathbf{b}\cdot\mathbf{n}|$ for the upwind flux

A priori error estimate

Theorem: Let $u \in H^{p+1}(\Omega)$ be the exact solution to the linear advection equation. Furthermore, let $u_h \in \tilde{V}_h^p$ be the solution to

$$L_h(u_h, v_h) = F(v_h), \quad \forall v_h \in \tilde{V}_h^p,$$

where

$$L_h(u,v) = -\int_\Omega (\mathbf{b}u)\cdot\nabla_h v\,\mathrm{d}\mathbf{x} + \int_\Omega cuv\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma}(\mathbf{b}\cdot\mathbf{n}\,\{u\} + b_0\,[u])\,v\,\mathrm{d}s + \int_{\Gamma_+}\mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s,$$

$$F(v) = \int_\Omega fv\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,gv\,\mathrm{d}s.$$

Then, for $b_0 = \frac{1}{2}|\mathbf{b}\cdot\mathbf{n}|$, i.e. when using the *upwind flux*, we have

$$\boxed{|||u-u_h|||_{b_0} \le Ch^{p+1/2}|u|_{H^{p+1}(\Omega)},}$$

and for $b_0 = 0$, i.e. when using the *mean value flux*, we have

$$\boxed{|||u-u_h|||_{b_0} \le Ch^{p}|u|_{H^{p+1}(\Omega)},}$$

where $|||v|||_{b_0}^2 = \|c_0 v\|^2 + \sum_{e\in\Gamma_\mathcal{I}}\int_e b_0\,[v]^2\,\mathrm{d}s + \frac{1}{2}\int_\Gamma |\mathbf{b}\cdot\mathbf{n}|\,v^2\,\mathrm{d}s.$

Adjoint consistency

Given the (compatible) target quantity $J(u)$ and its discretization $J_h(u_h)$,

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_+} j_\Gamma \, u \, \mathrm{d}s, \qquad J_h(u_h) = J(u_h) = \int_\Omega j_\Omega \, u_h \, \mathrm{d}\mathbf{x} + \int_{\Gamma_+} j_\Gamma \, u_h \, \mathrm{d}s,$$

then the discrete adjoint problem: find $z_h \in V_h^p$ such that

$$L_h(w_h, z_h) = J_h(w_h),$$

rewrites in adjoint residual form: find $z_h \in V_h^p$ such that

$$\int_\Omega w_h \, R^*(z_h) \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} w_h \, r^*(z_h) \, \mathrm{d}s + \int_\Gamma w_h \, r_\Gamma^*(z_h) \, \mathrm{d}s = 0 \quad \forall w_h \in V_h^p,$$

with

$$\begin{aligned} R^*(z_h) &= j_\Omega + \mathbf{b} \cdot \nabla_h z_h - c z_h && \text{in } \kappa, \kappa \in \mathcal{T}_h, \\ r^*(z_h) &= -\mathbf{b} \cdot \mathbf{n} \, [z_h] && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ r^*(z_h) &= j_\Gamma - \mathbf{b} \cdot \mathbf{n} \, z_h^+ && \text{on } \Gamma_+, \end{aligned}$$

The adjoint residuals vanish for the exact solution z to the adjoint equation

$$-\mathbf{b} \cdot \nabla z + cz = j_\Omega \quad \text{in } \Omega, \qquad \mathbf{b} \cdot \mathbf{n} \, z = j_\Gamma \quad \text{on } \Gamma_+.$$

$\Rightarrow$ discretization $L_h(u_h, v_h)$ in combination with $J_h(u_h)$ is **adjoint consistency**.

A priori error estimates for target functionals $J(\cdot)$

Corollary: Let $u_h \in V_h^p$ be the solution to the DG discretization with upwind flux. Assume that $u \in H^{p+1}(\Omega)$ and $z \in H^{p+1}(\Omega)$. Then, there is a constant $C > 0$ such that

$$|J(u) - J_h(u_h)| \leq Ch^{2p+1}|u|_{H^{p+1}(\Omega)}|z|_{H^{p+1}(\Omega)} \qquad \forall u \in H^{p+1}(\Omega). \tag{1}$$

Proof: See (Houston and Süli, 2001; Harriman et al., 2003).

Compare with

$$|||u - u_h|||_{b_0} \leq Ch^{p+1/2}|u|_{H^{p+1}(\Omega)},$$

and note the *order doubling* in (1) due to adjoint consistency.

Outline

The continuous adjoint problem to Poisson's equation

For $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \neq \emptyset$ consider the Dirichlet-Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N.$$

The continuous adjoint problem to Poisson's equation

For $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \neq \emptyset$ consider the Dirichlet-Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N.$$

Multiply left hand side by z and integrate by parts twice

$$(-\Delta u, z)_\Omega = (\nabla u, \nabla z)_\Omega - (\mathbf{n} \cdot \nabla u, z)_\Gamma = (u, -\Delta z)_\Omega + (u, \mathbf{n} \cdot \nabla z)_\Gamma - (\mathbf{n} \cdot \nabla u, z)_\Gamma.$$

The continuous adjoint problem to Poisson's equation

For $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \neq \emptyset$ consider the Dirichlet-Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N.$$

Multiply left hand side by z and integrate by parts twice

$$(-\Delta u, z)_\Omega = (\nabla u, \nabla z)_\Omega - (\mathbf{n} \cdot \nabla u, z)_\Gamma = (u, -\Delta z)_\Omega + (u, \mathbf{n} \cdot \nabla z)_\Gamma - (\mathbf{n} \cdot \nabla u, z)_\Gamma.$$

After splitting the boundary terms according to $\Gamma = \Gamma_D \cup \Gamma_N$ and shuffling terms

$$(-\Delta u, z)_\Omega + (u, -\mathbf{n} \cdot \nabla z)_{\Gamma_D} + (\mathbf{n} \cdot \nabla u, z)_{\Gamma_N} = (u, -\Delta z)_\Omega + (\mathbf{n} \cdot \nabla u, -z)_{\Gamma_D} + (u, \mathbf{n} \cdot \nabla z)_{\Gamma_N}.$$

The continuous adjoint problem to Poisson's equation

For $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \neq \emptyset$ consider the Dirichlet-Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n}\cdot\nabla u = g_N \quad \text{on } \Gamma_N.$$

Multiply left hand side by z and integrate by parts twice

$$(-\Delta u, z)_\Omega = (\nabla u, \nabla z)_\Omega - (\mathbf{n}\cdot\nabla u, z)_\Gamma = (u, -\Delta z)_\Omega + (u, \mathbf{n}\cdot\nabla z)_\Gamma - (\mathbf{n}\cdot\nabla u, z)_\Gamma.$$

After splitting the boundary terms according to $\Gamma = \Gamma_D \cup \Gamma_N$ and shuffling terms

$$(-\Delta u, z)_\Omega + (u, -\mathbf{n}\cdot\nabla z)_{\Gamma_D} + (\mathbf{n}\cdot\nabla u, z)_{\Gamma_N} = (u, -\Delta z)_\Omega + (\mathbf{n}\cdot\nabla u, -z)_{\Gamma_D} + (u, \mathbf{n}\cdot\nabla z)_{\Gamma_N}.$$

Comparing with the compatibility condition

$$(Lu, z)_\Omega + (Bu, C^*z)_\Gamma = (u, L^*z)_\Omega + (Cu, B^*z)_\Gamma.$$

we see that for $Lu = -\Delta u$ in Ω and

$$Bu = u, \qquad Cu = \mathbf{n}\cdot\nabla u \qquad \text{on } \Gamma_D,$$
$$Bu = \mathbf{n}\cdot\nabla u, \qquad Cu = u \qquad \text{on } \Gamma_N,$$

the adjoint operators are given by $L^*z = -\Delta z$ on Ω and

$$B^*z = -z, \qquad C^*z = -\mathbf{n}\cdot\nabla z \qquad \text{on } \Gamma_D,$$
$$B^*z = \mathbf{n}\cdot\nabla z, \qquad C^*z = z \qquad \text{on } \Gamma_N.$$

The continuous adjoint problem to Poisson's equation

Primal problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n}\cdot\nabla u = g_N \quad \text{on } \Gamma_N,$$

For the operators $Lu = -\Delta u$ in Ω and

$$\begin{aligned} Bu &= u, & Cu &= \mathbf{n}\cdot\nabla u & &\text{on } \Gamma_D,\\ Bu &= \mathbf{n}\cdot\nabla u, & Cu &= u & &\text{on } \Gamma_N, \end{aligned}$$

the adjoint operators are given by $L^*z = -\Delta z$ on Ω and

$$\begin{aligned} B^*z &= -z, & C^*z &= -\mathbf{n}\cdot\nabla z & &\text{on } \Gamma_D,\\ B^*z &= \mathbf{n}\cdot\nabla z, & C^*z &= z & &\text{on } \Gamma_N. \end{aligned}$$

The continuous adjoint problem to Poisson's equation

Primal problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n}\cdot\nabla u = g_N \quad \text{on } \Gamma_N,$$

For the operators $Lu = -\Delta u$ in Ω and

$$\begin{aligned} Bu &= u, & Cu &= \mathbf{n}\cdot\nabla u & &\text{on } \Gamma_D,\\ Bu &= \mathbf{n}\cdot\nabla u, & Cu &= u & &\text{on } \Gamma_N, \end{aligned}$$

the adjoint operators are given by $L^*z = -\Delta z$ on Ω and

$$\begin{aligned} B^*z &= -z, & C^*z &= -\mathbf{n}\cdot\nabla z & &\text{on } \Gamma_D,\\ B^*z &= \mathbf{n}\cdot\nabla z, & C^*z &= z & &\text{on } \Gamma_N. \end{aligned}$$

In particular,

$$\begin{aligned} J(u) &= \int_\Omega j_\Omega\, u \,\mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma\, Cu \,\mathrm{d}s \\ &= \int_\Omega j_\Omega\, u \,\mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D\, \mathbf{n}\cdot\nabla u \,\mathrm{d}s + \int_{\Gamma_N} j_N\, u \,\mathrm{d}s, \end{aligned}$$

is **compatible** and the continuous **adjoint problem** is given by

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad -z = j_D \quad \text{on } \Gamma_D, \qquad \mathbf{n}\cdot\nabla z = j_N \quad \text{on } \Gamma_N.$$

Outline

Derivation of the DG discretization

For $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \neq \emptyset$ consider the Dirichlet-Neumann problem

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^2, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N,$$

where $f \in L^2(\Omega)$, $g_D \in L^2(\Gamma_D)$ and $g_N \in L^2(\Gamma_N)$.
Rewrite this as a first-order system:

$$\boldsymbol{\sigma} = \nabla u, \quad -\nabla \cdot \boldsymbol{\sigma} = f \quad \text{in } \Omega, \quad u = g_D \quad \text{on } \Gamma_D, \quad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N.$$

Multiply the first and second equation by test functions $\boldsymbol{\tau}$ and v, respectively, integrate over $\kappa \in \mathcal{T}_h$ and integrate by parts

$$\int_\kappa \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, \mathrm{d}\mathbf{x} = -\int_\kappa u \nabla \cdot \boldsymbol{\tau} \, \mathrm{d}\mathbf{x} + \int_{\partial\kappa} u \, \mathbf{n} \cdot \boldsymbol{\tau} \, \mathrm{d}s,$$
$$\int_\kappa \boldsymbol{\sigma} \cdot \nabla v \, \mathrm{d}\mathbf{x} = \int_\kappa f v \, \mathrm{d}\mathbf{x} + \int_{\partial\kappa} \boldsymbol{\sigma} \cdot \mathbf{n} \, v \, \mathrm{d}s.$$

Derivation of the DG discretization

Sum over all elements $\kappa \in \mathcal{T}_h$

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, \mathrm{d}\mathbf{x} = -\int_\Omega u \, \nabla \cdot \boldsymbol{\tau} \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} u \, \mathbf{n} \cdot \boldsymbol{\tau} \, \mathrm{d}s,$$

$$\int_\Omega \boldsymbol{\sigma} \cdot \nabla v \, \mathrm{d}\mathbf{x} = \int_\Omega f v \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \boldsymbol{\sigma} \cdot \mathbf{n} \, v \, \mathrm{d}s,$$

Replace u and $\boldsymbol{\sigma}$ by discrete functions $u_h \in V_h^p$ and $\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h^p = [V_h^p]^2$ and by numerical flux functions $\hat{u}_h$ and $\hat{\boldsymbol{\sigma}}_h$ on interfaces $\partial\kappa \cap \partial\kappa'$ between elements

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}_h \, \mathrm{d}\mathbf{x} = -\int_\Omega u_h \nabla_h \cdot \boldsymbol{\tau}_h \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{u}_h \, \mathbf{n} \cdot \boldsymbol{\tau}_h \, \mathrm{d}s \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^p,$$

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v_h \, \mathrm{d}\mathbf{x} = \int_\Omega f v_h \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n} \, v_h \, \mathrm{d}s \qquad \forall v_h \in V_h^p.$$

$\hat{u}_h = \hat{u}(u_h) = \hat{u}(u_h^+, u_h^-)$ and $\hat{\boldsymbol{\sigma}}_h = \hat{\boldsymbol{\sigma}}(u_h, \nabla u_h) = \hat{\boldsymbol{\sigma}}(u_h^+, u_h^-, \nabla u_h^+, \nabla u_h^-)$ will be specified later.

The primal flux formulation

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}_h \, d\mathbf{x} = -\int_\Omega u_h \nabla_h \cdot \boldsymbol{\tau}_h \, d\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \hat{u}_h \, \mathbf{n} \cdot \boldsymbol{\tau}_h \, ds \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^p,$$

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v_h \, d\mathbf{x} = \int_\Omega f v_h \, d\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n} \, v_h \, ds \qquad \forall v_h \in V_h^p.$$

Replace $\boldsymbol{\tau}_h$ by $\nabla_h v_h$ and perform second integration by parts in the first equation:

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v_h \, d\mathbf{x} = \int_\Omega \nabla_h u \cdot \nabla_h v \, d\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} (\hat{u}_h - u_h)\mathbf{n} \cdot \nabla_h v \, ds.$$

Eliminate $\boldsymbol{\sigma}_h$ by substituting this into the second equation gives the **primal flux formulation:** find $u_h \in V_h^p$ such that

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, d\mathbf{x} - \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n} \, v_h \, ds + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} (\hat{u}_h - u_h)\mathbf{n} \cdot \nabla_h v_h \, ds = \int_\Omega f v_h \, d\mathbf{x},$$

for all $v_h \in V_h^p$.

Consistency and conservation property

The discretization

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h \,\mathrm{d}\mathbf{x} - \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n}\, v_h \,\mathrm{d}s + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} (\hat{u}_h - u_h)\mathbf{n} \cdot \nabla_h v_h \,\mathrm{d}s = \int_\Omega f v_h \,\mathrm{d}\mathbf{x},$$

is **consistent** if and only if the numerical flux functions $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ are **consistent**,

$$\begin{aligned}
&\hat{u}(v) = v, && \hat{\boldsymbol{\sigma}}(v,\nabla v) = \nabla v, && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h,\\
&\hat{u}(v) = g_D, && \hat{\boldsymbol{\sigma}}(v,\nabla v) = \nabla v, && \text{on } \Gamma_D,\\
&\hat{u}(v) = v, && \hat{\boldsymbol{\sigma}}(v,\nabla v)\cdot\mathbf{n} = g_N, && \text{on } \Gamma_N,
\end{aligned}$$

whenever v is a smooth function satisfying $v = g_D$ on Γ_D and $\mathbf{n}\cdot\nabla v = g_N$ on Γ_N.

It is **conservative** if and only if the numerical flux function $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}(u^+, u^-, \nabla u^+, \nabla u^-)$ is **conservative**, i.e.,

$$\hat{\boldsymbol{\sigma}}(u^+, u^-, \nabla u^+, \nabla u^-) = \hat{\boldsymbol{\sigma}}(u^-, u^+, \nabla u^-, \nabla u^+),$$

(also call "$\hat{\boldsymbol{\sigma}}$ is **single-valued**").

Adjoint consistency

For the (compatible) target quantity

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D \, \mathbf{n} \cdot \nabla u \, \mathrm{d}s + \int_{\Gamma_N} j_N \, u \, \mathrm{d}s,$$

and its discretization

$$J_h(u_h) = \int_\Omega j_\Omega \, u_h \, \mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D \, \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n} \, \mathrm{d}s + \int_{\Gamma_N} j_N \, \hat{u}_h \, \mathrm{d}s,$$

the DG discretization of Poisson's equation is **adjoint consistent** if and only if the numerical fluxes $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ are **single-valued**, i.e.,

$$\hat{\boldsymbol{\sigma}}(u^+, u^-, \nabla u^+, \nabla u^-) = \hat{\boldsymbol{\sigma}}(u^-, u^+, \nabla u^-, \nabla u^+), \qquad \hat{u}(u^+, u^-) = \hat{u}(u^-, u^+).$$

Derivation of various DG discretization methods

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h \,\mathrm{d}\mathbf{x} - \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n}\, v_h \,\mathrm{d}s + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} (\hat{u}_h - u_h)\mathbf{n} \cdot \nabla_h v_h \,\mathrm{d}s = \int_\Omega f v_h \,\mathrm{d}\mathbf{x},$$

where the numerical fluxes $\hat{u}_h$ and $\hat{\boldsymbol{\sigma}}_h$ are given by

	on $\Gamma_\mathcal{I}$		on Γ_D		on Γ_N	
	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$
BO	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\}$	$2u_h - g_D$	$\nabla_h u_h$	u_h	$g_N\mathbf{n}$
NIPG	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	$2u_h - g_D$	$\nabla_h u_h - \boldsymbol{\delta}^{\mathrm{ip}}_\Gamma(u_h)$	u_h	$g_N\mathbf{n}$
SIPG	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}^{\mathrm{ip}}_\Gamma(u_h)$	u_h	$g_N\mathbf{n}$
BR2	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{br2}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}^{\mathrm{br2}}_\Gamma(u_h)$	u_h	$g_N\mathbf{n}$

where $\{\!\{u_h\}\!\} = \frac{1}{2}(u_h^+ + u_h^-)$, $[\![u_h]\!] = u_h^+ \mathbf{n}_h^+ + u_h^- \mathbf{n}^-$.

Derivation of various DG discretization methods

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h \,\mathrm{d}\mathbf{x} - \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n}\, v_h \,\mathrm{d}s + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} (\hat{u}_h - u_h)\mathbf{n}\cdot\nabla_h v_h \,\mathrm{d}s = \int_\Omega f v_h \,\mathrm{d}\mathbf{x},$$

where the numerical fluxes $\hat{u}_h$ and $\hat{\boldsymbol{\sigma}}_h$ are given by

	on $\Gamma_\mathcal{I}$		on Γ_D		on Γ_N	
	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$
BO	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\}$	$2u_h - g_D$	$\nabla_h u_h$	u_h	$g_N\mathbf{n}$
NIPG	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\rm ip}(u_h)$	$2u_h - g_D$	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\rm ip}(u_h)$	u_h	$g_N\mathbf{n}$
SIPG	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\rm ip}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\rm ip}(u_h)$	u_h	$g_N\mathbf{n}$
BR2	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\rm br2}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\rm br2}(u_h)$	u_h	$g_N\mathbf{n}$

where $\{\!\{u_h\}\!\} = \frac{1}{2}(u_h^+ + u_h^-)$, $[\![u_h]\!] = u_h^+\mathbf{n}_h^+ + u_h^-\mathbf{n}^-$.

- Discretization is consistent if $\hat{u}(v) = v$ and $\hat{\boldsymbol{\sigma}}(v, \nabla v) = \nabla v$ for smooth v
- Discretization is adjoint consistent if $\hat{u}_h$ and $\hat{\boldsymbol{\sigma}}_h$ single-valued

Derivation of various DG discretization methods

	on $\Gamma_\mathcal{I}$		on Γ_D		on Γ_N	
	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$
BO	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\}$	$2u_h - g_D$	$\nabla_h u_h$	u_h	$g_N \mathbf{n}$
NIPG	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	$2u_h - g_D$	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
SIPG	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
BR2	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{br2}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{br2}}(u_h)$	u_h	$g_N \mathbf{n}$

- Discretization is consistent if $\hat{u}(v) = v$ and $\hat{\boldsymbol{\sigma}}(v, \nabla v) = \nabla v$ for smooth v
- Discretization is adjoint consistent if $\hat{u}_h$ and $\hat{\boldsymbol{\sigma}}_h$ single-valued

Derivation of various DG discretization methods

	on $\Gamma_{\mathcal{I}}$		on Γ_D		on Γ_N	
	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$	$\hat{u}_h$	$\hat{\boldsymbol{\sigma}}_h$
BO	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\}$	$2u_h - g_D$	$\nabla_h u_h$	u_h	$g_N \mathbf{n}$
NIPG	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	$2u_h - g_D$	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
SIPG	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
BR2	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{br2}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{br2}}(u_h)$	u_h	$g_N \mathbf{n}$

- Discretization is consistent if $\hat{u}(v) = v$ and $\hat{\boldsymbol{\sigma}}(v, \nabla v) = \nabla v$ for smooth v
- Discretization is adjoint consistent if $\hat{u}_h$ and $\hat{\boldsymbol{\sigma}}_h$ single-valued

Assume that $\boldsymbol{\delta}^{\mathrm{ip}}(v) = \boldsymbol{\delta}^{\mathrm{br2}}(v) = \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(v) = \boldsymbol{\delta}_\Gamma^{\mathrm{br2}}(v) = 0$ for smooth functions v and $\boldsymbol{\delta}^{\mathrm{ip}}(u_h)$, $\boldsymbol{\delta}^{\mathrm{br2}}(u_h)$ single-valued

Derivation of various DG discretization methods

	on $\Gamma_{\mathcal{I}}$ $\hat{u}_h$	on $\Gamma_{\mathcal{I}}$ $\hat{\boldsymbol{\sigma}}_h$	on Γ_D $\hat{u}_h$	on Γ_D $\hat{\boldsymbol{\sigma}}_h$	on Γ_N $\hat{u}_h$	on Γ_N $\hat{\boldsymbol{\sigma}}_h$
BO	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\}$	$2u_h - g_D$	$\nabla_h u_h$	u_h	$g_N \mathbf{n}$
NIPG	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	$2u_h - g_D$	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
SIPG	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
BR2	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{br2}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{br2}}(u_h)$	u_h	$g_N \mathbf{n}$

- Discretization is consistent if $\hat{u}(v) = v$ and $\hat{\boldsymbol{\sigma}}(v, \nabla v) = \nabla v$ for smooth v
- Discretization is adjoint consistent if $\hat{u}_h$ and $\hat{\boldsymbol{\sigma}}_h$ single-valued

Assume that $\boldsymbol{\delta}^{\mathrm{ip}}(v) = \boldsymbol{\delta}^{\mathrm{br2}}(v) = \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(v) = \boldsymbol{\delta}_\Gamma^{\mathrm{br2}}(v) = 0$ for smooth functions v and $\boldsymbol{\delta}^{\mathrm{ip}}(u_h)$, $\boldsymbol{\delta}^{\mathrm{br2}}(u_h)$ single-valued

method of Baumann-Oden	BO	consistent	adjoint **in**consistent
non-sym. interior penalty Galerkin	NIPG	consistent	adjoint **in**consistent
symmetric interior penalty Galerkin	SIPG	consistent	adjoint consistent
2nd scheme of Bassi & Rebay	BR2	consistent	adjoint consistent

Baumann-Oden, symmetric and non-symmetric interior penalty

Choose the interior penalty term:

$$\boldsymbol{\delta}^{\mathrm{ip}}(u_h) = \delta [\![u_h]\!] \quad \text{on } \Gamma_{\mathcal{I}}, \qquad \boldsymbol{\delta}_{\Gamma}^{\mathrm{ip}}(u_h) = \delta \left(u_h - g_D \right) \mathbf{n} \quad \text{on } \Gamma_D$$

Baumann-Oden, symmetric and non-symmetric interior penalty

Choose the interior penalty term:

$$\boldsymbol{\delta}^{\mathrm{ip}}(u_h) = \delta[\![u_h]\!] \quad \text{on } \Gamma_{\mathcal{I}}, \qquad \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h) = \delta\,(u_h - g_D)\,\mathbf{n} \quad \text{on } \Gamma_D$$

Find $u_h \in V_h^p$ such that

$$L_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h^p, \qquad \text{where}$$

$$\begin{aligned} L_h(u,v) = &\int_\Omega \nabla_h u \cdot \nabla_h v \,\mathrm{d}\mathbf{x} \\ &+ \int_{\Gamma_{\mathcal{I}} \cup \Gamma_D} (\theta[\![u]\!] \cdot \{\!\{\nabla_h v\}\!\} - \{\!\{\nabla_h u\}\!\} \cdot [\![v]\!])\,\mathrm{d}s + \int_{\Gamma_{\mathcal{I}} \cup \Gamma_D} \delta[\![u]\!] \cdot [\![v]\!]\,\mathrm{d}s, \end{aligned}$$

$$F_h(v) = \int_\Omega f v \,\mathrm{d}\mathbf{x} + \int_{\Gamma_D} \theta g_D\,\mathbf{n} \cdot \nabla v \,\mathrm{d}s + \int_{\Gamma_D} \delta g_D v \,\mathrm{d}s + \int_{\Gamma_N} g_N v \,\mathrm{d}s,$$

with

method of Baumann-Oden	BO	$\theta = 1$	$\delta = 0$
non-sym. interior penalty Galerkin	NIPG	$\theta = 1$	$\delta > 0$
symmetric interior penalty Galerkin	SIPG	$\theta = -1$	$\delta > 0$

2nd method of Bassi & Rebay

Choose the penalization term:

$$\boldsymbol{\delta}^{\mathrm{br2}}(u_h) = \boldsymbol{\delta}_{\Gamma}^{\mathrm{br2}}(u_h) = -C_{\mathrm{BR2}}\{\!\{\mathbf{L}_{g_D}^e(u_h)\}\!\} \quad \text{for } e \subset \Gamma_{\mathcal{I}} \cup \Gamma_D,$$

where the so-called *local lifting operator* including Dirichlet bc's is given by: $\mathbf{L}_{g_D}^e(w) \in \mathbf{\Sigma}_h^p$ is the solution to

$$\int_\Omega \mathbf{L}_{g_D}^e(w) \cdot \tau \, \mathrm{d}\mathbf{x} = \int_e (w - g_D)\, \mathbf{n} \cdot \tau \, \mathrm{d}s \quad \forall \tau \in \mathbf{\Sigma}_h^p, \qquad \text{for } e \subset \Gamma_D$$

$$\int_\Omega \mathbf{L}_{g_D}^e(w) \cdot \tau \, \mathrm{d}\mathbf{x} = \int_e [\![w]\!] \cdot \{\!\{\tau\}\!\} \, \mathrm{d}s \quad \forall \tau \in \mathbf{\Sigma}_h^p, \qquad \text{on } e \subset \Gamma_{\mathcal{I}},$$

and $\mathbf{L}_{g_D}^e(w)$ is defined to be zero for $e \subset \Gamma_N$.

Coercivity and stability

- Method of Baumann-Oden (BO):

$$L_h(v_h, v_h) = \|\nabla_h v_h\|^2_{L^2(\Omega)} \qquad \forall v_h \in V_h^p.$$

But $L_h(v_h, v_h) = 0$ for $v_h \in V_h^0$ and $v_h \not\equiv 0$, i.e. BO is **unstable**.

Coercivity and stability

- Method of Baumann-Oden (BO):

$$L_h(v_h, v_h) = \|\nabla_h v_h\|^2_{L^2(\Omega)} \qquad \forall v_h \in V_h^p.$$

But $L_h(v_h, v_h) = 0$ for $v_h \in V_h^0$ and $v_h \not\equiv 0$, i.e. BO is **unstable**.

- (Non-)Symmetric interior penalty Galerkin (NIPG and SIPG) with $\delta = C_{\mathrm{IP}} \frac{p^2}{h}$:

$$L_h(v_h, v_h) \geq \gamma |||v_h|||^2_\delta \qquad \forall v_h \in V_h^p.$$

NIPG **stable** for $C_{\mathrm{IP}} > 0$ and SIPG **stable** for $C_{\mathrm{IP}} > C_{\mathrm{IP}}^0 > 0$.
For C_{IP}^0 see (Shahbazi, 2005; Hillewaert, 2013).

Coercivity and stability

- Method of Baumann-Oden (BO):

$$L_h(v_h, v_h) = \|\nabla_h v_h\|^2_{L^2(\Omega)} \qquad \forall v_h \in V_h^p.$$

But $L_h(v_h, v_h) = 0$ for $v_h \in V_h^0$ and $v_h \not\equiv 0$, i.e. BO is **unstable**.

- (Non-)Symmetric interior penalty Galerkin (NIPG and SIPG) with $\delta = C_{\mathrm{IP}} \frac{p^2}{h}$:

$$L_h(v_h, v_h) \geq \gamma |||v_h|||^2_\delta \qquad \forall v_h \in V_h^p.$$

NIPG **stable** for $C_{\mathrm{IP}} > 0$ and SIPG **stable** for $C_{\mathrm{IP}} > C^0_{\mathrm{IP}} > 0$.
For C^0_{IP} see (Shahbazi, 2005; Hillewaert, 2013).

- 2nd method of Bassi & Rebay (BR2):

$$L_h(v_h, v_h) \geq \gamma |||v_h|||^2_{L^e_0} \qquad \forall v_h \in V_h^p.$$

BR2 **stable** for $C_{\mathrm{BR2}} > C^0_{\mathrm{BR2}}$ where C^0_{BR2} is the number of faces of an element ($C^0_{\mathrm{BR2}} = 3$ on triangles, $C^0_{\mathrm{BR2}} = 4$ on quadrilaterals).

A priori error estimate in DG-norm for NIPG and SIPG

Lemma 4.12: Let $u \in H^{p+1}(\Omega)$ be the exact solution to Poisson's equation. Furthermore, let $u_h \in V_h^p$ be the solution to

$$L_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h^p,$$

for NIPG ($\theta = 1$) and for SIPG($\theta = -1$), with $\delta = C_{\mathrm{IP}} \frac{p^2}{h}$, $C_{\mathrm{IP}} > C_{\mathrm{IP}}^0$. Then

$$\boxed{|||u - u_h|||_\delta \le C h^p |u|_{H^{p+1}(\Omega)}}$$

where $|||\cdot|||_\delta^2$ is the norm as defined in

$$|||v|||_\delta^2 = \|\nabla_h v\|_{L^2(\Omega)}^2 + \int_{\Gamma_{\mathcal{I}} \cup \Gamma_D} \delta^{-1} \left(\mathbf{n} \cdot \{\!\{\nabla v\}\!\}\right)^2 \, \mathrm{d}s + \int_{\Gamma_{\mathcal{I}} \cup \Gamma_D} \delta [v]^2 \, \mathrm{d}s.$$

Thereby, the discretization error of the NIPG and SIPG method in the H^1-norm behaves like $\mathcal{O}(h^p)$.

Example: Model problem

Consider $\Omega = (0,1)^2$ and Poisson's equation with forcing function f such that

$$u(\mathbf{x}) = \sin(\tfrac{1}{2}\pi x_1)\sin(\tfrac{1}{2}\pi x_2).$$

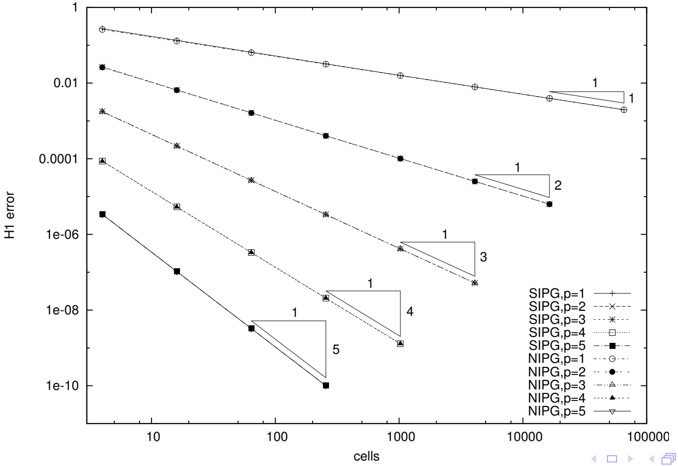
Dirichlet boundary conditions are based on the exact solution u.

Example: Model problem

Consider $\Omega = (0,1)^2$ and Poisson's equation with forcing function f such that

$$u(\mathbf{x}) = \sin(\tfrac{1}{2}\pi x_1)\sin(\tfrac{1}{2}\pi x_2).$$

Dirichlet boundary conditions are based on the exact solution u.



The H^1-error of the NIPG and SIPG methods with $p = 1, \ldots, 5$, behaves like $\mathcal{O}(h^p)$

A priori error estimate in DG-norm for NIPG and SIPG

Lemma 4.12: Let $u \in H^{p+1}(\Omega)$ be the exact solution to Poisson's equation. Furthermore, let $u_h \in V_h^p$ be the solution to

$$L_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h^p,$$

Then, for NIPG ($\theta = 1$):

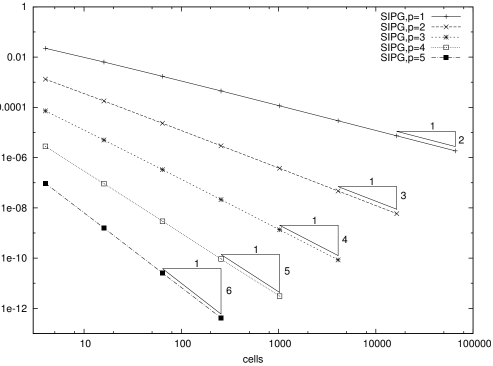
$$\|u - u_h\|_{L^2(\Omega)} \le Ch^p |u|_{H^{p+1}(\Omega)},$$

and for SIPG ($\theta = -1$):

$$\|u - u_h\|_{L^2(\Omega)} \le Ch^{p+1} |u|_{H^{p+1}(\Omega)}.$$

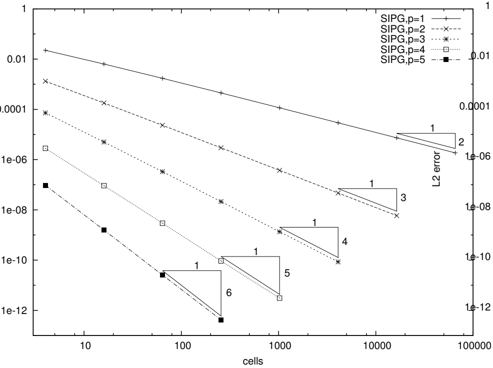
Due to adjoint consistency, the discretization error in L^2 of the SIPG method, $\mathcal{O}(h^{p+1})$, is one order higher than that of the NIPG method, $\mathcal{O}(h^p)$.

Example: Model problem

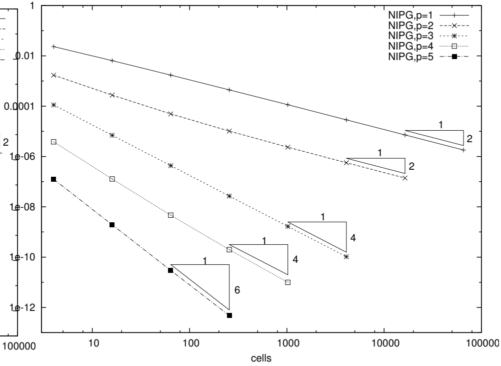


The L^2-error of the SIPG method with $p = 1, \dots, 5$, behaves like $\mathcal{O}(h^{p+1})$

Example: Model problem

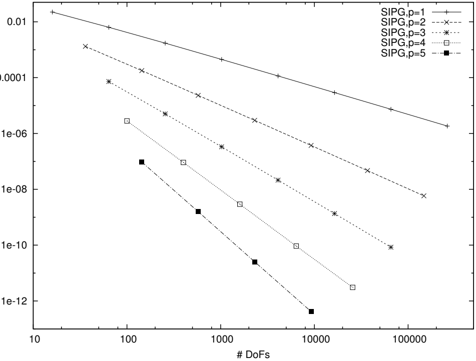


The L^2-error of the SIPG method with $p = 1, \dots, 5$, behaves like $\mathcal{O}(h^{p+1})$



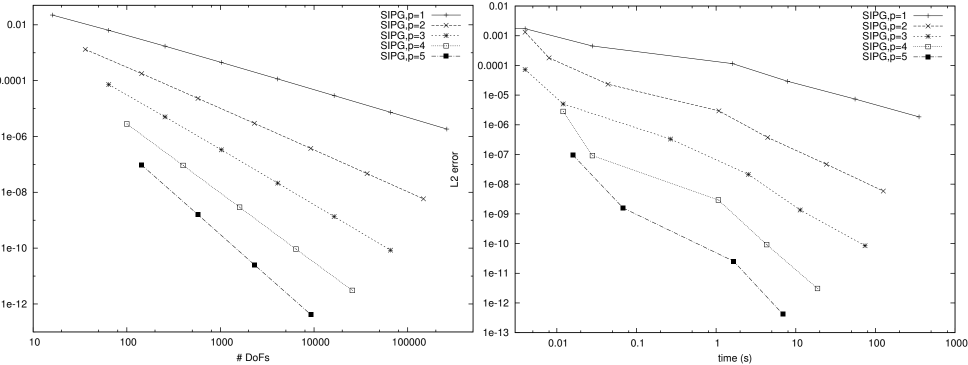
The L^2-error of the NIPG method with $p = 1, \dots, 5$, behaves like $\mathcal{O}(h^{p+1})$ for odd p and like $\mathcal{O}(h^{p})$ for even p

Example: Model problem, computational effort



The L^2-error of the SIPG method against the number of degrees of freedom (DoFs)

Example: Model problem, computational effort



The L^2-error of the SIPG method against the number of degrees of freedom (DoFs)

The L^2-error of the SIPG method against the computing time in seconds

Outline

A priori error estimates for target functionals $J(\cdot)$

Given an adjoint consistent discretization (e.g. SIPG): Find $u_h \in V_h^p$ such that

$$L_h(u_h, v_h) = F_h(v_h) \qquad v_h \in V_h^p.$$

Note, that L_h is continuous (cf. Appendix A.4.2):

$$L_h(w, v) \leq C_B |||w||| \; |||v||| \quad \forall w, v \in V.$$

Furthermore, we have following *a priori* error estimate:

$$|||u - u_h|||_\delta \leq Ch^p |u|_{H^{p+1}(\Omega)} \quad \forall u \in H^{p+1}(\Omega).$$

and following approximation estimate for the L^2-projection P_h^p:

$$|||v - P_h^p v|||_\delta \leq Ch^p |v|_{H^{p+1}(\Omega)} \qquad \forall v \in H^{p+1}(\Omega).$$

Let $z \in V$ be the solution to the adjoint problem. Due to **adjoint consistency** we have $L_h(w, z) = J_h(w)$ for all $w \in V$. Thus, for $|J(u) - J_h(u_h)| = |J_h(e)|$ we have

$$\begin{aligned} |J_h(e)| &= |L_h(e, z)| = |L_h(u - u_h, z - P_h^p z)| \leq C |||u - u_h||| \; |||z - P_h^p z||| \\ &\leq Ch^p |u|_{H^{p+1}(\Omega)} Ch^p |z|_{H^{p+1}(\Omega)} = Ch^{2p} |u|_{H^{p+1}(\Omega)} |z|_{H^{p+1}(\Omega)} \qquad \forall u \in H^{p+1}(\Omega), \end{aligned}$$

i.e., the error $|J(u) - J_h(u_h)|$ is of order $\mathcal{O}(h^{2p})$.

A priori error estimates for target functionals $J(\cdot)$

Same situation as before. But now consider a discretization which in combination with the discretized target functional $J_h(\cdot)$ is **adjoint inconsistent**.

Then the solution z to the adjoint problem does **not** satisfy

$$L_h(w, z) = J_h(w) \qquad \forall w \in V.$$

A priori error estimates for target functionals $J(\cdot)$

Same situation as before. But now consider a discretization which in combination with the discretized target functional $J_h(\cdot)$ is **adjoint inconsistent**.

Then the solution z to the adjoint problem does **not** satisfy

$$L_h(w, z) = J_h(w) \qquad \forall w \in V.$$

Instead define the solution ψ to following **mesh-dependent adjoint problem**:

$$L_h(w, \psi) = J_h(w) \quad \forall w \in V.$$

ψ is mesh-dependent and not smooth. We obtain

$$\begin{aligned} |J_h(e)| = |L_h(e, \psi)| = |L_h(u - u_h, \psi - P_h^p \psi)| &\le C |||u - u_h|||\, |||\psi - P_h^p \psi||| \\ &\le C h^p |u|_{H^{p+1}(\Omega)}, \end{aligned}$$

i.e., the error $|J(u) - J_h(u_h)|$ is of order $\mathcal{O}(h^p)$.

A priori error estimates for target functionals $J(\cdot)$

Target quantity which is compatible with Poisson's equation:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D \, \mathbf{n} \cdot \nabla u \, \mathrm{d}s + \int_{\Gamma_N} j_N \, u \, \mathrm{d}s,$$

- SIPG discretization in combination with

$$\begin{aligned} J_h(u_h) &= \int_\Omega j_\Omega \, u_h \, \mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D \, \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n} \, \mathrm{d}s + \int_{\Gamma_N} j_N \, \hat{u}_h \, \mathrm{d}s \\ &= \int_\Omega j_\Omega \, u_h \, \mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D \left(\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h) \right) \cdot \mathbf{n} \, \mathrm{d}s + \int_{\Gamma_N} j_N \, u_h \, \mathrm{d}s \\ &= J(u_h) - \int_{\Gamma_D} j_D \, \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h) \cdot \mathbf{n} \, \mathrm{d}s = J(u_h) - \int_{\Gamma_D} j_D \, \delta \, (u_h - g_D) \, \mathrm{d}s \end{aligned}$$

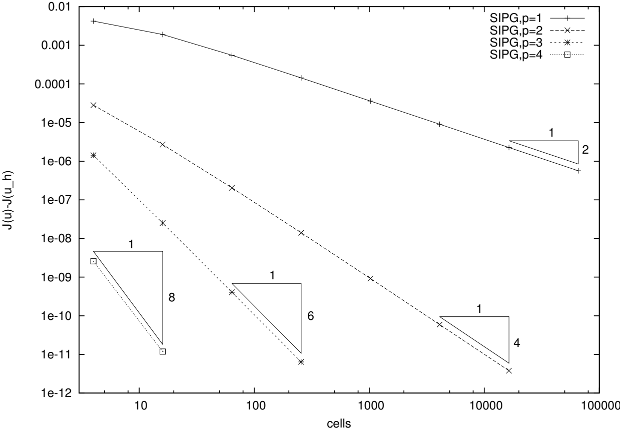
  is **adjoint consistent**. Thereby, $|J(u) - J_h(u_h)|$ is of order $\mathcal{O}(h^{2p})$.
- SIPG discretization with $J(u_h)$ is **adjoint inconsistent**. Thereby, $\mathcal{O}(h^p)$.
- NIPG discretization is **adjoint inconsistent**. Thereby, $\mathcal{O}(h^p)$.

Example 1: Model problem with SIPG

Dirichlet problem of Poisson's equation on $(0,1)^2$. Consider the target quantity

$$J_1(u) = \int_\Omega j_\Omega \, u \, d\mathbf{x}, \qquad \text{with} \quad j_\Omega(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2) \quad \text{on } \Omega$$

This target quantity is **compatible** with the model problem.



SIPG discretization of Poisson's equation:

The error $|J_1(u) - J_1(u_h)|$ of the DG(p), $p = 1, \ldots, 4$, discretization is of $\mathcal{O}(h^{2p})$

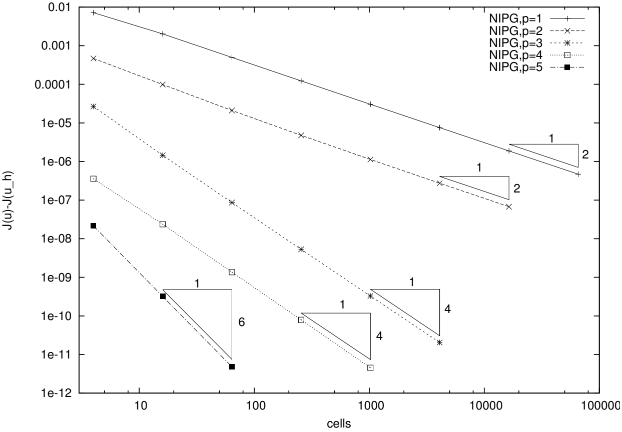
adjoint consistent

Example 1: Model problem with NIPG

Dirichlet problem of Poisson's equation on $(0,1)^2$. Consider the target quantity

$$J_1(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x}, \qquad \text{with} \quad j_\Omega(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2) \quad \text{on } \Omega$$

This target quantity is **compatible** with the model problem.



NIPG discretization of Poisson's equation:

The error $|J_1(u) - J_1(u_h)|$ of the DG(p), $p = 1, \ldots, 5$, discretization is of $\mathcal{O}(h^{p+1})$ for odd p and of $\mathcal{O}(h^p)$ for even p

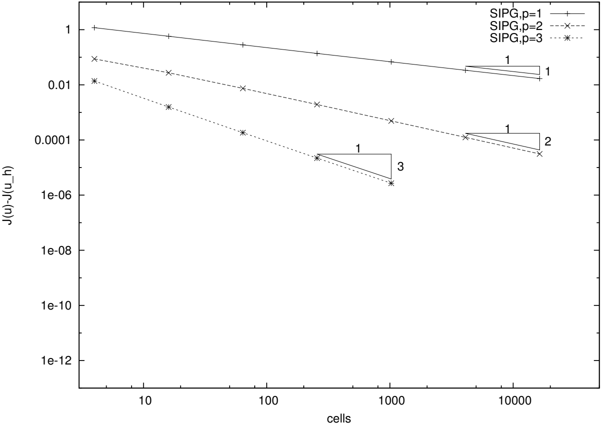
adjoint inconsistent

Example 2: Model problem with SIPG but adjoint inconsistent

Dirichlet problem of Poisson's equation on $(0,1)^2$. Consider the target quantity

$$J_2(u) = \int_\Gamma j_D \, \mathbf{n} \cdot \nabla_h u \, ds, \qquad \text{with} \quad j_D \equiv 1 \quad \text{on } \Gamma_D = \Gamma$$

This target quantity is also **compatible** with the model problem.



SIPG discretization of Poisson's equation:

The error $|J_2(u) - J_2(u_h)|$ of the DG(p), $p = 1, \ldots, 3$, discretization is of $\mathcal{O}(h^p)$

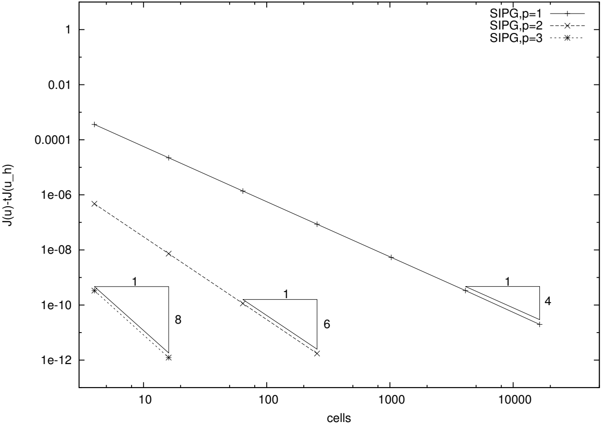
adjoint inconsistent

Example 2: Model problem with SIPG and adjoint consistent

Dirichlet problem of Poisson's equation on $(0,1)^2$. Consider

$$J_{2,h}(u_h) = \int_\Gamma j_D \, \mathbf{n} \cdot \nabla_h u_h \, ds - \int_{\Gamma_D} \delta(u_h - g_D) j_D \, ds \quad \text{with} \quad j_D \equiv 1 \quad \text{on } \Gamma_D = \Gamma$$

which is a consistent discretization of $J_2(u)$.



SIPG discretization of Poisson's equation:

The error $|J_2(u) - J_{2,h}(u_h)|$ of the DG(p), $p = 1, \ldots, 3$, discretization behaves like $\mathcal{O}(h^{2(p+1)})$

adjoint consistent

of even higher order than the expected $\mathcal{O}(h^{2p})$

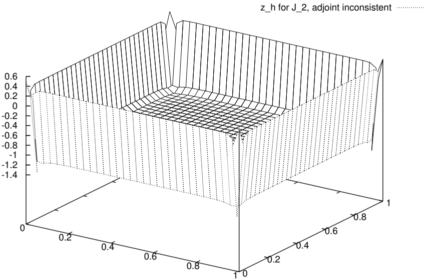
Example 2: Smoothness of the discrete adjoint solution

The exact solution to the adjoint problem

$$-\Delta z = 0 \quad \text{in } \Omega, \qquad\qquad -z = j_D \quad \text{on } \Gamma_D$$

with $j_D \equiv 1$ is given by $z \equiv -1$ on Ω.

Using the SIPG discretization in combination with $J_2(u_h)$ and $J_{2,h}(u_h)$:



discrete adjoint solution z_h
connected to $J_2(u_h)$
adjoint inconsistent

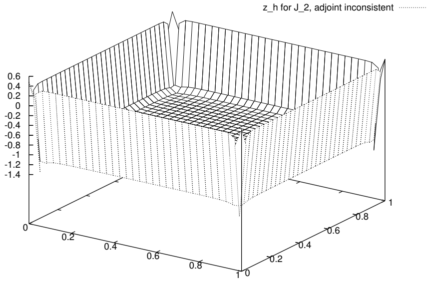
Example 2: Smoothness of the discrete adjoint solution

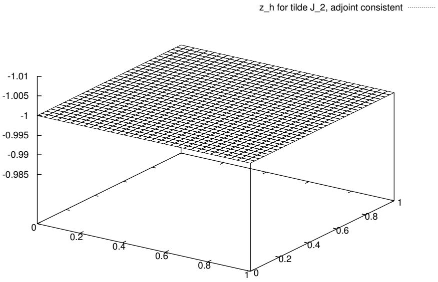
The exact solution to the adjoint problem

$$-\Delta z = 0 \quad \text{in } \Omega, \qquad\qquad -z = j_D \quad \text{on } \Gamma_D$$

with $j_D \equiv 1$ is given by $z \equiv -1$ on Ω.

Using the SIPG discretization in combination with $J_2(u_h)$ and $J_{2,h}(u_h)$:





discrete adjoint solution z_h
connected to $J_2(u_h)$
adjoint inconsistent

discrete adjoint solution z_h
connected to $J_{2,h}(u_h)$
adjoint consistent

Example 3: Another Dirichlet problem

Consider $\Omega = (0,1) \times (0.1,1)$ and Poisson's equation with forcing function f such that

$$u(\mathbf{x}) = \tfrac{1}{4}(1+x_1)^2 \sin(2\pi x_1 x_2).$$

Dirichlet boundary conditions are based on the exact solution u.
Consider the target quantity $J_3(u_h)$ and its adjoint consistent discretization $J_{3,h}(u_h)$:

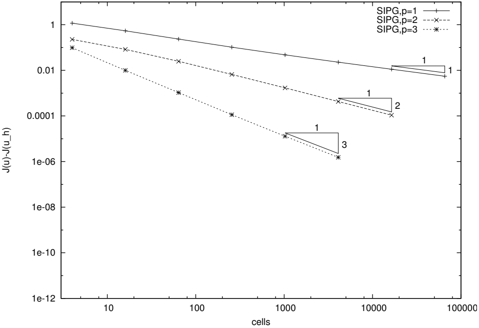
$$\begin{aligned} J_3(u_h) &= \int_\Gamma j_D\, \mathbf{n} \cdot \nabla_h u_h \, \mathrm{d}s, \\ J_{3,h}(u_h) &= J_3(u_h) - \int_\Gamma \delta (u_h - g_D) j_D \, \mathrm{d}s. \end{aligned}$$

and choose $j_D \in L^2(\Gamma)$ to be given by

$$j_D(\mathbf{x}) = \begin{cases} \exp\left(4 - \frac{1}{16}((x_1 - \frac{1}{4})^2 - \frac{1}{8})^{-2}\right) & \text{for } \mathbf{x} \in (0,\frac{1}{4}) \times (0.1,1), \\ \exp\left(4 - \frac{1}{16}((x_1 - \frac{3}{4})^2 - \frac{1}{8})^{-2}\right) & \text{for } \mathbf{x} \in (\frac{3}{4},1) \times (0.1,1), \\ 1 & \text{for } \mathbf{x} \in (\frac{1}{4},\frac{3}{4}) \times (0.1,1), \\ 0 & \text{elsewhere on } \Gamma. \end{cases}$$

Example 3: Another Dirichlet problem

Using the SIPG discretization in combination with $J_3(u_h)$ and $J_{3,h}(u_h)$:

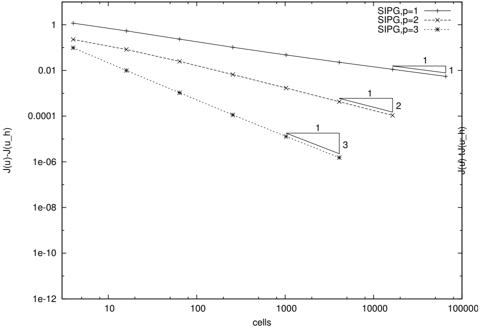


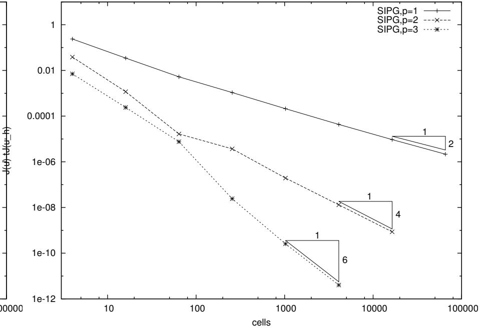
The error $|J_3(u) - J_3(u_h)|$
of the DG(p), $p = 1, \ldots, 3$,
discretization behaves like $\mathcal{O}(h^p)$

adjoint inconsistent

Example 3: Another Dirichlet problem

Using the SIPG discretization in combination with $J_3(u_h)$ and $J_{3,h}(u_h)$:





The error $|J_3(u) - J_3(u_h)|$ of the DG(p), $p = 1, \ldots, 3$, discretization behaves like $\mathcal{O}(h^p)$

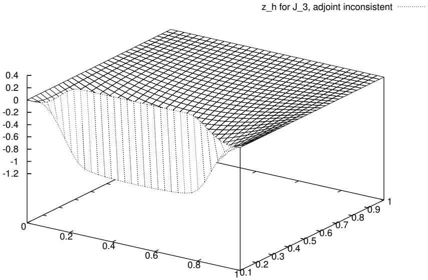
adjoint inconsistent

The error $|J_3(u) - J_{3,h}(u_h)|$ of the DG(p), $p = 1, \ldots, 3$, discretization behaves like $\mathcal{O}(h^{2p})$

adjoint consistent

Example 3: Smoothness of the discrete adjoint solution

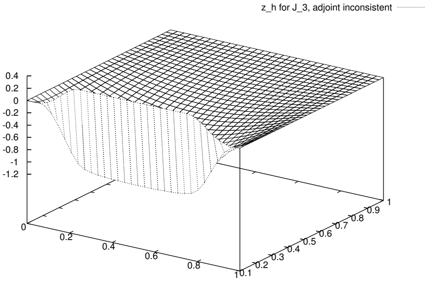
Using the SIPG discretization in combination with $J_3(u_h)$ and $J_{3,h}(u_h)$:

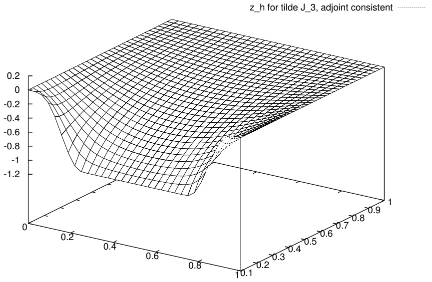


discrete adjoint solution z_h
connected to $J_3(u_h)$
adjoint inconsistent

Example 3: Smoothness of the discrete adjoint solution

Using the SIPG discretization in combination with $J_3(u_h)$ and $J_{3,h}(u_h)$:





discrete adjoint solution z_h
connected to $J_3(u_h)$
adjoint inconsistent

discrete adjoint solution z_h
connected to $J_{3,h}(u_h)$
adjoint consistent

Outline

Adjoint consistency: Questions covered today

- What is adjoint consistency?

Adjoint consistency: Questions covered today

- What is adjoint consistency?
  For an adjoint consistent discretization the discrete adjoint problem is a consistent discretization of the continuous adjoint problem.

Adjoint consistency: Questions covered today

- What is adjoint consistency?
  For an adjoint consistent discretization the discrete adjoint problem is a consistent discretization of the continuous adjoint problem.
- Take the adjoint of the discretized primal equations or discretize the continuous adjoint equations?

Adjoint consistency: Questions covered today

- What is adjoint consistency?
  For an adjoint consistent discretization the discrete adjoint problem is a consistent discretization of the continuous adjoint problem.
- Take the adjoint of the discretized primal equations or discretize the continuous adjoint equations?
  For an adjoint consistent discretization both approaches might lead to the same.

Adjoint consistency: Questions covered today

- What is adjoint consistency?
  For an adjoint consistent discretization the discrete adjoint problem is a consistent discretization of the continuous adjoint problem.
- Take the adjoint of the discretized primal equations or
  discretize the continuous adjoint equations?
  For an adjoint consistent discretization both approaches might lead to the same.
- What is the effect of adjoint consistency?

Adjoint consistency: Questions covered today

- What is adjoint consistency?
  For an adjoint consistent discretization the discrete adjoint problem is a consistent discretization of the continuous adjoint problem.
- Take the adjoint of the discretized primal equations or discretize the continuous adjoint equations?
  For an adjoint consistent discretization both approaches might lead to the same.
- What is the effect of adjoint consistency?
  Smooth adjoint solution. Improved order of convergence in L^2 and $J(\cdot)$.
- Can one derive an adjoint consistent discretization for any target quantity?

Adjoint consistency: Questions covered today

- What is adjoint consistency?
  For an adjoint consistent discretization the discrete adjoint problem is a consistent discretization of the continuous adjoint problem.
- Take the adjoint of the discretized primal equations or discretize the continuous adjoint equations?
  For an adjoint consistent discretization both approaches might lead to the same.
- What is the effect of adjoint consistency?
  Smooth adjoint solution. Improved order of convergence in L^2 and $J(\cdot)$.
- Can one derive an adjoint consistent discretization for any target quantity?
  No, only for target quantities which are *compatible* with the equations.

Adjoint consistency: Questions covered today

- So, what is a compatible target quantity?

Adjoint consistency: Questions covered today

- So, what is a compatible target quantity? Compatibility condition

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma,$$

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s$$

Adjoint consistency: Questions covered today

- So, what is a compatible target quantity? Compatibility condition

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma,$$

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s$$

  - For the linear advection equation:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_+} j_\Gamma \, u \, \mathrm{d}s.$$

Adjoint consistency: Questions covered today

- So, what is a compatible target quantity? Compatibility condition

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma,$$

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s$$

  - For the linear advection equation:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_+} j_\Gamma \, u \, \mathrm{d}s.$$

  - For Poisson's equation

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D \, \mathbf{n} \cdot \nabla u \, \mathrm{d}s + \int_{\Gamma_N} j_N \, u \, \mathrm{d}s.$$

Adjoint consistency: Questions covered today

- So, what is a compatible target quantity? Compatibility condition

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma,$$

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s$$

  - For the linear advection equation:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_+} j_\Gamma \, u \, \mathrm{d}s.$$

  - For Poisson's equation

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D \, \mathbf{n} \cdot \nabla u \, \mathrm{d}s + \int_{\Gamma_N} j_N \, u \, \mathrm{d}s.$$

- Given a discretization and a (compatible) target quantity. Does any discretization of the target quantity give an adjoint consistent discretization?

Adjoint consistency: Questions covered today

- So, what is a compatible target quantity? Compatibility condition

$$(Lu, z)_\Omega + (Bu, C^*z)_\Gamma = (u, L^*z)_\Omega + (Cu, B^*z)_\Gamma,$$

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s$$

  - For the linear advection equation:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_+} j_\Gamma \, u \, \mathrm{d}s.$$

  - For Poisson's equation

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D \, \mathbf{n} \cdot \nabla u \, \mathrm{d}s + \int_{\Gamma_N} j_N \, u \, \mathrm{d}s.$$

- Given a discretization and a (compatible) target quantity. Does any discretization of the target quantity give an adjoint consistent discretization? No. There may be arbitrarily many different consistent discretizations of the target quantity but only *one* may give an adjoint consistent discretization.

Adjoint consistency: Questions covered today

- So, what is a compatible target quantity? Compatibility condition

$$(Lu, z)_\Omega + (Bu, C^* z)_\Gamma = (u, L^* z)_\Omega + (Cu, B^* z)_\Gamma,$$

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, Cu \, \mathrm{d}s$$

  - For the linear advection equation:

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_+} j_\Gamma \, u \, \mathrm{d}s.$$

  - For Poisson's equation

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_{\Gamma_D} j_D \, \mathbf{n} \cdot \nabla u \, \mathrm{d}s + \int_{\Gamma_N} j_N \, u \, \mathrm{d}s.$$

- Given a discretization and a (compatible) target quantity. Does any discretization of the target quantity give an adjoint consistent discretization? No. There may be arbitrarily many different consistent discretizations of the target quantity but only *one* may give an adjoint consistent discretization.

Outline

Questions covered in next lecture

Adjoint consistency:

- What is a compatible target quantity . . . ?
  - . . . for the compressible Euler equations?
    The pressure-induced drag, lift and moment coefficients.

Questions covered in next lecture

Adjoint consistency:

- What is a compatible target quantity ...?
  - ...for the compressible Euler equations?
    The pressure-induced drag, lift and moment coefficients.
  - ...the compressible Navier-Stokes equations?

Questions covered in next lecture

Adjoint consistency:

- What is a compatible target quantity . . . ?
  - . . . for the compressible Euler equations?
    The pressure-induced drag, lift and moment coefficients.
  - . . . the compressible Navier-Stokes equations?
    The total drag, lift and moment coefficients.

Questions covered in next lecture

Adjoint consistency:

- What is a compatible target quantity . . . ?
  - . . . for the compressible Euler equations?
    The pressure-induced drag, lift and moment coefficients.
  - . . . the compressible Navier-Stokes equations?
    The total drag, lift and moment coefficients.
- Given a consistent DG discretization with adjoint consistent (interior) faces terms (like SIPG, BR2). For adjoint consistency: Is it possible to provide a discretization of the target quantity for any discretization of boundary terms?

Questions covered in next lecture

Error estimation and adaptivity for a compressible flow around an airfoil:

- I want to computed accurate drag, lift and moment coefficients.
  Where should I refine the mesh?
- How accurate are the drag and lift values I computed?
- I want a good resolution of the overall flow field (including e.g. vortical structures).
  Where should I refine the mesh?

Questions covered in next lecture

Error estimation and adaptivity for a compressible flow around an airfoil:

- I want to computed accurate drag, lift and moment coefficients.
  Where should I refine the mesh?
- How accurate are the drag and lift values I computed?
- I want a good resolution of the overall flow field (including e.g. vortical structures).
  Where should I refine the mesh?

To be continued...