Efficient multigrid algorithms for higher order Discontinuous Galerkin discretizations of turbulent flows

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DLR Braunschweig (AS - $C^2A^2S^2E$)

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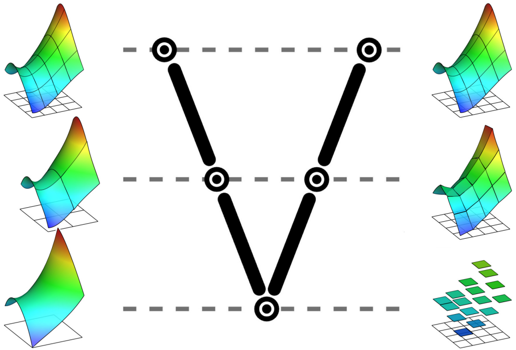
Multigrid



DG discretization

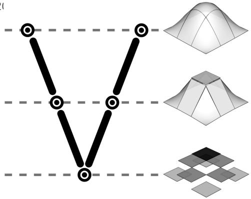
Basis functions

- non-parametric ortho-normal basis functions
- directly formulated in physical space
- also referred to as Taylor-DG
- need to be evaluated for each mesh element

RANS-$k\omega$ equations

- $k\omega$ turbulence model
- second scheme of Bassi and Rebay (BR2) for the viscous terms
- Roe flux as a convective flux

Non-linear multigrid method



nested hierarchy of linear spaces

$$\begin{array}{ccccccc} \mathbf{V}_{l_{\min}} & \subset & \mathbf{V}_{l_{\min}+1} & \subset \cdots \subset & \mathbf{V}_{l_{\max}-1} & \subset & \mathbf{V}_{l_{\max}} \\ \mathbb{R}^{n_{l_{\min}}} & & \mathbb{R}^{n_{l_{\min}+1}} & \cdots & \mathbb{R}^{n_{l_{\max}-1}} & & \mathbb{R}^{n_{l_{\max}}} \end{array}$$

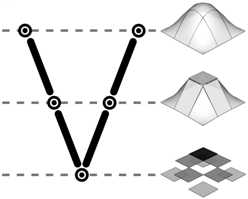
intergrid transfer operators:

- prolongation: natural injection $I_{l-1}^{l} : \mathbb{R}^{n_{l-1}} \to \mathbb{R}^{n_l}$
- canonical restriction operator $I_l^{l-1} := \left(I_{l-1}^{l}\right)^{\top}$

non-linear multigrid algorithm also requires:

- restricted non-linear state vector:
  orthogonal L^2-projection $\hat{I}_l^{l-1}$ on the space $\mathbf{V}_{l-1}$

Non-linear multigrid method



Let the non-linear problem to be solved on the fine level $l_{\max}$ be given by

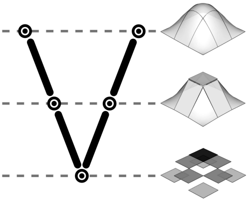
$$\mathbf{L}_{l_{\max}}(\mathbf{u}_{l_{\max}}) = \mathbf{f}_{l_{\max}}.$$

- restrict solution approximation $\mathbf{u}_{l-1} := \hat{I}_l^{l-1} \mathbf{u}_l$
- compute forcing function for the coarse level:

$$\mathbf{f}_{l-1} \leftarrow \mathbf{f}_{l-1} + I_l^{l-1} \left(\mathbf{f}_l - \mathbf{L}_l(\mathbf{u}_l)\right) - \Big(\mathbf{f}_{l-1} - \mathbf{L}_{l-1}(\mathbf{u}_{l-1}^0)\Big)$$

- Galerkin-transfer for the Jacobian: $\underline{R}_{l-1} = I_l^{l-1} \; \underline{R}_l \; I_{l-1}^l$

Non-linear smoother / solver



smoother / solver

- linearized Backward-Euler
  - Solve $\left[(\alpha_i \Delta t)^{-1}\underline{\mathrm{M}} + \underline{\mathrm{R}}_l\right](\mathbf{u}_{l,i} - \mathbf{u}_{l,i-1}) = \left[\mathbf{f}_l - \mathbf{L}_l(\mathbf{u}_{l,i-1})\right]$,
  - where $\underline{\mathrm{R}}_l$ is the fully implicit Jacobian matrix and $\underline{\mathrm{M}}$ is the mass matrix. In addition to that $\mathbf{u}_{l,j}$ is a state vector, with $\mathbf{u}_{l,j} \in \mathbf{V}_l \; \forall \; j \in \mathbb{N}$.
- local pseudo-time steps, adaptive CFL number

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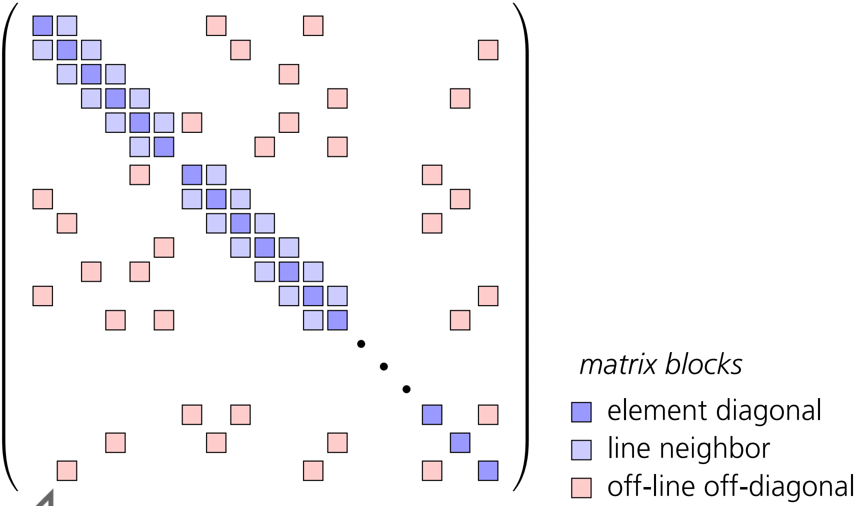
Linear smoother / solver

- Krylov method as linear solver (GMRES method)
- line-Jacobi as preconditioner / smoother
  - let $\mathcal{L}_{l,k}(\mathfrak{u}_{l,k}) = \mathfrak{f}_{l,k}$ the underlying linear problem on line k,
  - solve $\delta\mathfrak{u}_{l,k,i} := \mathfrak{u}_{l,k,i} - \mathfrak{u}_{l,k,i-1} = \mathsf{R}_{l,k}^{-1}(\mathfrak{f}_{l,k} - \mathcal{L}_{l,k}\mathfrak{u}_{l,k,i-1})$
  - set $\mathfrak{u}_{l,k,i} := \mathfrak{u}_{l,k,i-1} + \delta\mathfrak{u}_{l,k,i}$,
  - where $\mathsf{R}_{l,k}^{-1}$ is the inverse of the Jacobian matrix computed one line k in the mesh

Relaxation scheme

Jacobian / system matrix structure



Numerical algorithms

possible solver choices

- single grid Backward-Euler
- start up strategy in mesh or order sequencing for improved initial conditions
- linear MG as preconditioner
- non-linear MG to accelerate process in pseudo-time
- non-linear MG with linear MG on each level

Numerical parameters for the non-linear problems

non-linear multigrid

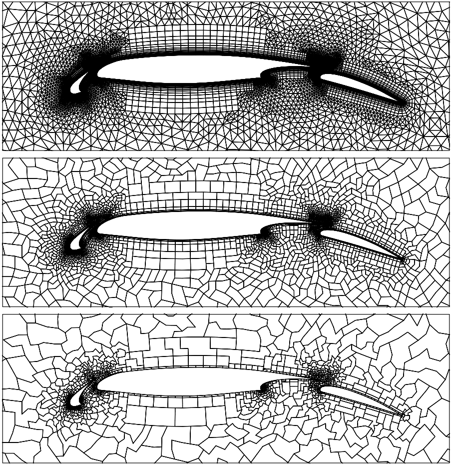
- only V-cycles will be presented
- one pre- and post-smoothing iteration on each level
- one smoothing iteration on the lowest level
- a linearized Backward-Euler scheme as smoother
- using an SER time steping scheme for the Backward-Euler
- Galerkin-transfer to obtain the Jacobian on the lower levels

Numerical parameters for the linear problems

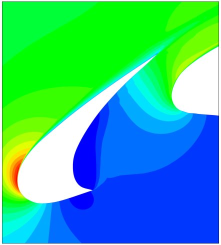
parameters for solving the resulting linear problems on every level from the Backward-Euler linearization

- GMRES method with a fixed number of max steps on every level
- linear multigrid as a preconditioner for the GMRES method
- four smoothing iterations
- line-Jacobi scheme as smoother
- Galerkin-transfer to obtain the lower level matrices

L1T2 high-lift configuration

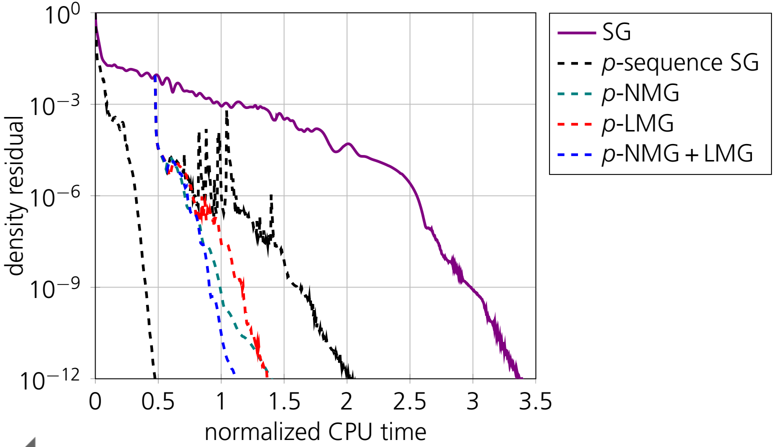


- Mach: 0.197
- Reynolds number: 3,520,000
- $\alpha = 20.18°$
- Testcase from EC funded ADIGMA project.



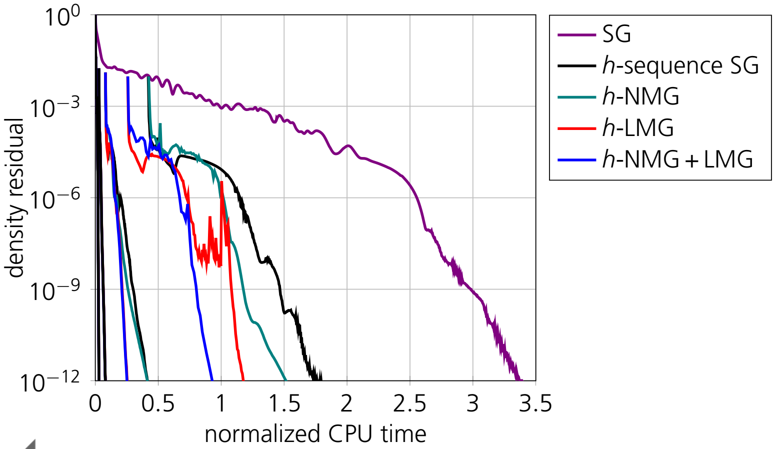
L1T2 high-lift configuration

run time comparison: *p*-MG



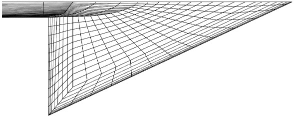
L1T2 high-lift configuration

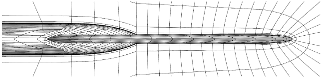
run time comparison: *h*-MG

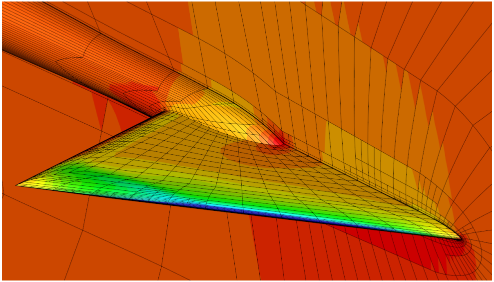


VFE-2 Delta-Wing with rounded leading edge

- Mach: 0.4
- Reynolds number: 3,000,000
- $\alpha = 13.3°, \ \beta = 0°$
- Testcase from EC funded IDIHOM project.



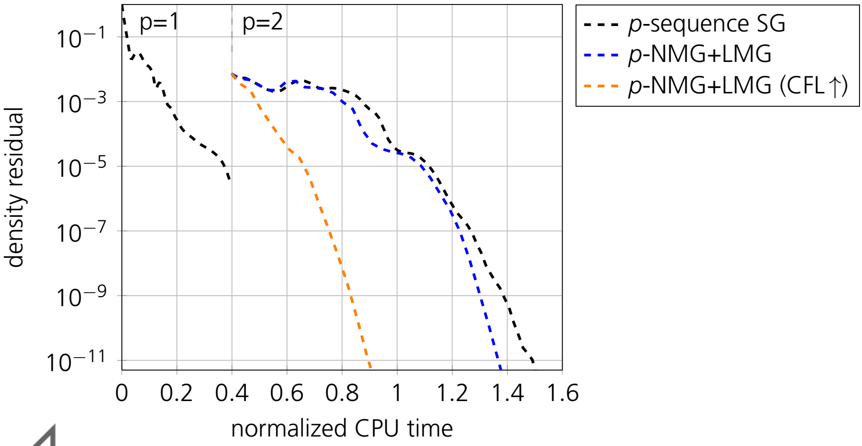




- Top: geometry/mesh
- Left: Surface pressure plot of a $p = 2$ solution on an adjoint-based refined mesh with 23877 elements.

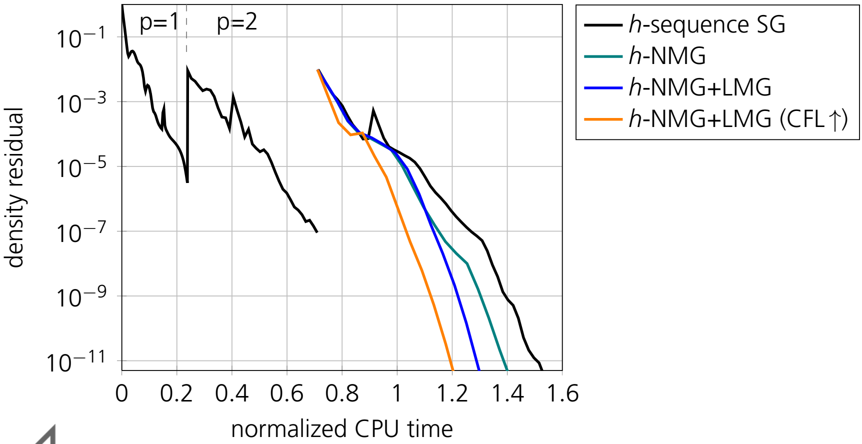
VFE2 Delta-Wing with rounded leading edge

run time comparison: *p*-MG



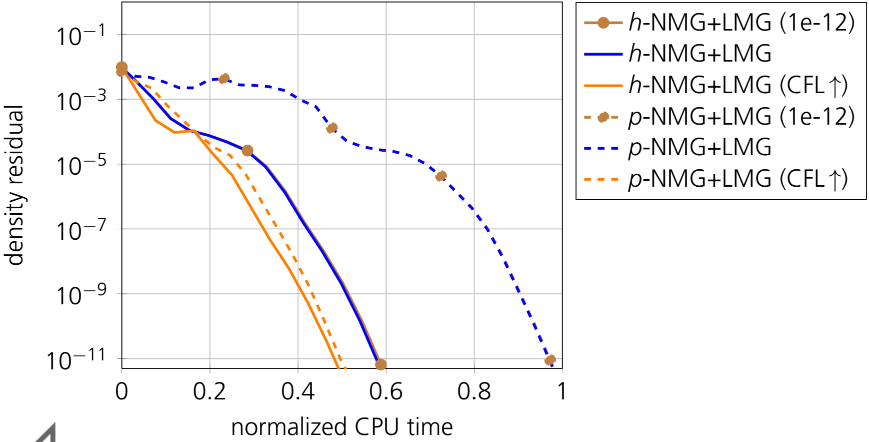
VFE2 Delta-Wing with rounded leading edge

run time comparison: *h*-MG



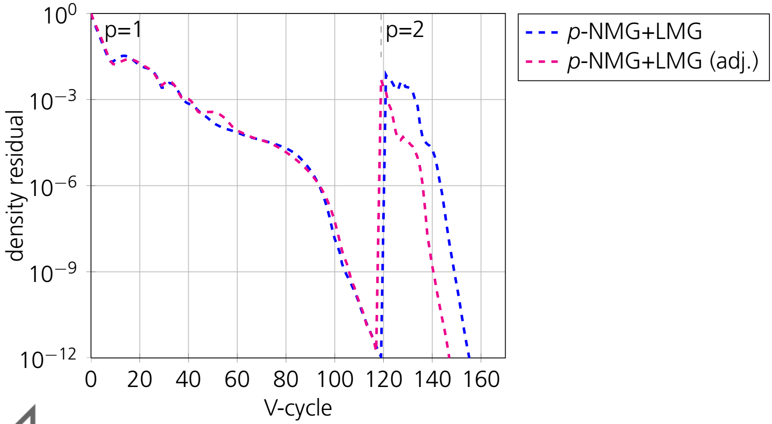
VFE2 Delta-Wing with rounded leading edge

top level time comparison



VFE-2 Delta-Wing with rounded leading edge

The *blue* computation on the mesh with 13816 elements and the *magenta* computation on an adjoint refined mesh with 23877 elements.



Conclusion

- Development of a nonlinear p-multigrid algorithm for turbulent flows (as proposed for DGHPOPT) was successful
- Additionally, we implemented
  - a linear p-multigrid as preconditioner of a Backward-Euler smoothing step,
  - and the corresponding nonlinear and linear h-multigrid algorithms based on agglomeration